

Tetraquark-adequate QCD sum rules for quark-exchange processes

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We consider quark-hadron duality relations and QCD sum rules for correlators involving exotic tetraquark currents, here specializing to the case of quark-exchange processes. We point out the differences they exhibit with respect to the cases involving ordinary bilinear quark currents. Based on the observation that only diagrams possessing at least four-quark singularities can contribute to the formation of tetraquark states, we show that the quark-hadron duality relations and the corresponding sum rules split into two nonoverlapping relations. The ultimate tetraquark-adequate QCD sum rule is concerned only with one of these relations, in which the operator product expansion starts with diagrams of order $O(\alpha_s^2)$.

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I. MOTIVATION

In recent years, experimental data provided increasing evidence for near-threshold hadron resonances with a favorable interpretation as tetraquark and pentaquark hadrons, i.e., hadrons with minimal parton configurations consisting of four and five quarks, respectively [1–5]. The experimental progress has been escorted by extensive theoretical studies aimed at understanding the possible nature and structure of such exotic hadrons in QCD.

Numerous works deal with the application of Shifman-Vainshtein-Zakharov (SVZ) sum rules [6] to tetraquark and pentaquark states [7,8] (and references therein). The method of SVZ sum rules (or QCD sum rules) makes use of dispersion representations to calculate QCD Green functions, or correlators, in two different ways: first, by applying the Wilson operator product expansion (OPE), which gives the OPE (theoretical) side of a QCD sum rule, and second, by calculating the same Green function by the insertion of a complete set of hadronic intermediate states; this yields the hadron (phenomenological) side of the QCD sum rule. For Green functions of bilinear meson or trilinear baryon interpolating currents, the hadron continuum is counterbalanced by the contribution of perturbative QCD

diagrams above an appropriate effective threshold. Making use of this property, one relates parameters of the ordinary hadrons to the low-energy region of perturbative QCD diagrams starting at order $O(\alpha_s^0)$ [hereafter referred to as $O(1)$ diagrams, α_s being the strong coupling constant] and supplemented by appropriate condensate contributions [6].

Previous applications of SVZ sum rules to exotic states [7,8] have followed the same route by calculating the $O(1)$ QCD diagrams (and in some cases also radiative corrections) and the corresponding power corrections and borrowing the same criteria for continuum subtraction as prescribed for the ordinary hadrons [6]. As a result, the tetraquark or pentaquark properties have been related to $O(1)$ QCD diagrams.¹

However, the case of exotic multi-quark currents has a fundamental difference compared to the ordinary currents: Namely, for correlators of exotic currents, the OPE side as well as the hadron side of SVZ sum rules may be split into two nonoverlapping classes of contributions, with respect to their singularity structure. Moreover, diagrams of each of these two classes on the OPE side and on the hadron side satisfy QCD sum rules of their own; i.e., the quark-hadron duality relation for the exotic correlator leads to two independent SVZ sum rules for each class of contributions.

¹A caveat about subtleties in the application of the SVZ sum rules to exotic states has been raised in [9], where it was noticed that, depending on the way one treats the contributions of two-hadron states on the phenomenological side of the QCD sum rule, one arrives at different assignments of the pentaquark quantum numbers. The origin of this problem, however, was not clarified.

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Exotic states contribute to only one of these sum rules, which we name, in the case of tetraquarks, T -adequate QCD sum rule, and do not contribute to the other sum rule. The OPE side of the T -adequate sum rule contains contributions of only those diagrams which may participate in the formation of the tetraquark state. These are called T -phile diagrams and are selected according to their singularity structure in the Feynman diagrams of the four-point function, Γ_{4j} , of quark bilinear color-singlet currents. T -phile diagrams should have a four-quark s -channel cut; such diagrams emerge at order $O(\alpha_s^2)$ and higher [10]. Obviously, in order to obtain meaningful estimates of the parameters of exotic states, one needs to use the T -adequate QCD sum rule.

In fact, in the context of large- N_c QCD, it is well known that some classes of Feynman diagrams are not related to exotic hadrons [11–20]. However, only recently have the consequences of this property for the formulation of sum rules for exotic states been worked out: In [10], we have focused on quark-hadron duality relations for correlation functions of the tetraquark currents and have explicitly demonstrated for specific “direct” Green functions, corresponding to processes where the initial and final states have the same quark-antiquark bilinear flavor structures, that the $O(1)$ and the $O(\alpha_s)$ contributions on the QCD side precisely cancel against the two-hadron continuum contributions on the hadronic side, as soon as SVZ sum rules for correlators of ordinary currents are used. The OPE side of the T -adequate QCD sum rule has been shown to start with specific nonfactorizable diagrams of order $O(\alpha_s^2)$.

In this paper, we derive the T -adequate QCD sum rules for “recombination” diagrams, corresponding to quark-exchange processes, and prove that also in this case the OPE side of the T -adequate QCD sum rule starts with diagrams of order $O(\alpha_s^2)$. For the sake of clarity, we discuss the case of exotic currents composed of quarks of four different flavors. The topology of recombination diagrams being different from that of direct diagrams, the derivation of T -adequate sum rules necessitates a more detailed study. The proof given here is based on the analysis of four-quark singularities of Feynman diagrams. This paper therefore completes the derivation of tetraquark-adequate QCD sum rules.

The paper is organized as follows: Section II highlights some necessary properties of tetraquark interpolating currents and their Green functions. Section III focuses on recombination Green functions, in which quark flavors in the initial and final tetraquark currents are arranged differently. Our conclusions follow in Sec. IV.

II. TETRAQUARK INTERPOLATING CURRENTS AND CORRELATION FUNCTIONS

We discuss properties of tetraquarks consisting of two quarks of flavors a and c and two antiquarks of flavors b and d . The Dirac structure of the currents is of no relevance

for the arguments of this paper and will not be specified; we therefore do not explicitly write the appropriate combinations of γ matrices between the quark fields.

We will exploit two properties of the exotic currents and their Green functions:

- (i) QCD sum rules adopt local multiquark interpolating currents, and any local tetraquark current may be brought to the form of a linear combination of products of color-singlet combinations of quark fields with two different flavor structures $\theta_{\bar{a}b\bar{c}d} = j_{\bar{a}b}j_{\bar{c}d}$ and $\theta_{\bar{a}d\bar{c}b} = j_{\bar{a}d}j_{\bar{c}b}$ with $j_{ab} = \bar{q}_a q_b$ [21].² For instance, a triplet-antitriplet tetraquark current may be written as

$$(e^{ijk}\bar{q}_a^j\bar{q}_c^k)(e^{i'j'k'}q_b^{j'}q_d^{k'}) = -\theta_{\bar{a}b\bar{c}d} - \theta_{\bar{a}d\bar{c}b}, \quad (1)$$

and for an octet-octet current one finds

$$(\bar{q}_a T^A q_b)(\bar{q}_c T^A q_d) = -\frac{1}{2}\theta_{\bar{a}d\bar{c}b} - \frac{1}{6}\theta_{\bar{a}b\bar{c}d}. \quad (2)$$

Here, $T^A = \lambda^A/2$, $A = 1, 2, \dots, 8$, λ^A are the Gell-Mann matrices, and we made use of the anticommutativity of quark fields. Taking into account the Dirac structure of quark bilinears, one needs to perform also the Fierzing with respect to the spinor indices (see e.g., Refs. [1,7]).

It is therefore sufficient to study QCD sum rules for exotic interpolating currents taken as products of two color-singlet quark bilinears. (When needed, the momentum of a four-quark current will be designated by p .)

- (ii) Any diagram involving the tetraquark currents $\theta_{\bar{a}b\bar{c}d}$ and/or $\theta_{\bar{a}d\bar{c}b}$ can be obtained from a diagram involving only the bilinear quark currents $j_{\bar{a}b}$, $j_{\bar{c}d}$, $j_{\bar{a}d}$, and $j_{\bar{c}b}$ by merging the appropriate vertices. In particular, the two-point function of tetraquark currents,

$$\Pi_{\theta\theta} = \langle T\{\theta(x)\theta^\dagger(0)\} \rangle, \quad (3)$$

can be obtained from the four-point function of ordinary bilinear currents,

$$\Gamma_{4j} = \langle T\{j(x_1)j(x_2)j^\dagger(x_3)j^\dagger(0)\} \rangle, \quad (4)$$

by merging two pairs of vertices. The three-point function involving one tetraquark current and two bilinear interpolating currents,

$$\Gamma_{\theta jj} = \langle T\{\theta(0)j^\dagger(x)j^\dagger(y)\} \rangle, \quad (5)$$

²More generally, any gauge-invariant multiquark operator can be reduced to a combination of products of colorless clusters [22].

can be obtained from the same four-point function Γ_{4j} by merging only one pair of vertices. The functions $\Pi_{\theta\theta}$ and $\Gamma_{\theta jj}$ have been the subject of QCD sum rule analyses extensively presented in the literature.

According to property (ii), the quark-hadron duality relations for $\Pi_{\theta\theta}$ and $\Gamma_{\theta jj}$ and the corresponding QCD sum rules follow directly from the duality relations for Γ_{4j} . Analytic properties of the latter, in particular, the structure of its four-quark singularities, relevant for the derivation of consistent QCD sum rules for tetraquarks, have been studied in detail in [16,17].

For a given global flavor content of the tetraquark current $\bar{a}\bar{c}bd$, we have at our disposal two different flavor combinations of two color singlets, $\theta_{\bar{a}\bar{b}c\bar{d}}$ and $\theta_{\bar{a}\bar{d}c\bar{b}}$, and therefore one should distinguish between the diagrams where quark flavors in the initial and final states are combined in the same way (direct diagrams) and in a different way (quark-exchange or recombination diagrams). The Feynman diagrams for the corresponding four-point functions have different topologies and structures in their four-quark singularities and therefore necessitate separate studies. For the case of direct diagrams, the reader is referred to Ref. [10], where a detailed analysis has been presented. In the following, we concentrate on the case of quark-exchange diagrams.

III. RECOMBINATION GREEN FUNCTIONS INVOLVING TETRAQUARK CURRENTS

We now discuss diagrams with a recombination or quark-exchange topology, where the quark flavors in the initial and the final currents are arranged differently.

For the direct Green functions

$$\Gamma_{4j}^{\text{dir}} = \langle T\{j_{\bar{a}b}(x_1)j_{\bar{c}d}(x_2)j_{\bar{a}b}^\dagger(x_3)j_{\bar{c}d}^\dagger(0)\} \rangle, \quad (6)$$

due to the factorization property of lowest-order QCD diagrams, use was made of conventional SVZ sum rules for correlators of bilinear quark currents to show the cancellation of the non- T -phile diagrams on the OPE side against factorizable two-meson contributions on the hadron side [10]. This transparent procedure is not applicable in the recombination case and therefore one needs to follow a different line of argument, based on the singularity analysis of Feynman diagrams, for the related study.

Let us focus on the quark-hadron duality relations for the recombination four-point function

$$\Gamma_{4j}^{\text{rec}} = \langle T\{j_{\bar{a}b}(x_1)j_{\bar{c}d}(x_2)j_{\bar{a}d}^\dagger(x_3)j_{\bar{c}b}^\dagger(0)\} \rangle. \quad (7)$$

As emphasized in Sec. II, the understanding of the duality relations for Γ_{4j}^{rec} immediately leads to the understanding of the duality relations for $\Pi_{\theta\theta}$ and $\Gamma_{\theta jj}$, since the diagrams for

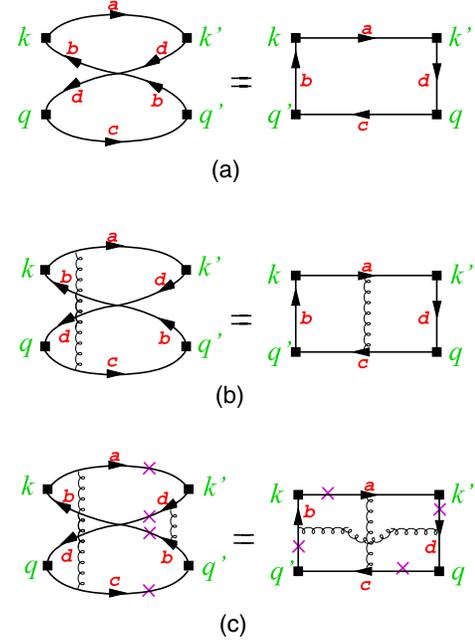


FIG. 1. Singularities of Feynman diagrams describing Γ_{4j}^{rec} . The left-column (right-column) diagrams (a) and (b) do not contain a four-particle s -channel (u -channel) cut. The lowest-order diagram that has the four-particle s -channel cut is the left-column diagram (c). This cut [and the corresponding u -channel cut in the diagram (c) in the right column] emerges when the crossed quark propagators go on the mass shell.

the latter are obtained from the diagrams for Γ_{4j}^{rec} by merging the appropriate vertices.

The analytic properties of the four-point functions of bilinear currents have been studied in detail in [16,17]. Here, we would like to show one example which is particularly important for the understanding of quark-hadron duality relations for the recombination diagrams.

Figure 1 presents the recombination diagrams of lowest orders in an unfolded form as box diagrams: The s -channel singularities of the diagrams in the left column correspond to u -channel singularities of the diagrams in the right column. The absence of u -channel cuts (not only four-quark cuts, but any cuts) in the diagrams in the right column of Figs. 1(a) and 1(b) is evident; this means the absence of s -channel singularities in the diagrams Figs. 1(a) and 1(b) in the left column. In order to understand the structure of singularities of the diagram with two-gluon exchanges, Fig. 1(c), one needs to solve the Landau equations [23]. The corresponding equations and their solutions are presented in the Appendix. With the help of the Landau equations one finds that the left-column diagram Fig. 1(c) has the four-quark s -channel threshold at $s = (m_a + m_b + m_c + m_d)^2$; this threshold and the corresponding four-quark cut are related to the configuration of quark momenta when the crossed quark lines go on their mass shell. With this knowledge at hand, we turn to the analysis of QCD sum rules for the recombination Green functions. Figure 2

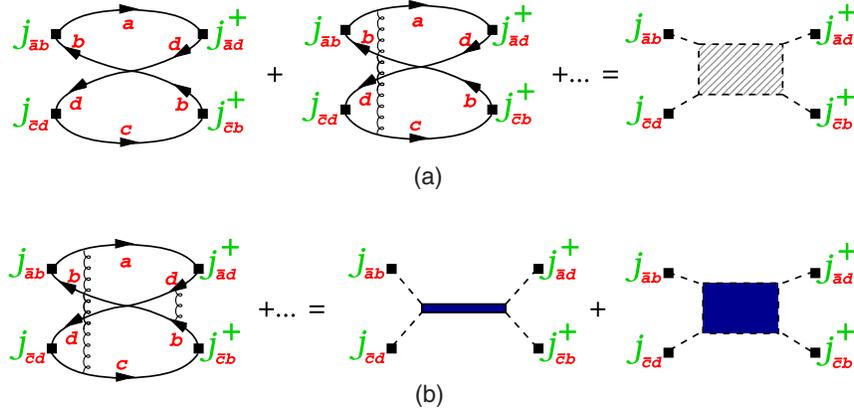


FIG. 2. Duality relations for two classes of contributions to Γ_{4j}^{rec} : (a) OPE representation for those contributions to Γ_{4j}^{rec} that do not contain four-quark s -channel cuts (non- T -phile contributions) and the corresponding representation for such contributions using hadron degrees of freedom (d.o.f.): The dashed rectangle denotes the sum of all contributions to Γ_{4j}^{rec} that do not have two-meson s -channel cuts. (b) OPE representation for those contributions to Γ_{4j}^{rec} that do have four-quark s -channel cuts (class- T contributions) and the corresponding representation for class- T contributions using meson d.o.f.: The solid blue rectangle denotes the sum of all hadronic contributions to Γ_{4j}^{rec} that have two-meson s -channel cuts. A possible isolated T pole has been added.

presents in diagrammatic form the quark-hadron duality relations for Γ_{4j}^{rec} .

Several remarks are in order.

- (i) All QCD diagrams for Γ_{4j}^{rec} may be divided into two nonoverlapping classes according to the structure of their s -channel singularities: The diagrams of the first class, referred to as non- T -phile diagrams, do not have a four-quark s -channel cut. All diagrams that contain a four-quark s -channel cut belong to the second class, referred to as T -phile diagrams. If a tetraquark pole emerges in the s channel, then it can emerge only through the infinite set of T -phile diagrams. Any s -channel singularity in the set of non- T -phile diagrams is not related to the tetraquark.
- (ii) If we look at the representation of Green functions in the hadron picture, then Green functions of the non- T -phile class do not contain two-meson s -channel intermediate states, whereas the hadron representation for T -phile contributions does contain such two-meson intermediate states and also a possible tetraquark state.
- (iii) Quark-hadron duality relations are fulfilled for non- T -phile diagrams and for T -phile diagrams separately. It is therefore straightforward to write down the corresponding QCD sum rules, as in Fig. 2.

In Fig. 2(a), the non- T -phile diagrams on the OPE side are dual to the specific meson diagrams without two-meson s -channel cuts and without a possible tetraquark pole on the hadron side of the QCD sum rule for Γ_{4j}^{rec} . (Two-meson cuts appear in the t and u channels.) Obviously, non- T -phile diagrams cannot be related to the tetraquark properties.

In Fig. 2(b), the QCD sum rule for the T -phile part of Γ_{4j}^{rec} represents the desired T -adequate QCD sum rule.

Having at hand the latter relation, one easily constructs T -adequate QCD sum rules for $\Pi_{\theta\theta}^{\text{rec}}$ and $\Gamma_{\theta jj}^{\text{rec}}$ by merging the appropriate vertices in Γ_{4j}^{rec} , as shown in Figs. 3 and 4. In the end, only the T -phile diagrams with, at least, two gluon exchanges of the type shown in Figs. 3(c) and 4(c)

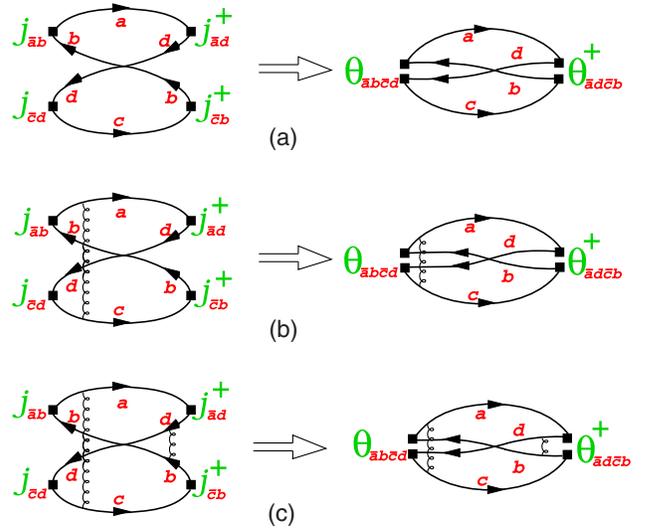


FIG. 3. Feynman diagrams for a recombination two-point function of tetraquark currents (rhs). They are obtained by merging vertices in the recombination four-point function of bilinear quark currents (lhs). Diagram (c) on the lhs is the lowest-order diagram that contains a four-quark s -channel cut. Diagrams (a) and (b) on the lhs do not contain four-quark singularities in the s channel. Only diagram (c) on the lhs is a T -phile diagram. Similarly, among the diagrams on the rhs, it is only diagram (c) that contributes to the OPE side of the tetraquark SVZ sum rule (8).

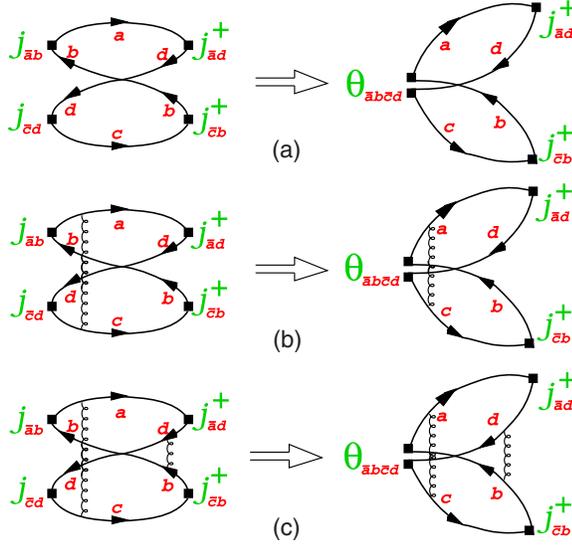


FIG. 4. Feynman diagrams for a recombination three-point function (rhs). They are obtained by merging vertices in the four-point function of bilinear quark currents (lhs). Diagram (a) in the rhs has at most a polynomial dependence on the variable p^2 (p , the momentum of the four-quark current) and does not have a four-quark cut in it. Diagram (b) in the rhs has a nontrivial p^2 dependence, but it drops out from the T -adequate QCD sum rule: Its contribution cancels against the hadron contributions not related to the tetraquark properties by virtue of the duality relations for Γ_{4j}^{rec} . Therefore, diagrams (a) and (b) are not related to tetraquark properties. Only the diagram with the two-gluon exchange (c) is a T -phile diagram, contributing to the tetraquark SVZ sum rule (9).

appear in the T -adequate QCD sum rules for the tetraquark properties.

Here, one should bear in mind a subtlety related to the difference between confined (compact) tetraquarks and molecular tetraquark states: The compact tetraquarks are poles in full QCD Green functions, but they do not emerge in the effective low-energy theory described in terms of meson d.o.f.; molecular states, on the contrary, are not only poles in full QCD Green functions, but also emerge as poles in the effective meson low-energy theory. Therefore, in the case of a molecular tetraquark state, the corresponding pole is contained in the infinite sum of diagrams denoted by the solid blue rectangle of Fig. 2(b). The decomposition in the rhs of Fig. 2(b) should then be understood as the sum of meson diagrams, from which the T pole has already been subtracted. In case the tetraquark pole emerges as a compact four-quark state, it is not present in the effective meson theory and should be added separately, as in Fig. 2(b).

As a final step, assuming that the high-momentum tail (corresponding to the s integration above an effective threshold s_{eff} [24–26]) of the T -phile Feynman diagrams on the OPE side of QCD sum rules cancels against the hadron continuum contributions on the hadron side of QCD

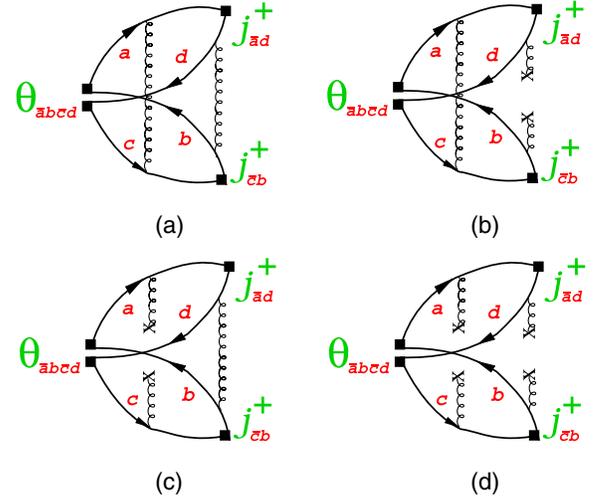


FIG. 5. Typical T -phile $O(\alpha_s^2)$ perturbative diagram (a) and the corresponding power corrections of order $O(\alpha_s(\alpha_s GG))$ (b), (c) which contribute to the OPE side of the T -adequate sum rule for the $T \rightarrow M_{\bar{a}d}M_{\bar{c}b}$ coupling. Diagram (d), of order $O(\langle\alpha_s^2 GGGG\rangle)$, has at most a polynomial dependence on p^2 and thus does not contribute to the Borel sum rule for the $T \rightarrow M_{\bar{a}d}M_{\bar{c}b}$ coupling.

sum rules, we arrive at the ultimate T -adequate QCD sum rules for recombination Green functions:

$$f_T^{\bar{a}b\bar{c}d} f_T^{\bar{a}d\bar{c}b} \exp(-M_T^2 \tau) = \int_{(4m_q)^2}^{s_{\text{eff}}} ds \exp(-s\tau) \rho_T^{\text{rec}}(s) + \text{BPC}, \quad (8)$$

$$f_T^{\bar{a}b\bar{c}d} A(T \rightarrow j_{\bar{a}d} j_{\bar{c}b}) \exp(-M_T^2 \tau) = \int_{(4m_q)^2}^{s_{\text{eff}}} ds \exp(-s\tau) \Delta_T^{\text{rec}}(s) + \text{BPC}, \quad (9)$$

$$f_T^{\bar{a}b\bar{c}d} = \langle T | \theta_{\bar{a}b\bar{c}d} | 0 \rangle, f_T^{\bar{a}d\bar{c}b} = \langle T | \theta_{\bar{a}d\bar{c}b} | 0 \rangle, \quad (10)$$

where BPC is short for Borelized power corrections, $4m_q \equiv m_a + m_b + m_c + m_d$, and $A(T \rightarrow j_{\bar{a}d} j_{\bar{c}b})$ is the amplitude $\langle 0 | T \{ j_{\bar{a}d}(x) j_{\bar{c}b}(0) \} | T(p) \rangle$ in momentum space; $\rho_T^{\text{rec}}(s)$ and $\Delta_T^{\text{rec}}(s)$ are the spectral densities in the variable s of the $O(\alpha_s^2)$ diagrams with two-gluon exchanges, of the type shown in the rhs of Figs. 3(c) and 4(c), respectively. Power corrections in the rhs of Eqs. (8) and (9) correspond to condensate insertions in these diagrams. The typical lowest-order diagrams contributing to the OPE side of Eq. (9) are shown in Figs. 5(a)–5(c); the power correction given by Fig. 5(d) does not depend on p^2 and thus vanishes under the Borel transformation and does not contribute to the Borel sum rule (9).

Typical diagrams that do not contribute to the OPE side of the T -adequate QCD sum rules for $\Gamma_{\theta_{jj}}^{\text{rec}}$ are shown in Fig. 6. These have often been considered in the literature as

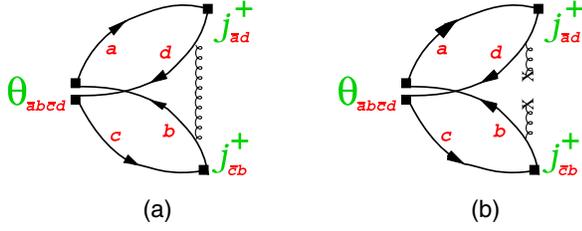


FIG. 6. A typical perturbative $O(\alpha_s)$ diagram and $O(\langle\alpha_s GG\rangle)$ power correction that both do not contribute to the OPE side of the T -adequate QCD sum rule for the $T \rightarrow M_{\bar{a}d}M_{\bar{c}b}$ coupling, as these diagrams are obtained by merging vertices in the non- T -phile diagram for Γ_{4j}^{rec} .

the appropriate OPE diagrams for the coupling of a tetraquark to two mesons, $T \rightarrow M_{\bar{a}d}M_{\bar{c}b}$ [7]. However, as it is clear from the duality relation depicted in Fig. 2(a), in spite of a nontrivial p^2 dependence of the diagram in Fig. 6(a), it does not contribute to the T -adequate QCD sum rule for $\Gamma_{\theta jj}^{\text{rec}}$. For the same reason, the condensate diagram Fig. 6(b) also cannot contribute to the $T \rightarrow M_{\bar{a}d}M_{\bar{c}b}$ coupling. [Moreover, the diagram Fig. 6(b) does not depend on p^2 at all and thus its Borel transform vanishes.]

We have discussed the case of flavor-exotic tetraquark currents. It is clear, however, that the same arguments apply also to the case of crypto-exotic flavor structure of the tetraquark current (i.e., $\bar{q}_a q_b \bar{q}_c q_d$, with a, b, c denoting quark flavors). Compared to the flavor-exotic case, the crypto-exotic case exhibits an extended set of T -phile diagrams (see a detailed discussion in [17]). Apart from this feature, the splitting of all diagrams in the OPE for Γ_{4j} into two nonoverlapping classes of non- T -phile and T -phile diagrams is valid also in the crypto-exotic case. Obviously, the arguments given in this section remain unchanged; the only modification appears in the way of selecting the appropriate set of T -phile diagrams.

Considering tetraquark interpolating currents in the form of products of two color-singlet bilinears allowed us to formulate a clear criterion for selecting T -phile diagrams that contribute to the tetraquark-adequate QCD sum rules for $\Pi_{\theta\theta}$ and $\Gamma_{\theta jj}$. One may use other color structures of the quark bilinears forming colorless local tetraquark interpolating currents: For instance, one may work with the triplet-antitriplet, i.e., diquark-antidiquark, structure $\bar{D}^i D_i$ with $D_i = \epsilon_{ijk} q^j q^k$, $i, j, k = 1, 2, 3$ being color indices, or the octet-octet structure $\bar{q} \lambda^A q \bar{q} \lambda^A q$, λ^A being the Gell-Mann matrices, $A = 1, \dots, 8$. However, for such color structures of the tetraquark interpolating currents, it is difficult to provide explicit criteria for selecting the appropriate T -phile diagrams. A consistent way to select such T -phile diagrams is to rearrange the diquark-antidiquark or octet-octet local currents into the singlet-singlet color structures and make use of the criteria for selecting the T -phile diagrams already formulated for this case. In this way, the derivation of the tetraquark-adequate QCD sum rules also

works for other choices of the color structure of the local tetraquark interpolating currents.

IV. CONCLUSION

We have considered the SVZ sum rules for correlation functions involving exotic tetraquark currents θ , namely, the two-point function $\Pi_{\theta\theta}$ and the three-point function $\Gamma_{\theta jj}$, specifically related in the present paper to quark-exchange processes.

It turns out that the duality relations and QCD sum rules for these correlators have different properties compared with the duality relations for the correlators of bilinear quark currents. The OPE part as well as the hadron part of the sum rules split into two nonoverlapping classes of contributions; each of them satisfies its own QCD sum rule: One QCD sum rule contains, on the OPE side, tetraquark-concerning diagrams, also named T -phile diagrams, which start at order $O(\alpha_s^2)$, and, on the hadron side, the tetraquark contribution. This sum rule has been named T -adequate sum rule. The second QCD sum rule, which contains non- T -phile diagrams on the OPE side, does not contain the tetraquark contribution on the hadron side and therefore has no relation to the tetraquark properties.

The T -phile diagrams for any correlator involving the tetraquark currents can be obtained from the corresponding T -phile diagrams of the four-point function Γ_{4j} of quark color-singlet bilinear currents; the latter are defined as those diagrams which have four-quark s -channel cuts. The T -phile diagrams can be identified by solving the Landau equations [23].

The present work completes the proof provided in [10] for direct-type processes, where the quark flavors are arranged in color-singlet bilinears in the same way in the initial and in the final states.

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APPENDIX: LANDAU EQUATIONS FOR RECOMBINATION DIAGRAMS

A generic expression of a Feynman diagram has the form

$$I(p) = \int \prod_{\ell=1}^L \frac{d^4 k_\ell}{(2\pi)^4} \prod_{i=1}^I \frac{1}{(q_i^2 - m_i^2 + i\epsilon)}, \quad (\text{A1})$$

where p represents a set of external momenta and q_i are linear functions of the p 's and of the independent loop variables k .

The Landau equations are [23]

$$\lambda_i(q_i^2 - m_i^2) = 0, \quad i = 1, \dots, I, \quad (\text{A2})$$

$$\sum_{i=1}^I \lambda_i q_i \cdot \frac{\partial q_i}{\partial k_\ell} = 0, \quad \ell = 1, \dots, L, \quad (\text{A3})$$

where the λ 's are parameters (Lagrange multipliers) to be determined.

This system of equations may have independent subsystems, corresponding to the vanishing of a certain number of parameters λ .

Here, we present the Landau equations for the Feynman diagrams shown in the left column of Fig. 1. We are interested in the singularities produced by the quark propagators. We therefore do not consider gluon propagators in the Landau equations; this amounts to taking the corresponding λ 's equal to zero.

We start with the recombination diagram of leading order of Fig. 7(a). The external momenta are p_{ab} , p_{cd} , p_{ad} , p_{cb} , with

$$\begin{aligned} P &= p_{ab} + p_{cd} = p_{ad} + p_{cb}, & s &= P^2, \\ t &= (p_{ab} - p_{ad})^2, & u &= (p_{ab} - p_{cb})^2. \end{aligned} \quad (\text{A4})$$

The Landau equations read

$$\begin{aligned} \lambda_a(k^2 - m_a^2) &= 0, \\ \lambda_b((k - p_{ab})^2 - m_b^2) &= 0, \\ \lambda_c((p_{cd} - p_{ad} + k)^2 - m_c^2) &= 0, \\ \lambda_d((k - p_{ad})^2 - m_d^2) &= 0, \\ \lambda_a k + \lambda_b(k - p_{ab}) + \lambda_c(k + p_{cd} - p_{ad}) + \lambda_d(k - p_{ad}) &= 0. \end{aligned} \quad (\text{A5})$$

This system of equations has several independent subsystems. Choosing $\lambda_b = \lambda_d = 0$, one obtains $u = (m_a \pm m_c)^2$. (Only physical singularities, corresponding to + signs between the masses, are relevant.) Choosing $\lambda_a = \lambda_c = 0$, one obtains $t = (m_b \pm m_d)^2$. Choosing $\lambda_c = \lambda_d = 0$, one obtains $p_{ab}^2 = (m_a \pm m_b)^2$, and so forth. The property that the singularities in u and t involve only two quark masses shows that there are no four-quark singularities in this diagram. In the variable s no singularities at all are found.

The second diagram, Fig. 7(b), is a two-loop (the loops shown in different colors) diagram with one-gluon exchange between quarks a and c . It contains seven propagators and thus the full system of Landau equations includes seven parameters λ_i . One has two four-momentum conservation laws related to two loops. The full system of

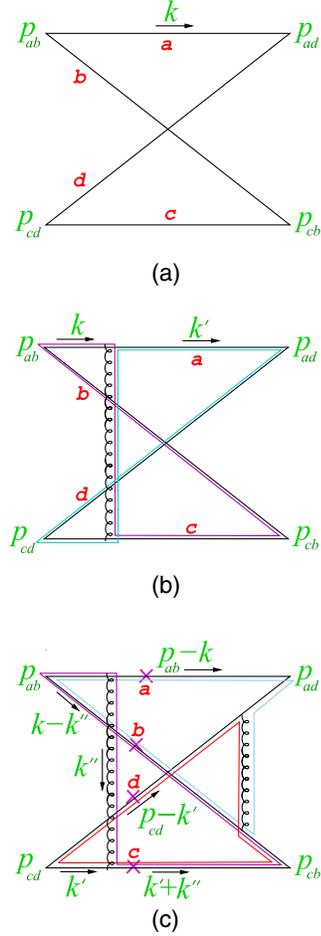


FIG. 7. Diagrams of Fig. 1 with explicit momentum flow.

Landau equations splits into several subsystems related to setting to zero some of the parameters λ_i . These subsystems lead to thresholds at $p_{ij}^2 = (m_i \pm m_j)^2$, $t = (m_b \pm m_d)^2$ and $u = (m_a \pm m_c)^2$ (only physical singularities related to + signs between the masses are relevant). None of the subsystems leads to the solution corresponding to the threshold in the variable s , indicating the absence of s cuts.

Finally, we turn to the diagram Fig. 7(c) with two-gluon exchanges between quarks a and c , and b and d . It is a three-loop diagram (each loop shown in a different color), containing three independent integration momenta k , k' , and k'' , and ten propagators. Therefore, the general system of Landau equations contains ten parameters λ_i (equal to the number of the propagators) and three independent momentum conservation relations, equal to the number of loops. We present a subsystem of Landau equations corresponding to the crossed propagators in Fig. 7(c) (i.e., all other λ 's are set to zero) which leads to the four-quark threshold in the variable s :

$$\begin{aligned} \lambda_a((p_{ab} - k)^2 - m_a^2) &= 0, & \lambda_b((k - k'')^2 - m_b^2) &= 0, \\ \lambda_c((k' + k'')^2 - m_c^2) &= 0, & \lambda_d((p_{cd} - k')^2 - m_d^2) &= 0, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned}
\lambda_a(p_{ab} - k) - \lambda_b(k - k'') &= 0, \\
\lambda_c(k' + k'') - \lambda_d(p_{cd} - k') &= 0, \\
-\lambda_b(k - k'') + \lambda_c(k' + k'') &= 0.
\end{aligned}
\tag{A7}$$

This system of four equations can be solved and leads to a nontrivial solution $P^2 = s = (m_a + m_b + m_c + m_d)^2$, thus indicating the presence of an s -channel four-quark threshold. (The unphysical singularities, corresponding to changes of sign in front of the masses, exist as well.)

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- [1] A. Esposito, A. Pilloni, and A. D. Polosa, Multiquark resonances, *Phys. Rep.* **668**, 1 (2017).
- [2] A. Ali, J. S. Lange, and S. Stone, Exotics: Heavy pentaquarks and tetraquarks, *Prog. Part. Nucl. Phys.* **97**, 123 (2017).
- [3] S. L. Olsen, T. Skwarnicki, and D. Zieminska, Nonstandard heavy mesons and baryons: Experimental evidence, *Rev. Mod. Phys.* **90**, 015003 (2018).
- [4] F.-K. Guo, C. Hanhart, U.-G. Meissner, Q. Wang, Q. Zhao, and B.-S. Zou, Hadronic molecules, *Rev. Mod. Phys.* **90**, 015004 (2018).
- [5] N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C.-P. Shen, C. E. Thomas, A. Vairo, and C.-Z. Yuan, The XYZ states: Experimental and theoretical status and perspectives, [arXiv:1907.07583](https://arxiv.org/abs/1907.07583).
- [6] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, QCD and resonance physics. Theoretical foundations, *Nucl. Phys.* **B147**, 385 (1979).
- [7] M. Nielsen, F. S. Navarra, and S. H. Lee, New charmonium states in QCD sum rules: A concise review, *Phys. Rep.* **497**, 41 (2010).
- [8] R. M. Albuquerque, J. M. Dias, K. P. Khemchandani, A. Martínez Torres, F. S. Navarra, M. Nielsen, and C. M. Zanetti, QCD sum rules approach to the X, Y and Z states, *J. Phys. G* **46**, 093002 (2019).
- [9] Y. Kondo, O. Morimatsu, and T. Nishikawa, Two-hadron-irreducible QCD sum rule for pentaquark baryon, *Phys. Lett. B* **611**, 93 (2005).
- [10] W. Lucha, D. Melikhov, and H. Sazdjian, Tetraquark-adequate formulation of QCD sum rules, *Phys. Rev. D* **100**, 014010 (2019).
- [11] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985), Chap. 8.
- [12] S. Weinberg, Tetraquark Mesons in Large- N Quantum Chromodynamics, *Phys. Rev. Lett.* **110**, 261601 (2013).
- [13] M. Knecht and S. Peris, Narrow tetraquarks at large N , *Phys. Rev. D* **88**, 036016 (2013).
- [14] T. D. Cohen and R. F. Lebed, Are there tetraquarks at large N_c in QCD(F)? *Phys. Rev. D* **90**, 016001 (2014).
- [15] L. Maiani, A. D. Polosa, and V. Riquer, Tetraquarks in the $1/N$ expansion and meson-meson resonances, *J. High Energy Phys.* **06** (2016) 160.
- [16] W. Lucha, D. Melikhov, and H. Sazdjian, Narrow exotic tetraquark mesons in large- N_c QCD, *Phys. Rev. D* **96**, 014022 (2017).
- [17] W. Lucha, D. Melikhov, and H. Sazdjian, Tetraquark and two-meson states at large N_c , *Eur. Phys. J. C* **77**, 866 (2017).
- [18] L. Maiani, A. D. Polosa, and V. Riquer, Tetraquarks in the $1/N$ expansion: A new appraisal, *Phys. Rev. D* **98**, 054023 (2018).
- [19] W. Lucha, D. Melikhov, and H. Sazdjian, Exotic states and their properties from large- N_c QCD, *Proc. Sci., EPS-HEP2017* (2017) 390.
- [20] W. Lucha, D. Melikhov, and H. Sazdjian, Are there narrow flavor-exotic tetraquarks in large- N_c QCD? *Phys. Rev. D* **98**, 094011 (2018).
- [21] R. L. Jaffe, Two types of hadrons, *Nucl. Phys.* **A804**, 25 (2008).
- [22] W. Lucha, D. Melikhov, and H. Sazdjian, Cluster reducibility of multiquark operators, [arXiv:1908.10164](https://arxiv.org/abs/1908.10164).
- [23] L. D. Landau, On analytic properties of vertex parts in quantum field theory, *Nucl. Phys.* **13**, 181 (1959).
- [24] W. Lucha, D. Melikhov, and S. Simula, Effective continuum threshold in dispersive sum rules, *Phys. Rev. D* **79**, 096011 (2009).
- [25] W. Lucha, D. Melikhov, and S. Simula, Bound-state parameters from dispersive sum rules for vacuum-to-vacuum correlators, *J. Phys. G* **37**, 035003 (2010).
- [26] W. Lucha, D. Melikhov, and S. Simula, Extraction of ground-state decay constant from dispersive sum rules: QCD vs potential models, *Phys. Lett. B* **687**, 48 (2010).