Lepton and quark mixing from a stepwise breaking of flavor and CP

Claudia Hagedorn^{*} and Johannes König[†]

CP³-Origins, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark

(Received 1 June 2019; published 30 October 2019)

We explain all features of lepton and quark mixing in a scenario with the flavor symmetry $\Delta(384)$ and a charge conjugation-parity (CP) symmetry, where these are broken in several steps. The residual symmetry in the neutrino and up quark sector is a Klein group and CP , while a Z_3 and a Z_{16} symmetry are preserved among charged leptons and down quarks, respectively. If the Klein group in the neutrino sector is further broken down to a single Z_2 symmetry, we obtain predictions for all lepton mixing parameters in terms of one real quantity, whose size is determined by the value of the reactor mixing angle. The Dirac and Majorana phases are fixed, in particular $\sin \delta \approx -0.936$. A sum rule, relating these CP phases and the reactor and atmospheric mixing angles, θ_{13} and θ_{23} , is given. In the quark sector, we have for the Cabibbo angle $\theta_C = \sin \pi/16 \approx 0.195$ after the first step of symmetry breaking. This is brought into full accordance with experimental data with the second step of symmetry breaking, where either the Z_{16} group is broken to a Z_8 symmetry in the down quark sector or the Klein group to one Z_2 symmetry only among up quarks. The other two quark mixing angles are generated in the third and last symmetry breaking step, when the residual symmetries in the up and/or down quark sector are further broken. If this step occurs among both up and down quarks, the amount of CP violation in the quark sector is determined by the lepton sector and explaining the current neutrino oscillation data entails that the Jarlskog invariant J_{CP}^q is in very good agreement with experimental findings. Last, a sum rule is derived that contains the CP phase δ^q and θ_C of the quark sector and the lepton mixing parameters θ_{13} , θ_{23} , and δ .

DOI: [10.1103/PhysRevD.100.075036](https://doi.org/10.1103/PhysRevD.100.075036)

I. INTRODUCTION

Most of the free parameters in the Standard Model (SM) of particle physics are related to fermion masses and mixing. In particular, the size of the mixing angles in the quark and lepton sectors and their striking difference as well as the amount of charge conjugation-parity (CP) violation remain unexplained. Approaches with discrete, non-Abelian flavor symmetries G_f have been successfully employed in order to determine the mixing angles of leptons and quarks; for reviews on flavor symmetries, see [1–4]. Those, where different residual symmetries remain preserved in the charged lepton (down quark) and neutrino (up quark) sectors, can determine all mixing angles and the Dirac-type CP phase [5–7]. Even more predictive are approaches with a flavor and a CP symmetry [8–10], since these allow to fix all CP phases with the help of the choice of the residual symmetries. Many of the

analyses focus on the lepton sector [7,10–18] and only rather few attempt to (also) determine the mixing in the quark sector by the mismatch of residual symmetries in up and down quark sectors [5,6,19–24].

We consider as G_f the discrete group $\Delta(384)$ and a CP symmetry that corresponds to an automorphism of G_f [10,25–27]. Being primarily interested in fermion mixing in this study, we focus on the three generations of left-handed (LH) lepton doublets L_i and quark doublets Q_i , $i = 1, 2, 3$, not specifying the transformation properties of the right-handed (RH) fields. Furthermore, we assume that neutrinos are Majorana particles, whose mass is generated from the Weinberg operator. The three generations of LH lepton doublets and quark doublets both transform as a faithful, irreducible, complex three-dimensional representation **3** of $\Delta(384)$. The symmetries G_f and CP are broken by an unspecified mechanism to different residual symmetries in the neutrino and up quark sector, charged lepton as well as down quark sector. This first step of symmetry breaking leads to tri-bimaximal (TB) mixing [28,29] in the lepton sector and generates the Cabibbo angle $\theta_C = \sin \pi/16 \approx 0.195$ in the quark sector [6,11]. In a further step of symmetry breaking, where the residual symmetry among neutrinos is reduced, the reactor mixing angle is induced and all lepton mixing parameters become functions of one real quantity, whose size is determined by the reactor

^{*}hagedorn@cp3.sdu.dk

[†]konig@cp3.sdu.dk

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

mixing angle θ_{13} . Accommodating the measured value of the atmospheric mixing angle θ_{23} well and the indication for close to maximal CP violation in neutrino oscillations [30], one CP symmetry is singled out that leads to $\sin^2\theta_{23} \approx 0.579$ and $\sin\delta \approx -0.936$. For this choice of CP symmetry, the Majorana phases α and β have to fulfil $|\sin\alpha| = |\sin\beta| = 1/\sqrt{2}$. Both Majorana phases fulfil a nontrivial relation, involving the CP phase δ and the two mixing angles θ_{13} and θ_{23} . The quantity m_{ee} , measurable in neutrinoless double beta decay, becomes strongly constrained, e.g., for positive $\sin\alpha$ and $\sin\beta$ we find $m_{ee} \gtrsim 2.86 \times 10^{-3}$ eV for neutrino masses with normal ordering and for inverted ordering that m_{ee} is detectable with the next generation of experiments [31]. In the quark sector, either the residual group in the up or the down quark sector is reduced at the second step of symmetry breaking, so that the Cabibbo angle is brought into full accordance with experimental data [32]. Breaking the residual symmetries among up and/or down quarks even further eventually gives rise to the remaining two quark mixing angles. If this breaking occurs in only one of the two sectors, the amount of CP violation in quark mixing will depend on phases, that are in general not specified further by the residual symmetries. If instead the latter are broken in the down as well as the up quark sector, a strong correlation between CP violation in the quark and in the lepton sector can be established and the Jarlskog invariant J_{CP}^q [33] can be determined to be $J_{CP}^q \approx 3.29 \times 10^{-5}$, in very good agreement with experimental data [32]. This strong correlation leads to a sum rule, relating the CP phase δ^q and the Cabibbo angle θ_C in the quark sector to the lepton mixing parameters δ , θ_{13} , and θ_{23} . These symmetry breaking sequences differ in several aspects from the symmetry breaking pattern that has been proposed in [23,24] in order to describe lepton and quark mixing with the help of a flavor group $\Delta(6n^2)$, n integer, and CP .

The remainder of the paper is organized as follows: in Sec. II, we present basic information about $\Delta(384)$ and the employed CP symmetry. In Secs. III and IV, we show the results for fermion mixing, arising from the different steps of symmetry breaking, in the lepton and quark sector, respectively. We summarize and conclude in Sec. V.

II. BASICS ABOUT $\Delta(384)$ AND CP

We use as G_f the group $\Delta(384)$, which belongs to the series $\Delta(6n^2)$, n integer. This group can be written as $(Z_8 \times Z_8) \rtimes S_3$ with S_3 being the permutation group of three distinct objects. It can be described with four generators, a , b , c , and d , that fulfil the relations

$$\begin{aligned} a^3 &= e, & b^2 &= e, & c^8 &= e, & d^8 &= e, \\ (ab)^2 &= e, & cd &= dc, & aca^{-1} &= c^{-1}d^{-1}, \\ ada^{-1} &= c, & bcb^{-1} &= d^{-1}, & bdb^{-1} &= c^{-1}, \end{aligned} \quad (1)$$

with e being the neutral element of the group [34]. The group $\Delta(384)$ has several subgroups, among them Klein groups $Z_2 \times Z_2$, Z_3 and Z_{16} symmetries [11]. It possesses in total 13 Klein groups: 12 of these are conjugate to each other and can be generated by c^4 and abc^k , d^4 and a^2bd^k , and $(cd)^4$ and bc^kd^k for $k = 0, 1, 2, 3$, respectively, while the remaining one is normal and can be obtained from c^4 and d^4 as generators. There are 64 Z_3 symmetries contained in $\Delta(384)$, and these can be described by $ac^i d^j$ for $0 \leq i, j \leq 7$. The 12 Z_{16} subgroups can be generated by bd^{2l+1} , abd^{2l+1} , and a^2bc^{2l+1} for $l = 0, 1, 2, 3$. All Z_3 and Z_{16} groups are conjugate to each other, respectively. For a comprehensive list of subgroups and their properties, see [11,16].

As is known, CP symmetries are automorphisms of G_f [10,25–27] and a large class of these have been discussed in [15] for the groups $\Delta(6n^2)$. In the present study, we consider the ones corresponding to the automorphisms composed by

$$a \rightarrow a, \quad b \rightarrow b, \quad c \rightarrow c^{-1}, \quad \text{and} \quad d \rightarrow d^{-1} \quad (2)$$

and the group transformation $abc^s d^{2s}$, $0 \leq s \leq 7$.

The three generations of LH lepton doublets L_i and quark doublets Q_i , $i = 1, 2, 3$ are assigned to the same faithful, irreducible, complex three-dimensional representation **3**, which can be represented by the generators $a(\mathbf{3})$, $b(\mathbf{3})$, $c(\mathbf{3})$,

$$a(\mathbf{3}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b(\mathbf{3}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c(\mathbf{3}) = \begin{pmatrix} \omega_8 & 0 & 0 \\ 0 & \omega_8^7 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

and $d(\mathbf{3}) = a(\mathbf{3})^2 c(\mathbf{3}) a(\mathbf{3})$ with $\omega_8 = e^{2\pi i/8}$.¹ In the representation **3**, the mentioned type of CP symmetry corresponds to the CP transformation $X(\mathbf{3})$,

$$X(\mathbf{3})(s) = a(\mathbf{3})b(\mathbf{3})c(\mathbf{3})^s d(\mathbf{3})^{2s} X_0(\mathbf{3}), \quad 0 \leq s \leq 7, \quad (4)$$

with $X_0(\mathbf{3})$ representing the CP symmetry induced by the automorphism in Eq. (2) and being of the form of the identity matrix in the used basis.

Residual symmetries in the different fermion sectors of the theory are Abelian subgroups of G_f , possibly together with the CP symmetry. From their mismatch, fermion mixing arises. In particular, quark mixing is due to the mismatch of the residual group G_u in the up quark and the one, called G_d , in the down quark sector, while lepton

¹We could have chosen any of the other faithful, irreducible, complex three-dimensional representations of $\Delta(384)$. Note that this can lead to the residual symmetries having different generators in terms of the elements of the flavor group in order to achieve the same results for mixing angles and CP phases; compare also [11,12].

mixing comes from the mismatch of G_ν and G_l , the residual symmetries in the neutrino and charged lepton sector, respectively.

III. LEPTON SECTOR

In the first step of symmetry breaking, we preserve a Klein group contained in $\Delta(384)$ together with a CP symmetry, represented by a CP transformation in Eq. (4), in the neutrino sector, while a Z_3 symmetry is conserved among charged leptons,

$$G_{\nu,1} = Z_2 \times Z_2 \times CP \quad \text{and} \quad G_l = Z_3; \quad (5)$$

see Fig. 1. We choose as generators of the Klein group c^4 and abc^k and leave k as well as the parameter s , labeling the CP symmetry, unspecified for the moment. As can be checked explicitly, the generators of the Klein group and the CP transformation commute in the representation **3**, i.e.,

$$\begin{aligned} X(\mathbf{3})(s)(c(\mathbf{3})^4)^* - c(\mathbf{3})^4 X(\mathbf{3})(s) &= 0 \quad \text{and} \\ X(\mathbf{3})(s)(a(\mathbf{3})b(\mathbf{3})c(\mathbf{3})^k)^* - a(\mathbf{3})b(\mathbf{3})c(\mathbf{3})^k X(\mathbf{3})(s) &= 0, \end{aligned} \quad (6)$$

with $*$ denoting complex conjugation. This implies that also the other elements of the Klein group commute with the CP transformation. For G_l , we use the generator a . The contributions U_ν and U_l to the lepton mixing matrix U_{PMNS} , arising from the neutrino and charged lepton sector, respectively, are determined by the unitary matrices diagonalizing the generators of the residual symmetries $G_{\nu,1}$ and G_l , respectively.² They are of the form

$$\begin{aligned} U_\nu(k, s) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_{16}^{s-k} & 0 & -i\omega_{16}^{s-k} \\ \omega_{16}^{k+s} & 0 & i\omega_{16}^{k+s} \\ 0 & \sqrt{2}\omega_8^{8-s} & 0 \end{pmatrix} \quad \text{and} \\ U_l &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{pmatrix}, \end{aligned} \quad (7)$$

with $\omega_{16} = e^{2\pi i/16}$ and $\omega = e^{2\pi i/3}$, where we have chosen a particular ordering of the columns, and thus ordering of the neutrino and charged lepton masses, that can entail a viable description of the data on lepton mixing parameters.³ The form of the lepton mixing matrix is then

$$\begin{aligned} U_{\text{PMNS},1}(k, s) &= U_l^\dagger U_\nu(k, s) \\ &= \frac{1}{\sqrt{6}} \begin{pmatrix} 2\omega_{16}^s \cos \phi_k & \sqrt{2}\omega_8^{8-s} & -2\omega_{16}^s \sin \phi_k \\ -\omega_{16}^s (\cos \phi_k + \sqrt{3} \sin \phi_k) & \sqrt{2}\omega_8^{8-s} & -\omega_{16}^s (\sqrt{3} \cos \phi_k - \sin \phi_k) \\ -\omega_{16}^s (\cos \phi_k - \sqrt{3} \sin \phi_k) & \sqrt{2}\omega_8^{8-s} & \omega_{16}^s (\sqrt{3} \cos \phi_k + \sin \phi_k) \end{pmatrix} \quad \text{with} \quad \phi_k = \frac{\pi k}{8}. \end{aligned} \quad (8)$$

We see, in particular, that the sine of the reactor mixing angle θ_{13} is given by

$$\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \phi_k. \quad (9)$$

Taking into account the admitted choices of k , $k = 0, 1, 2, 3$, we find that the matrix in Eq. (8) becomes the TB mixing matrix for the choice⁴

²We note that we constrain U_ν additionally such that the CP transformation $X(\mathbf{3})(s)$ becomes the identity matrix after applying U_ν , i.e., $U_\nu^\dagger X(\mathbf{3})(s) U_\nu^*$ is the identity matrix.

³As is well known, exchanging the second and third columns of U_l leads to the atmospheric mixing angle changing octant and the Dirac phase sign.

⁴This result can be understood by considering the group arising from the combination of the generators of the Klein group and of the Z_3 symmetry, preserved among neutrinos and charged leptons, respectively, as one generates with the elements c^4 , ab , and a the group $\Delta(24)$ ($n = 2$) that is isomorphic to S_4 . The latter group is well known for leading to TB mixing [7], if a residual Klein group among neutrinos and a Z_3 symmetry in the charged lepton sector is present.

$$k = 0. \quad (10)$$

This can be considered as a reasonable result after the first step of symmetry breaking.

In the second step, the Klein group in the neutrino sector is broken to a Z_2 symmetry, generated by abc^4 , while the CP symmetry is preserved

$$G_{\nu,2} = Z_2 \times CP. \quad (11)$$

The residual symmetry among charged leptons remains untouched; see Fig. 1. In this way, the contribution to lepton mixing, arising from the neutrino sector, becomes less constrained, i.e., an additional rotation in the (23)-plane through an angle θ , undetermined by the symmetries of the theory and in the range between 0 and π , has to be taken into account. The plane of the rotation is fixed by the fact that the unbroken generator abc^4 in the representation **3** reads after applying $U_\nu(k = 0, s)$,

$$U_\nu(k = 0, s)^\dagger a(\mathbf{3})b(\mathbf{3})c(\mathbf{3})^4 U_\nu(k = 0, s) = \text{diag}(-1, 1, 1). \quad (12)$$

FIG. 1. Stepwise breaking of $\Delta(384)$ and CP . The stepwise breaking of the flavor and CP symmetry in the different sectors of the theory and the corresponding results for lepton and quark mixing are shown. There are two steps of symmetry breaking in the lepton sector, while three steps are necessary in the quark sector in order to generate all three quark mixing angles. The particular choice of the residual symmetries in the second and third steps in the quark sector corresponds to the minimal viable case that leads to a strong correlation between the amount of CP violation in the lepton and in the quark sector. The results for quark mixing angles and the Jarlskog invariant J_{CP}^q can be found in Sec. IV, while those for the lepton mixing parameters are detailed in Sec. III.

We thus arrive at a lepton mixing matrix of the form

$$U_{\text{PMNS},2}(s) = U_{\text{PMNS},1}(k=0, s) R_{23}(\theta) \quad \text{with} \quad (13)$$

$$R_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}.$$

When using $U_{\text{PMNS},2}(s)$ as lepton mixing matrix, we assume that the diagonal elements of the neutrino mass matrix are positive semidefinite after the application of $U_\nu(k=0, s) R_{23}(\theta)$. If this is not given, additional signs will appear in the Majorana invariants I_1 and I_2 , defined in Eq. (20); compare discussion in [10,15]. From $U_{\text{PMNS},2}(s)$, we read off that

$$\sin^2 \theta_{13} = \frac{1}{3} \sin^2 \theta. \quad (14)$$

Thus, the experimental data on the reactor mixing angle are accommodated well, $\sin^2 \theta_{13} \approx 0.022$ [30], for⁵

$$\theta \approx 0.26. \quad (15)$$

Also the value of the solar mixing angle is fixed, since it is strongly correlated with the reactor one

⁵We note that there exists a second solution with θ being replaced by $\pi - \theta$. This is known from preceding analyses [15].

$$\sin^2 \theta_{12} = \frac{\cos^2 \theta}{2 + \cos^2 \theta} = \frac{1}{3} \left(\frac{1 - 3 \sin^2 \theta_{13}}{1 - \sin^2 \theta_{13}} \right) \approx 0.318. \quad (16)$$

This sum rule is well known [35] and follows from the fact that the lepton mixing matrix in Eq. (13) leads to TM₁ mixing [36]. In contrast, the atmospheric mixing angle θ_{23} also depends on the value of s ,

$$\sin^2 \theta_{23} = \frac{1}{2} \left(1 - \left(\frac{2\sqrt{6} \sin 2\theta}{5 + \cos 2\theta} \right) \cos \left(\frac{3\pi s}{8} \right) \right). \quad (17)$$

Using Eq. (14) and that θ is small, see Eq. (15), we can relate θ_{23} and θ_{13} ,

$$\sin^2 \theta_{23} \approx \frac{1}{2} - \sqrt{2} \cos \left(\frac{3\pi s}{8} \right) \sin \theta_{13}. \quad (18)$$

This relation can be used to express $\cos(\frac{3\pi s}{8})$ in terms of the two lepton mixing angles θ_{13} and θ_{23} . Requesting $\sin^2 \theta_{23}$ to be within the experimentally preferred 3σ range [30] for $\theta \approx 0.26$ excludes several choices of s : $s = 0$, $s = 2$, $s = 3$, $s = 5$, and $s = 6$, while $s = 1$ leads to $\sin^2 \theta_{23}$ very close to the lower 3σ bound. The remaining two values of s , $s = 4$ and $s = 7$, lead to maximal atmospheric mixing and $\sin^2 \theta_{23} \approx 0.579$, respectively.

The results for the amount of CP violation in the lepton sector can be quantified with the Jarlskog invariant J_{CP} and the Majorana invariants I_1 and I_2 [37] (see also [38–40]). These are defined as

$$\begin{aligned}
J_{CP} &= \text{Im}[(U_{\text{PMNS}})_{11}(U_{\text{PMNS}})_{13}^*(U_{\text{PMNS}})_{31}^*(U_{\text{PMNS}})_{33}] \\
&= \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta, \\
I_1 &= \text{Im}[(U_{\text{PMNS}})_{12}^2((U_{\text{PMNS}})_{11}^*)^2] \\
&= \sin^2 \theta_{12} \cos^2 \theta_{12} \cos^4 \theta_{13} \sin \alpha, \\
I_2 &= \text{Im}[(U_{\text{PMNS}})_{13}^2((U_{\text{PMNS}})_{11}^*)^2] \\
&= \sin^2 \theta_{13} \cos^2 \theta_{12} \cos^2 \theta_{13} \sin \beta
\end{aligned} \tag{19}$$

and read for $U_{\text{PMNS}} = U_{\text{PMNS},2}(s)$ in Eq. (13)

$$\begin{aligned}
J_{CP} &= -\frac{1}{6\sqrt{6}} \sin 2\theta \sin\left(\frac{3\pi s}{8}\right), \\
I_1 &= -\frac{2}{9} \cos^2 \theta \sin\left(\frac{3\pi s}{4}\right) \quad \text{and} \quad I_2 = -\frac{2}{9} \sin^2 \theta \sin\left(\frac{3\pi s}{4}\right).
\end{aligned} \tag{20}$$

For $\theta \ll 1$, we can extract simple formulae for the CP phases δ , α , and β ,

$$\sin \delta \approx -\sin\left(\frac{3\pi s}{8}\right), \quad \sin \alpha = \sin \beta = -\sin\left(\frac{3\pi s}{4}\right), \tag{21}$$

compare also [15]. With the help of Eq. (18) and the first relation in Eq. (21), we can relate the Dirac phase δ and the two lepton mixing angles θ_{13} and θ_{23} ,

$$(1 - 2\sin^2 \theta_{23})^2 \approx 8\sin^2 \theta_{13} \cos^2 \delta. \tag{22}$$

This sum rule turns out to be a very good approximation of the exact one, given in [35] for TM_1 mixing. Furthermore, we can express the Majorana phases α and β in terms of the Dirac phase δ and the two lepton mixing angles θ_{13} and θ_{23} ,

$$\begin{aligned}
\sin \alpha = \sin \beta &= -2 \sin\left(\frac{3\pi s}{8}\right) \cos\left(\frac{3\pi s}{8}\right) \\
&\approx \sin \delta \left(\frac{1 - 2\sin^2 \theta_{23}}{\sqrt{2} \sin \theta_{13}}\right).
\end{aligned} \tag{23}$$

Using Eq. (20) and $\theta \approx 0.26$ together with the choice $s = 7$, we find

$$\sin \delta \approx -0.936, \quad \sin \alpha = \sin \beta = \frac{1}{\sqrt{2}}. \tag{24}$$

The value of δ is hence close to the best fit value, obtained from the global fit in [30]. The choice $s = 4$ would in contrast correspond to maximal CP violation $\delta = \pi/2$

which is disfavored at more than the 3σ level.⁶ We thus consider

$$s = 7 \tag{25}$$

as most suitable choice for s . Using the information on the two Majorana phases α and β , we can compute the range of the quantity m_{ee} , accessible in neutrinoless double beta decay. Given that this scenario does not make predictions for the neutrino mass spectrum, we parametrize the latter in terms of the lightest neutrino mass m_0 , $0 \leq m_0 \lesssim 4 \times 10^{-2}$ eV in agreement with limits from cosmology [41], and the best fit values of the two mass squared differences Δm_{sol}^2 and Δm_{atm}^2 , which we take for the two different neutrino mass orderings from [30]. Assuming neutrinos follow normal ordering, we obtain

$$\begin{aligned}
2.86 \times 10^{-3} \text{ eV} &\lesssim m_{ee} \lesssim 1.94 \times 10^{-2} \text{ eV} \quad \text{and} \\
2.43 \times 10^{-2} \text{ eV} &\lesssim m_{ee} \lesssim 3.11 \times 10^{-2} \text{ eV}
\end{aligned} \tag{26}$$

for neutrino masses with inverted ordering. We note, in particular, that for neutrinos with normal ordering m_{ee} cannot vanish and has a minimum value for $m_0 \approx 3.62 \times 10^{-3}$ eV and that for inverted ordering the minimum of m_{ee} is also larger than the lower bound expected when using the current information from neutrino oscillation experiments at the 3σ level, cf. also Fig. 9 in [42].

This mixing pattern belongs, according to the classification in [15], to Case 3 b.1). In [15], very similar numerical results have been found for $\Delta(6n^2)$ with $n = 8$, $G_l = Z_3$ generated by a and $Z_2 \times CP$, preserved in the neutrino sector, with bc^4d^4 being the generator of Z_2 and the CP transformation given as $b(\mathbf{3})c(\mathbf{3})^7d(\mathbf{3})X_0(\mathbf{3})$ in the representation $\mathbf{3}$. This choice of residual symmetries is related to the one used in the present study by the similarity transformation a ; see [15] for details, and thus the results coincide, up to a different definition of the angle θ .

As all data on lepton mixing can be satisfactorily described and predictions for CP phases are made after the second step of symmetry breaking, we conclude the study of the lepton sector at this point. We note, however, that in concrete models that realize this symmetry breaking sequence we expect that higher order corrections at a certain point break all residual symmetries; see, e.g., [43,44].

IV. QUARK SECTOR

In the first step of symmetry breaking, the same residual symmetry is preserved in the up quark sector as in the

⁶A possibility to cure this is to include an additional permutation of the second and third generations of charged leptons. Since the atmospheric mixing angle is maximal for $s = 4$, this permutation does not affect its value.

neutrino one and a Z_{16} subgroup remains intact in the down quark sector, i.e.,

$$G_{u,1} = G_{\nu,1} = Z_2 \times Z_2 \times CP \quad \text{and} \quad G_{d,1}^\pm = Z_{16}, \quad (27)$$

as also shown in Fig. 1. In order to obtain a value for the Cabibbo angle close to the measured one at this step of symmetry breaking, we have to choose the generator of $G_{d,1}^\pm$ as $abd^{2(4-k)\pm 1}$ with $k = 0, 1, 2, 3$ for $G_{u,1} = G_{\nu,1}$. The form of the contribution of the up quark sector to the quark mixing matrix is the same as $U_\nu(k, s)$ in Eq. (7), up to the ordering of the columns of the mixing matrix that depends on the up quark masses. Indeed, we choose in the following a slightly different ordering for the columns of $U_u(k, s)$ than of $U_\nu(k, s)$, i.e.,

$$U_u(k, s) = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_{16}^{s-k} & -i\omega_{16}^{s-k} & 0 \\ \omega_{16}^{k+s} & i\omega_{16}^{k+s} & 0 \\ 0 & 0 & \sqrt{2}\omega_8^{8-s} \end{pmatrix}. \quad (28)$$

In addition, the form of the mixing matrix U_d that encodes the contribution from the down quark sector to the quark mixing matrix is

$$U_{d,\pm 1}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_{16}^{\pm 1-2k} & -\omega_{16}^{\pm 1-2k} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \quad (29)$$

where $U_{d,\pm 1}(k)$ corresponds to the choice of $G_{d,1}^\pm$ as residual symmetry among down quarks. The absolute values of the elements of the resulting quark mixing matrix read for both choices

$$|V_{\text{CKM},1}| = |U_u(k, s)^\dagger U_{d,\pm 1}(k)| = \begin{pmatrix} \cos \pi/16 & \sin \pi/16 & 0 \\ \sin \pi/16 & \cos \pi/16 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (30)$$

and are independent of k . As can be read off, the size of the Cabibbo angle θ_C is $\sin \pi/16 \approx 0.195$ which represents a reasonably good leading order approximation to the experimentally measured value [32]. Combining the generators of $G_{u,1}$ and $G_{d,1}^\pm$ leads for all choices of k to a group that has 128 elements and is called $[[128, 67]]$ in the computer program GAP [45,46]. This group can be written as $(Z_8 \times Z_8) \rtimes Z_2$ and only has irreducible representations of dimension one and two. We have explicitly checked that the representation **3** of $\Delta(384)$ decomposes into a complex one- and a faithful, irreducible, complex two-dimensional representation in the group $[[128, 67]]$. The observation that breaking $\Delta(384)$ to a Z_{16} subgroup and a Klein group can lead to one mixing angle $\sin \pi/16 \approx 0.195$, suitable for a

leading order description of quark mixing, has already been made in [11].

In the second step of symmetry breaking in the quark sector, the leading order result of the Cabibbo angle is brought into full agreement with the experimental data [32]. This can be achieved in two different ways. One possibility is that the residual symmetry $G_{u,1}$ is broken to

$$G_{u,2} = Z_2 \times CP, \quad (31)$$

where the remaining Z_2 symmetry is generated by c^4 ; see Fig. 1. The latter element reads in the representation **3** after applying the matrix $U_u(k, s)$,

$$U_u(k, s)^\dagger c(\mathbf{3})^4 U_u(k, s) = \text{diag}(-1, -1, 1), \quad (32)$$

meaning that it allows for a rotation $R_{12}(\theta_u)$ in the (12)-plane through an angle θ_u .⁷ The absolute values of the elements of the resulting quark mixing matrix become

$$|V_{\text{CKM},2,u}| = |(U_u(k, s)R_{12}(\theta_u))^\dagger U_{d,\pm 1}(k)| = \begin{pmatrix} |\cos(\pi/16 \mp \theta_u)| & |\sin(\pi/16 \mp \theta_u)| & 0 \\ |\sin(\pi/16 \mp \theta_u)| & |\cos(\pi/16 \mp \theta_u)| & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (33)$$

Choosing $\theta_u \approx \mp 0.030$ depending on the sign in Eq. (33), we achieve $|V_{us}| = 0.22452$, corresponding to the experimental best fit value [32]. The other possibility is to reduce $G_{d,1}^\pm$ to⁸

$$G_{d,2}^\pm = Z_8, \quad (34)$$

generated by $(abd^{2(4-k)\pm 1})^2 = (cd^2)^{\pm 1-2k}$, respectively, while the residual symmetry $G_{u,1}$ in the up quark sector remains intact. The form of these generators in the representation **3** is after applying $U_{d,\pm 1}(k)$

$$U_{d,\pm 1}(k)^\dagger (c(\mathbf{3})d(\mathbf{3})^2)^{\pm 1-2k} U_{d,\pm 1}(k) = \begin{pmatrix} \omega_8^{\pm 1-2k} & 0 & 0 \\ 0 & \omega_8^{\pm 1-2k} & 0 \\ 0 & 0 & i(-1)^{k+(1\pm 1)/2} \end{pmatrix}. \quad (35)$$

This shows that two of their eigenvalues are degenerate for all admitted values of k , such that the contribution of the

⁷Like in the case of $G_{\nu,2}$, the fact that $G_{u,2}$ contains CP as symmetry constrains the additional unitary matrix, leaving Eq. (32) invariant, to be a rotation with one free real parameter only.

⁸We could also consider $G_{d,2}^\pm = Z_4$ or $G_{d,2}^\pm = Z_2$, since these choices will lead to the same additional matrix $U_{12}(\theta_d, \psi_d)$, as also their generators have two degenerate eigenvalues in the representation **3**.

down quark sector to quark mixing includes an additional unitary transformation $U_{12}(\theta_d, \psi_d)$ of the form

$$U_{12}(\theta_d, \psi_d) = \begin{pmatrix} \cos \theta_d & \sin \theta_d e^{-i\psi_d} & 0 \\ -\sin \theta_d e^{i\psi_d} & \cos \theta_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (36)$$

with free parameters θ_d and ψ_d . The resulting quark mixing matrix is

$$|V_{\text{CKM},2,d}| = |U_u(k, s)^\dagger U_{d,\pm 1}(k) U_{12}(\theta_d, \psi_d)| \\ = \begin{pmatrix} |\cos \zeta| & |\sin \zeta| & 0 \\ |\sin \zeta| & |\cos \zeta| & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{with } \cos 2\zeta = \cos \pi/8 \cos 2\theta_d \mp \sin \pi/8 \sin 2\theta_d \sin \psi_d. \quad (37)$$

This shows that effectively only one real parameter is responsible for achieving $|V_{us}|$ in full accordance with the experimental results and possible numerical values of θ_d and ψ_d can be obtained in a similar way as for θ_u . We thus conclude that both versions of the second step of symmetry breaking in the quark sector, together with the two different possible choices $G_{d,1}^-$ and $G_{d,1}^+$, lead to the same phenomenology. For reasons of minimality, we do not consider the case where $G_{u,1}$ and $G_{d,1}^\pm$ are both broken to $G_{u,2}$ and $G_{d,2}^\pm$, respectively.

In the third step of symmetry breaking the two remaining quark mixing angles are generated together with the CP phase such that all experimental data can be accommodated well [32]. We thus assume that this step of symmetry breaking induces in the quark mixing matrix two additional angles acting in the (13) and (23) planes. We can envisage different scenarios for this last step:

- (i) only the residual flavor symmetry in the up quark sector is broken, while leaving the CP symmetry in this sector together with the residual group among down quarks intact;
- (ii) the residual symmetry in the up quark sector is broken completely, while the one in the down quark sector is still preserved;
- (iii) the symmetry in the up quark sector remains untouched, while the residual one among down quarks is broken; as well as
- (iv) the residual symmetries in both, up and down quark, sectors are broken.

With one of the four different quark mixing matrices, that can possibly arise from the second step of symmetry breaking, see Eqs. (33) and (37), the different versions of the third step lead to one of the following results for quark mixing: if the residual symmetry is only broken in one of the two sectors in the third step of symmetry

breaking, we find that J_{CP}^q crucially depends on the phases that are contained in the unitary matrices, arising from the last step of symmetry breaking (and possibly the second one, if in that the residual symmetry of the down quark sector is reduced). The value of such phases is in general not constrained by the residual symmetries. However, in a concrete model, these can take specific values due to fixed phases, originating from the vacuum expectation values of some flavor (and CP) symmetry breaking fields; see [44]. One example is to assume that in the second and third steps only the residual symmetry in the up quark sector is broken, i.e., after the second step the quark mixing matrix is of the form as in Eq. (33) and scenario (ii) is realized in the third step. We find

$$\sin \theta_{12}^q \approx \left| \sin \left(\frac{\pi}{16} \mp \theta_u \right) \right|, \quad \theta_{23}^q \approx |\theta_{u,23}|, \quad \theta_{13}^q \approx |\theta_{u,13}|, \\ \text{and } J_{CP}^q = \pm \frac{1}{8} \sin \left(\frac{\pi}{8} \mp 2\theta_u \right) \sin 2\theta_{u,13} \sin 2\theta_{u,23} \cos \theta_{u,23} \sin \psi \\ \text{with } \psi = \psi_{u,13} - \psi_{u,23}, \quad (38)$$

where $\theta_{u,13}$, $\psi_{u,13}$ and $\theta_{u,23}$, $\psi_{u,23}$ parametrize the unitary transformations, arising from the third step of symmetry breaking, analogously to θ_d , ψ_d for the matrix $U_{12}(\theta_d, \psi_d)$ in Eq. (36). We note that only the difference ψ of the two phases $\psi_{u,13}$ and $\psi_{u,23}$ enters and that its value is essential for the determination of the amount of CP violation in the quark sector. In order to accommodate the best fit values for the quark mixing angles θ_{ij}^q and the Jarlskog invariant [32], we choose the parameters as⁹

$$\theta_u \approx \mp 0.030, \quad |\theta_{u,23}| \approx 0.042, \\ |\theta_{u,13}| \approx 0.00365 \quad \text{and} \quad |\sin \psi| \approx 0.95, \quad (39)$$

showing that the breaking in the second and third steps is small. Indeed, it is almost an order of magnitude smaller than in the lepton sector; compare Eq. (15).

If instead one of the remaining quark mixing angles comes from the breaking of the residual symmetry in the up quark and the other one from that in the down quark sector, J_{CP}^q always depends on the choice of the parameters k and s . Minimal viable cases, in which only one phase is due to the symmetry breaking, are encountered, if the second step of symmetry breaking takes place in the up quark sector. Out of the four possible cases, depending on the choice of $G_{d,1}^\pm$ and on whether θ_{23}^q or θ_{13}^q is dominantly generated in the up quark sector, two are particularly interesting. In these cases, θ_{23}^q is due to the symmetry breaking in the up quark sector. They lead to a viable quark mixing matrix, even if no phase

⁹The signs of $\theta_{u,23}$ and $\theta_{u,13}$ are not uniquely fixed, but constrained through the requirement to eventually obtain the correct sign for J_{CP}^q for a fixed value of ψ .

is induced by the symmetry breaking in the down quark sector in the third step. We display this symmetry breaking scenario in Fig. 1. We get as result for the mixing angles and J_{CP}^q ,

$$\begin{aligned} \sin \theta_{12}^q &\approx \left| \sin \left(\frac{\pi}{16} \mp \theta_u \right) \right|, & \theta_{23}^q &\approx |\theta_{u,23}|, \\ \theta_{13}^q &\approx \left| \theta_{d,13} \cos \left(\frac{\pi}{16} \mp \theta_u \right) \right|, \\ J_{CP}^q &= \mp \frac{1}{8} \sin \left(\frac{\pi}{8} \mp 2\theta_u \right) \cos \left(\frac{\pi}{16} \mp \theta_u \right) \\ &\quad \times \sin 2\theta_{d,13} \sin 2\theta_{u,23} \sin \left(\phi_k + \frac{3\pi s}{8} \mp \frac{\pi}{16} + \psi_{d,13} \right) \end{aligned} \quad (40)$$

depending on $G_{d,1}^\pm$. Extracting a formula for the CP phase δ^q itself, we find

$$|\sin \delta^q| \approx \left| \sin \left(\phi_k + \frac{3\pi s}{8} \mp \frac{\pi}{16} + \psi_{d,13} \right) \right|. \quad (41)$$

This relation is very interesting, since it allows to relate the CP phase δ^q with the lepton mixing parameters δ , θ_{13} , and θ_{23} . Since k is set to zero after the first step of symmetry breaking in the lepton sector, see Eq. (10), we focus on this case. Using Eq. (18) and the first relation in Eq. (21), we find

$$\begin{aligned} |\sin \delta^q| &\approx \left| \sin \left(\frac{3\pi s}{8} \right) \cos \left(\mp \frac{\pi}{16} + \psi_{d,13} \right) \right. \\ &\quad \left. + \cos \left(\frac{3\pi s}{8} \right) \sin \left(\mp \frac{\pi}{16} + \psi_{d,13} \right) \right| \\ &\approx \left| \sin \delta \cos(\theta_C \mp \psi_{d,13}) \right. \\ &\quad \left. \pm \left(\frac{1 - 2\sin^2 \theta_{23}}{2\sqrt{2} \sin \theta_{13}} \right) \sin(\theta_C \mp \psi_{d,13}) \right|, \end{aligned} \quad (42)$$

where we have also inserted the Cabibbo angle θ_C , whose value is given by $\theta_C = \sin \pi/16 \approx \pi/16$ after the first step of symmetry breaking in the quark sector; compare Eq. (30). This relation simplifies further, if we consider a case, where $\psi_{d,13}$ vanishes, namely

$$|\sin \delta^q| \approx \left| \sin \delta \cos(\theta_C) \pm \left(\frac{1 - 2\sin^2 \theta_{23}}{2\sqrt{2} \sin \theta_{13}} \right) \sin(\theta_C) \right|. \quad (43)$$

The experimental best fit values of the quark mixing angles are achieved for

$$\theta_u \approx \mp 0.030, \quad |\theta_{u,23}| \approx 0.042 \quad \text{and} \quad |\theta_{d,13}| \approx 0.00375, \quad (44)$$

demonstrating that also in this case only a small breaking is needed at the second and third steps in the quark sector. Using the values of k and s that are most suitable for the description of the lepton sector, $k=0$ and $s=7$, and choosing the signs of the parameters $\theta_{u,23}$ and $\theta_{d,13}$ accordingly, we arrive at

$$J_{CP}^q \approx 3.35 \times 10^{-5} \times \cos \left((2 \mp 1) \frac{\pi}{16} + \psi_{d,13} \right). \quad (45)$$

This formula shows that even in the limit, in which no phase $\psi_{d,13}$ is generated in the third step of symmetry breaking, the value of J_{CP}^q is correctly accommodated

$$\begin{aligned} J_{CP}^q &\approx 3.29 \times 10^{-5} \quad \text{for } G_{d,1}^+, \\ \text{and } J_{CP}^q &\approx 2.79 \times 10^{-5} \quad \text{for } G_{d,1}^-, \end{aligned} \quad (46)$$

that is in the experimentally preferred 1σ range for $G_{d,1}^+$ and 3σ range for $G_{d,1}^-$, compare $J_{CP}^q = (3.18 \pm 0.15) \times 10^{-5}$ [32]. The vanishing of the phase $\psi_{d,13}$ can be explained in a concrete model, if, e.g., the operators relevant for the symmetry breaking in the down quark sector do only contain flavor (and CP) symmetry breaking fields that acquire real vacuum expectation values. In order to achieve the current experimental best fit value of J_{CP}^q , the phase $\psi_{d,13}$ should take a value

$$\psi_{d,13} \approx 0.13 \quad \text{for } G_{d,1}^+ \quad \text{and} \quad \psi_{d,13} \approx -0.27 \quad \text{for } G_{d,1}^-, \quad (47)$$

respectively. All shown numerical estimates agree well with the results of an χ^2 analysis, based on the experimental best fit values and 1σ errors found in [32]. In this χ^2 analysis, we have also studied quark mixing, arising from the other possible realizations of the second and third symmetry breaking step, and found that all of them can lead to a viable description of the experimental data except for one case, where no CP violation can be induced, since the only symmetry that is reduced in the second and third steps is the residual flavor symmetry in the up quark sector.

We have thus shown that quark mixing can be successfully described with three steps of symmetry breaking. While the breaking in the first step is unique, different options arise at the second and third steps. As explained, these can lead to different results. In particular, the case, where only the residual symmetry in either the up quark or the down quark sector is broken at the third step, has to be distinguished from the case, where symmetry breaking occurs in both sectors in the third step; see Fig. 1. In the former case, J_{CP}^q depends on phases that originate from the (second and) third step of symmetry breaking, whereas in

the latter J_{CP}^q crucially depends on the parameters k and s that determine the amount of CP violation in the lepton sector. This second case thus leads to a sum rule between the CP phase δ^q and the Cabibbo angle θ_C in the quark sector and the Dirac phase δ and the mixing angles θ_{13} and θ_{23} in the lepton sector; see Eqs. (42) and (43). In the limit, in which the second and third steps of symmetry breaking do not introduce any phases as free parameters, the most suitable choice of k and s for leptons can indeed lead to J_{CP}^q in accordance with experimental data. Corrections to the entertained symmetry breaking scenario might occur in an explicit model, e.g., through higher-dimensional operators, but are usually suppressed.

V. SUMMARY AND CONCLUSIONS

We have presented a scenario with a flavor and a CP symmetry, where the mixing patterns for leptons and quarks arise from the stepwise breaking of these symmetries to different residual subgroups in the different sectors of the theory. In particular, the lepton mixing pattern originates from two steps of symmetry breaking, where the first one gives rise to TB mixing and the second one introduces one real quantity, whose size is determined by the requirement to correctly accommodate the reactor mixing angle. Furthermore, a sum rule is derived that relates the Dirac and Majorana phases with the reactor and atmospheric mixing angles. One CP symmetry is singled out by the experimental data on the atmospheric mixing angle and the indications for δ close to $3\pi/2$. In the quark sector, the Cabibbo angle is fixed to $\theta_C = \sin \pi/16 \approx 0.195$

after the first step of symmetry breaking and brought into full accordance with experimental data with the second one. In the third step, eventually the two other quark mixing angles are generated. For certain sequences of symmetry breaking, we find a close correlation between the amount of CP violation in the lepton and in the quark sector. This correlation can be expressed as a sum rule among the CP phase δ^q and the Cabibbo angle θ_C in the quark sector and the Dirac phase δ and the reactor and atmospheric mixing angles θ_{13} and θ_{23} in the lepton sector. The most constrained viable sequence is shown in Fig. 1, where determining the generators of the residual symmetries, common to the neutrino and the up quark sector, with the help of neutrino oscillation data can lead to a value of J_{CP}^q that lies in the experimentally preferred 1σ range, independent of further free parameters.

A realization of this symmetry-based scenario in a concrete model will be presented elsewhere [44]. This model will have salient features beyond the successful description of lepton and quark mixing in terms of residual symmetries and their breaking, such as the explanation of the charged fermion mass hierarchies via higher-dimensional operators (fixing the ordering of rows and columns of the mixing matrices), the generation of light neutrino masses via the type-I seesaw mechanism, as well as the spontaneous breaking of the flavor and CP symmetry.

ACKNOWLEDGMENTS

The CP3-Origins centre is partially funded by the Danish National Research Foundation, Grant No. DNRF90.

-
- [1] G. Altarelli and F. Feruglio, Discrete flavor symmetries and models of neutrino mixing, *Rev. Mod. Phys.* **82**, 2701 (2010).
 - [2] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, Non-Abelian discrete symmetries in particle physics, *Prog. Theor. Phys. Suppl.* **183**, 1 (2010).
 - [3] S. F. King and C. Luhn, Neutrino mass and mixing with discrete symmetry, *Rep. Prog. Phys.* **76**, 056201 (2013).
 - [4] W. Grimus and P. O. Ludl, Finite flavour groups of fermions, *J. Phys. A* **45**, 233001 (2012).
 - [5] C. S. Lam, Symmetry of Lepton mixing, *Phys. Lett. B* **656**, 193 (2007).
 - [6] A. Blum, C. Hagedorn, and M. Lindner, Fermion masses and mixings from Dihedral flavor symmetries with preserved subgroups, *Phys. Rev. D* **77**, 076004 (2008).
 - [7] C. S. Lam, Determining Horizontal Symmetry from Neutrino Mixing, *Phys. Rev. Lett.* **101**, 121602 (2008).
 - [8] P. F. Harrison and W. G. Scott, Symmetries and generalizations of tri-bimaximal neutrino mixing, *Phys. Lett. B* **535**, 163 (2002).
 - [9] W. Grimus and L. Lavoura, A nonstandard CP transformation leading to maximal atmospheric neutrino mixing, *Phys. Lett. B* **579**, 113 (2004).
 - [10] F. Feruglio, C. Hagedorn, and R. Ziegler, Lepton mixing parameters from discrete and CP symmetries, *J. High Energy Phys.* **07** (2013) 027.
 - [11] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Finite modular groups and lepton mixing, *Nucl. Phys.* **B858**, 437 (2012).
 - [12] S. F. King, T. Neder, and A. J. Stuart, Lepton mixing predictions from $\Delta(6n^2)$ family symmetry, *Phys. Lett. B* **726**, 312 (2013).
 - [13] C. Hagedorn, A. Meroni, and L. Vitale, Mixing patterns from the groups $\Sigma(n\varphi)$, *J. Phys. A* **47**, 055201 (2014).
 - [14] R. M. Fonseca and W. Grimus, Classification of lepton mixing matrices from finite residual symmetries, *J. High Energy Phys.* **09** (2014) 033.
 - [15] C. Hagedorn, A. Meroni, and E. Molinaro, Lepton mixing from $\Delta(3n^2)$ and $\Delta(6n^2)$ and CP , *Nucl. Phys.* **B891**, 499 (2015).

- [16] G.-J. Ding, S. F. King, and T. Neder, Generalised CP and $\Delta(6n^2)$ family symmetry in semi-direct models of leptons, *J. High Energy Phys.* **12** (2014) 007.
- [17] J. Talbert, [Re]constructing finite flavour groups: Horizontal symmetry scans from the bottom-up, *J. High Energy Phys.* **12** (2014) 058.
- [18] L. L. Everett, T. Garon, and A. J. Stuart, A bottom-up approach to Lepton flavor and CP symmetries, *J. High Energy Phys.* **04** (2015) 069.
- [19] M. Holthausen and K. S. Lim, Quark and leptonic mixing patterns from the breakdown of a common discrete flavor symmetry, *Phys. Rev. D* **88**, 033018 (2013).
- [20] T. Araki, H. Ishida, H. Ishimori, T. Kobayashi, and A. Ogasahara, CKM matrix and flavor symmetries, *Phys. Rev. D* **88**, 096002 (2013).
- [21] Ch.-Y. Yao and G.-J. Ding, Lepton and quark mixing patterns from finite flavor symmetries, *Phys. Rev. D* **92**, 096010 (2015).
- [22] I. de Medeiros Varzielas, R. W. Rasmussen, and J. Talbert, Bottom-up discrete symmetries for Cabibbo mixing, *Int. J. Mod. Phys. A* **32**, 1750047 (2017).
- [23] C.-C. Li, J.-N. Lu, and G.-J. Ding, Toward a unified interpretation of quark and lepton mixing from flavor and CP symmetries, *J. High Energy Phys.* **02** (2018) 038.
- [24] J.-N. Lu and G.-J. Ding, Quark and lepton mixing patterns from a common discrete flavor symmetry with a generalized CP symmetry, *Phys. Rev. D* **98**, 055011 (2018).
- [25] W. Grimus and M. N. Rebelo, Automorphisms in gauge theories and the definition of CP and P , *Phys. Rep.* **281**, 239 (1997).
- [26] M. Holthausen, M. Lindner, and M. A. Schmidt, CP and discrete flavour symmetries, *J. High Energy Phys.* **04** (2013) 122.
- [27] M.-C. Chen, M. Fallbacher, K. T. Mahanthappa, M. Ratz, and A. Trautner, CP violation from finite groups, *Nucl. Phys.* **B883**, 267 (2014).
- [28] P. F. Harrison, D. H. Perkins, and W. G. Scott, Tri-bimaximal mixing and the neutrino oscillation data, *Phys. Lett. B* **530**, 167 (2002).
- [29] Z.-z. Xing, Nearly tri bimaximal neutrino mixing and CP violation, *Phys. Lett. B* **533**, 85 (2002).
- [30] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, and T. Schwetz, Updated fit to three neutrino mixing: Exploring the accelerator-reactor complementarity, *J. High Energy Phys.* **01** (2017) 087.
- [31] S. Dell’Oro, S. Marcocci, M. Viel, and F. Vissani, Neutrinoless double beta decay: 2015 review, *Adv. High Energy Phys.* **2016**, 2162659 (2016).
- [32] M. Tanabashi *et al.*, Review of particle physics, *Phys. Rev. D* **98**, 030001 (2018).
- [33] C. Jarlskog, Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation, *Phys. Rev. Lett.* **55**, 1039 (1985).
- [34] J. A. Escobar and C. Luhn, The flavor group $\Delta(6n^2)$, *J. Math. Phys. (N.Y.)* **50**, 013524 (2009).
- [35] C. H. Albright and W. Rodejohann, Comparing trimaximal mixing and its variants with deviations from tri-bimaximal mixing, *Eur. Phys. J. C* **62**, 599 (2009).
- [36] C. S. Lam, Mass independent textures and symmetry, *Phys. Rev. D* **74**, 113004 (2006).
- [37] E. E. Jenkins and A. V. Manohar, Rephasing invariants of quark and Lepton mixing matrices, *Nucl. Phys.* **B792**, 187 (2008).
- [38] G. C. Branco, L. Lavoura, and M. N. Rebelo, Majorana neutrinos and CP violation in the Leptonic sector, *Phys. Lett. B* **180**, 264 (1986).
- [39] J. F. Nieves and P. B. Pal, Minimal rephasing invariant CP violating parameters with Dirac and Majorana fermions, *Phys. Rev. D* **36**, 315 (1987).
- [40] F. del Aguila and M. Zralek, CP violation in the lepton sector with Majorana neutrinos, *Nucl. Phys.* **B447**, 211 (1995).
- [41] N. Aghanim *et al.*, Planck 2018 results. VI. Cosmological parameters, *arXiv:1807.06209*.
- [42] C. Hagedorn and E. Molinaro, Flavor and CP symmetries for leptogenesis and $0\nu\beta\beta$ decay, *Nucl. Phys.* **B919**, 404 (2017).
- [43] F. Feruglio, C. Hagedorn, and R. Ziegler, A realistic pattern of lepton mixing and masses from S_4 and CP , *Eur. Phys. J. C* **74**, 2753 (2014).
- [44] C. Hagedorn and J. König, Lepton and quark masses and mixing in a SUSY model with $\Delta(384)$ and CP , *arXiv:1811.09262*.
- [45] GAP, GAP—Groups, algorithms, and programming, version 4.5.5, <http://www.gap-system.org> (2012).
- [46] H. U. Besche, B. Eick, and E. O’Brien, SmallGroups-library of all ‘small’ groups, GAP package, Version included in GAP 4.5.5, <http://www.gap-system.org/Packages/sgl.html> (2002).