Interface in Kerr-AdS black hole spacetime

Koichi Nagasaki^{*}

*School of Physics, University of Electronic Science and Technology of China,
No. 4, Section II, North Jianshe Road, Chengdu 610054, China*

(Received 4 May 2019; published 30 September 2019)

A defect solution in the $\text{AdS}_5 \times S^5$ black hole spacetime is given. This is a generalization of the previous work [K. Nagasaki, H. Tanida, and S. Yamaguchi, *J. High Energy Phys.* **01** (2012) 139.] to another spacetime which includes rotating black holes. The equation of motion for a sort of nonlocal operator, “an interface,” is given, and we solve it by the numerical calculation. This result gives a new example of nonequilibrium systems in the AdS/CFT including the gauge flux on the probe brane.

DOI: [10.1103/PhysRevD.100.066032](https://doi.org/10.1103/PhysRevD.100.066032)

I. INTRODUCTION

Nonlocal operators are a useful tool for studying the AdS/CFT correspondence. An interface is an example of this sort of operator realized in the D3/D5-brane system. We studied this interface in the flat $\text{AdS}_5 \times S^5$ spacetime in Ref. [1]. These brane systems are related to nonequilibrium systems in the AdS/CFT correspondence [2]. In the present work, we would like to consider the space times which has a structure $\text{AdS}_5 \times S^5$ with a gauge flux on the D5-brane. In our past work [1], we studied the interface solution in anti-de Sitter (AdS) spacetime $\text{AdS}_5 \times S^5$ formed by D3-branes. That work showed that nonlocal operators introduce a new parameter which enables us to compare the results from a gauge theory and gravity theory and then actually proved that the results from both theories agree in the first order. This parameter corresponds to the flux on the probe brane. Introducing the flux is important also in the context of the compactification [3]. In this paper, we treat an interface in black hole spacetime. This analysis reveals the behavior of the interface in black hole spacetimes.

Recently, “complexity” was studied to reveal black hole physics [4]. The complexity-action relation [5,6] is a very interesting conjecture. This conjecture was recently studied in various spacetimes [7]. According to this conjecture, the quantity, holographic complexity, which describes the development of the inside of the black hole, is equal to the action calculated in a region called the Wheeler-DeWitt patch. This is a certain spacetime region defined by a given time at each boundary of the black hole spacetime. This is a

holographic relation. Then, there should be a quantity which describes the counterpart in the boundary gauge theories. By this motivation, recent works have tried to find the meaning of complexity in gauge theories [8–15]. Nonlocal properties are found in the context of complexity [16]. Then, nonlocal operators studied in this paper are also related to this topic.

By these motivations, we would like to study more types of nonlocal operators. We studied the effect of the fundamental string for the complexity growth in the previous works [17,18]. So, we are next interested in more various dimensional nonlocal operators. In this paper, the goal is to find the embedding of the probe D5-brane which realizes a nonlocal operator, an interface, including a gauge flux.

The interface was introduced in gauge theories as a defect to separate the boundary space into two gauge theories and was actively studied recently, for example, in Refs. [19,20]. In our previous work [21], we found that the interface can penetrate the horizon only for the case in which the D5-brane has no gauge flux. In this paper, we study a different kind of black hole which has the angular momentum.

In our assumption, the probe brane occupies the S^3 subspace of AdS_5 . There are two ways to parametrize this space. One is hyperspherical coordinates in which the S^3 metric is written as

$$ds^2 = d\vartheta^2 + \sin^2 \vartheta (d\psi_1^2 + \sin^2 \psi_1 d\varphi_2^2). \quad (1)$$

On the other hand, we can choose other polar coordinates (Hopf coordinates) as

$$\begin{aligned} x^1 &= \sin \theta \sin \phi_1, & x^2 &= \sin \theta \cos \phi_1, \\ x^3 &= \cos \theta \sin \phi_2, & x^4 &= \cos \theta \cos \phi_2, \end{aligned} \quad (2)$$

where the metric is expressed as

^{*}koichi.nagasaki24@gmail.com

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2. \quad (3)$$

We used the former coordinates system (1) in the previous analysis [21]. If we take the former case (hyperspherical coordinates), the equation of motion is

$$\begin{aligned} \frac{d}{dz} \left(\frac{\varphi' z^2 f(z)}{\sqrt{1 + \varphi'^2 z^2 f(z)}} \right) &= \frac{4}{z} \frac{\varphi' z^2 f(z)}{\sqrt{1 + \varphi'^2 z^2 f(z)}} - \frac{4K}{z} \\ &\quad - \frac{2 \tan \varphi}{\sqrt{1 + \varphi'^2 z^2 f(z)}}. \end{aligned} \quad (4)$$

We use the latter coordinates system (2) in this paper. This coordinate system includes two circles parametrized by ϕ_1 and ϕ_2 . The radii of these circles change depending on the latitude angle θ .

This paper is organized as follows. In Sec. II, we consider the interface in $\text{AdS}_5 \times S^5$ black hole spacetime without the angular momentum. We use the Hopf coordinates in which the S^3 metric is written as (3). This black hole spacetime corresponds to the case in which the angular momentum is zero in the Kerr-AdS black hole solution. In Sec. III, we introduce the angular momentum. This is the Kerr-AdS black hole spacetime. We investigate the gauge flux dependence of the interface solution. The D5-brane does not penetrate the horizon for nonzero gauge flux as expected from the previous analysis. We also find the angular momentum dependence of the interface solution.

II. INTERFACES IN THE ADS-SCHWARZSCHILD BLACK HOLE SPACETIME

In this section, we find the interface solution in $\text{AdS}_5 \times S^5$ black hole spacetime. This is the solution in the same background to Ref. [21] with other coordinates and assumptions.

A. Action and equation of motion

We set the coordinates as

$$\begin{aligned} \text{AdS}_5: t, \phi_1, \phi_2, r, \vartheta; \quad \vartheta \in [0, \pi/2], \quad \phi_1, \phi_2 \in [0, 2\pi), \\ S^5: \theta, \phi, \varphi_1, \varphi_2, \varphi_3; \quad \theta, \phi, \varphi_1, \varphi_2 \in [0, \pi], \quad \phi_3 \in [0, 2\pi). \end{aligned} \quad (5)$$

The world volume of the D5-brane stretches in the subspace spanned by $(t, \phi_1, \phi_2, r, \vartheta, \theta, \phi)$. It forms one-dimensional subspace in the (r, ϑ) plane. The compact part,

S^5 , is induced to $ds_{S^2}^2 = d\theta^2 + \sin^2 \theta d\phi^2$ on the D5-brane. For the noncompact part, AdS_5 , the bulk metric is

$$\begin{aligned} ds_{\text{AdS}_5}^2 &= -f(r)dt^2 + r^2(\sin^2 \vartheta d\phi_1^2 + \cos^2 \vartheta d\phi_2^2) \\ &\quad + \frac{dr^2}{f(r)} + r^2 d\vartheta^2, \\ f(r) &= 1 + r^2 - \frac{r_m^2}{r^2}. \end{aligned} \quad (6)$$

In the above, the parameter r_m is related to the black hole mass through

$$r_m^2 = \frac{16\pi GM}{5\text{vol}[S^3]} = \frac{8GM}{5\pi}. \quad (7)$$

In the (r, ϑ) plane, ϑ is a function of r :

$$\vartheta = \vartheta(r). \quad (8)$$

This system has the $\text{SO}(3)$ symmetry. Then, by the symmetry, this flux must be proportional to the volume form of S^2 subspace:

$$\mathcal{F} = -\kappa d\theta \wedge \sin \theta d\phi. \quad (9)$$

There is the Ramond-Ramond 4-form defined as

$$C_4 = -r^4 dt \wedge d\vartheta \wedge (\sin \vartheta d\phi_1) \wedge (\cos \vartheta d\phi_2) \quad (10)$$

to satisfy $dC_4 = 4d\text{vol}[\text{AdS}_5]$. From these Ansätze, the induced metric is

$$\begin{aligned} ds_{\text{ind}}^2 &= -f(r)dt^2 + \left(\frac{1}{f(r)} + r^2 \vartheta'(r)^2 \right) dr^2 \\ &\quad + r^2(\sin^2 \vartheta d\phi_1^2 + \cos^2 \vartheta d\phi_2^2) + d\theta^2 + \sin^2 \theta d\phi^2. \end{aligned} \quad (11)$$

Taking the summation with the gauge flux, its determinant is

$$G_{\text{ind}} + \mathcal{F} = \begin{bmatrix} -f(r) & & & & & \\ & 1/f(r) + r^2 \vartheta'^2 & & & & \\ & & r^2 \sin^2 \vartheta & & & \\ & & & r^2 \cos^2 \vartheta & & \\ & & & & 1 & -\kappa \sin \theta \\ & & & & \kappa \sin \theta & \sin^2 \theta \end{bmatrix},$$

$$\Rightarrow \sqrt{-\det[G + \mathcal{F}]} = \sqrt{1 + \kappa^2 \sin \theta r^2 \sin \vartheta \cos \vartheta} \sqrt{1 + \vartheta'^2 r^2 f(r)}. \quad (12)$$

The Wess-Zumino term is, from the gauge flux and the Ramond-Ramond 4-form,

$$S_{\text{WZ}}/T_5 = \int \mathcal{F} \wedge C_4 = (2\pi)^2 \text{vol}[S^2] T \kappa \int dr \sin \vartheta \cos \vartheta r^4 \vartheta'(r), \quad (13)$$

where $\text{vol}[S^2]$ is the volume spanned by θ and ϕ and $(2\pi)^2$ comes from the integral on the two circles spanned by ϕ_1 and ϕ_2 . The D5-brane action is

$$\begin{aligned} S_{\text{D5}}/T_5 &= - \int \sqrt{-\det[G_{\text{ind}} + \mathcal{F}]} + \int \mathcal{F} \wedge C_4, \\ &= -(2\pi)^2 \text{vol}[S^2] T \sqrt{1 + \kappa^2} \int dr \sin \vartheta \cos \vartheta r^2 \left(\sqrt{1 + r^2 f(r) \vartheta'(r)^2} - K r^2 \vartheta'(r) \right), \end{aligned} \quad (14)$$

where we use $K := \kappa/\sqrt{1 + \kappa^2}$ for later convenience. This action is rewritten in coordinates $z = 1/r$ and $\varphi = \pi/4 - \vartheta$ as

$$\begin{aligned} \frac{S_{\text{D5}}}{T_5 (2\pi)^2 \text{vol}[S^2] T} &= -\sqrt{1 + \kappa^2} \int dr \sin \vartheta \cos \vartheta r^2 \left(\sqrt{1 + r^2 f(r) \vartheta'(r)^2} - K r^2 \vartheta'(r) \right) \\ &= -\sqrt{1 + \kappa^2} \int \frac{dz}{2z^4} \cos(2\varphi) \left(\sqrt{1 + z^2 f(z) \varphi'(z)^2} - K \varphi'(z) \right) \\ &= -\frac{\sqrt{1 + \kappa^2}}{2} \int dz \frac{\cos(2\varphi)}{z^4} \left(\sqrt{1 + g(z) \varphi'(z)^2} - K \varphi'(z) \right), \end{aligned}$$

where we defined a function, $g(z) := z^2 f(z) = 1 + z^2 - r_m^2 z^4$. From this action, the equation of motion is

$$\frac{d}{dz} \left(\frac{\cos(2\varphi)}{z^4} \left(\frac{g(z) \varphi'(z)}{\sqrt{1 + g(z) \varphi'(z)^2}} - K \right) \right) + \frac{2 \sin(2\varphi)}{z^4} \left(\sqrt{1 + g(z) \varphi'(z)^2} - K \varphi'(z) \right) = 0. \quad (15)$$

By operating the differentiation of $\cos(2\varphi)/z^4$, this equation is rewritten as follows:

$$\frac{d}{dz} \left(\frac{\varphi' g(z)}{\sqrt{1 + \varphi'^2 g(z)}} \right) = \frac{4}{z} \frac{\varphi' g(z)}{\sqrt{1 + \varphi'^2 g(z)}} - \frac{4K}{z} - \frac{2 \tan(2\varphi)}{\sqrt{1 + \varphi'^2 g(z)}}. \quad (16)$$

Equation (16) differs from the hyperspherical coordinates case [Eq. (4)] only by the factor 2 in the argument of the tangent function.

Introducing a parameter expression $(r(\tau), \varphi(\tau))$, the Lagrangian is

$$\frac{S_{\text{D5}}}{T_5 (2\pi)^2 \text{vol}[S^2] T} = - \int d\tau \mathcal{L}, \quad \mathcal{L} = \frac{\sqrt{1 + \kappa^2}}{2} \frac{\cos(2\varphi)}{z^4} \left(\sqrt{\dot{z}^2 + g \dot{\varphi}^2} - K \dot{\varphi} \right). \quad (17)$$

Since there is a gauge invariance, we can choose the gauge in which $\dot{z}^2 + g\dot{\varphi}^2 = 1$. The equations of motion are

$$z: \frac{d}{d\tau} \left(\cos(2\varphi) \frac{\dot{z}}{z^4} \right) + \frac{4 \cos(2\varphi)}{z^5} (1 - K\dot{\varphi}) - \frac{\cos(2\varphi)}{2z^4} g' \dot{\varphi}^2 = 0, \quad (18a)$$

$$\varphi: \frac{d}{d\tau} \left(\frac{\cos(2\varphi)}{z^4} (g\dot{\varphi} - K) \right) + \frac{2 \sin(2\varphi)}{z^4} (1 - K\dot{\varphi}) = 0. \quad (18b)$$

Let us define $u_z := dz/d\tau$, $u_\varphi := d\varphi/d\tau$. Then, the equations are separated into the following four equations:

$$\frac{dz}{d\tau} = u_z, \quad (19a)$$

$$\frac{d\varphi}{d\tau} = u_\varphi, \quad (19b)$$

$$\frac{du_z}{d\tau} = 2u_z u_\varphi \tan(2\varphi) + \frac{4}{z} (K u_\varphi - g u_\varphi^2) + \frac{g' u_\varphi^2}{2}, \quad (19c)$$

$$\frac{du_\varphi}{d\tau} = -\frac{g'}{g} u_z u_\varphi + 2 \tan(2\varphi) \left(u_\varphi^2 - \frac{1}{g} \right) + \frac{4u_z}{z} \left(u_\varphi - \frac{K}{g} \right). \quad (19d)$$

1. Initial condition

Since near the boundary $z \approx 0$ the solution looks the same as the flat case [1], then $\varphi = \arcsin(\kappa z)$. Taking account of the gauge condition, $u_z^2 + u_\varphi^2 g = 1$, the initial condition is for small z_0 ,

$$\varphi(z_0) = \arcsin(\kappa z_0), \quad (20a)$$

$$u_z(z_0) = \sqrt{\frac{1 - \kappa^2 z_0^2}{1 + \kappa^2 (1 - r_m^2 z_0^4)}}, \quad (20b)$$

$$u_\varphi(z_0) = \frac{\kappa}{\sqrt{1 + \kappa^2 (1 - r_m^2 z_0^4)}}. \quad (20c)$$

Here, we have to confirm the validity of the constraint. Let us examine the evolution of the constraint. By substituting the Eq. (19),

$$\begin{aligned} \frac{d}{d\tau} (u_z^2 + g u_\varphi^2) &= 2u_z \dot{u}_z + 2g u_\varphi \dot{u}_\varphi + u_\varphi^2 g' u_z = 2u_z \left(2u_z u_\varphi \tan(2\varphi) + \frac{4}{z} (K u_\varphi + u_z^2 - 1) + \frac{g' u_\varphi^2}{2} \right) \\ &\quad + 2g u_\varphi \left(-\frac{g'}{g} u_z u_\varphi + 2 \tan(2\varphi) \left(u_\varphi^2 - \frac{1}{g} \right) + \frac{4u_z}{z} \left(u_\varphi - \frac{K}{g} \right) \right) + u_\varphi^2 g' u_z \\ &= 4u_z^2 u_\varphi \tan(2\varphi) + \frac{8u_z u_\varphi}{z} K + \frac{8u_z^3}{z} - \frac{8u_z}{z} + g' u_z u_\varphi^2 \\ &\quad - 2g' u_z u_\varphi^2 + 4 \tan(2\varphi) (g u_\varphi^3 - u_\varphi) + \frac{8g u_z u_\varphi^2}{z} - \frac{8u_z u_\varphi K}{z} + u_\varphi^2 g' u_z \\ &= 4u_\varphi (u_z^2 + g u_\varphi^2 - 1) \tan(2\varphi) + \frac{8u_z}{z} (u_z^2 + g u_\varphi^2 - 1) \\ &= \left(4u_\varphi \tan(2\varphi) + \frac{8u_z}{z} \right) (u_z^2 + g u_\varphi^2 - 1) = 0, \end{aligned} \quad (21)$$

where in the last equality we use the constraint $u_z^2 + g u_\varphi^2 = 1$. Then, this constraint is valid.

B. Interface solution

By the numerical calculation, we solve the equations of motion (19) under the initial condition (20). The result is shown in the following three figures. The interface touches the boundary at the position $\varphi = 0$ and extends toward the inside of the AdS spacetime. The slope of the line, $\varphi_0 = \kappa z_0$, depends on the gauge flux κ near the boundary.

Figure 1 shows the flux dependence of the D5-brane solution. The black hole mass is fixed to $r_m = 10$. The

degree of the slope at the boundary changes depending on the strength of gauge flux. We can see that the D5-brane can penetrate the horizon only for the no gauge flux case.

Figure 2 shows the mass dependence for fixed gauge flux $K = 0.2$. The D5-brane curves tightly for large masses to avoid the horizon. Note that the position of the horizon in z coordinates is given by

$$z^2 f(z) = 1 + z^2 - r_m^2 z^4 = 0, \quad z_h = \frac{1 + \sqrt{1 + 4r_m^2}}{2r_m^2}. \quad (22)$$

FIG. 1. Flux dependence: mass $r_m = 10$. FIG. 2. Mass dependence: $K = 0.2$.

For examples, $z_h = 0.324$ for $r_m = 10$ and $z_h = 0.105$ for $r_m = 90$.

The whole sketch of the D5-brane is drawn in Fig. 3. The metric (11) includes the two cyclic coordinates, ϕ_1 and ϕ_2 , which are coordinates on the two circles with radii $\sin \vartheta$ and $\cos \vartheta$. Then, this induced metric has the structure which includes the product of these two circles. The radii of two circles change depending on $\varphi = \pi/4 - \vartheta$. At the edges $\varphi = \pm\pi/4$, one of these circles shrinks to zero. Since the solution which penetrates the horizon is $\varphi = 0$ and we are interested in how this solution changes according to the gauge flux, we assumed that one of the edge of the brane touches the horizon at $\varphi = 0$ and changes the gauge flux. Then, we found that if the brane has the nonzero gauge flux the brane cannot touch the horizon. The smaller the value of the gauge flux, the closer the D5-brane approaches the horizon.

FIG. 3. Form of the D5-brane in (r, ϑ) ; the area shaded in gray is inside of the horizon.

III. INTERFACE IN THE KERR-ADS BLACK HOLE SPACETIME

Here, we add the black hole angular momentum to the black hole spacetime considered in the previous section. Its metric is called the Kerr-AdS black hole.

A. Action and equation of motion

We set the coordinates on the AdS subspace and S^5 subspace as follows:

$$\begin{aligned} \text{AdS}_5: t, \phi_1, \phi_2, r, \vartheta; \quad \vartheta \in [0, \pi/2], \quad \phi_1, \phi_2 \in [0, 2\pi), \\ S^5: \theta, \phi, \varphi_1, \varphi_2, \varphi_3, \quad \theta, \phi, \varphi_1, \varphi_2 \in [0, \pi], \quad \varphi_3 \in [0, 2\pi). \end{aligned} \quad (23)$$

The world volume of the D5-brane extends over the subspace spanned by $(t, r, \vartheta, \phi_1, \phi_2, \theta, \phi)$, where it forms a curve in the (r, ϑ) plane. Then, the embedding of the D5-brane is expressed as a function ϑ of r :

$$\vartheta = \vartheta(r). \quad (24)$$

We take the Ansatz for the gauge flux as before:

$$\mathcal{F} = -\kappa d\theta \wedge \sin \theta d\phi. \quad (25)$$

The compact part is induced to $ds_{S^5}^2 = d\theta^2 + \sin^2 \theta d\phi^2$ on the D5-brane. The noncompact part is the five-dimensional Kerr-AdS black hole [22]. Its metric is

$$ds_{\text{KA5}}^2 = -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi_1 - \frac{b \cos^2 \theta}{\Xi_b} d\phi_2 \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi_a} d\phi_1 \right)^2 + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left(b dt - \frac{r^2 + b^2}{\Xi_b} d\phi_2 \right)^2 \\ + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{1+r^2}{r^2 \rho^2} \left(ab dt - \frac{b(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi_1 - \frac{a(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\phi_2 \right)^2. \quad (26)$$

In the above,

$$\rho^2(r) = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \\ \Delta_r(r) = \frac{1}{r^2} (r^2 + a^2)(r^2 + b^2)(r^2 + 1) - 2m, \quad \Delta_\theta(\theta) = 1 - a^2 \cos^2 \theta - b^2 \sin^2 \theta, \\ \Xi_a = 1 - a^2, \quad \Xi_b = 1 - b^2.$$

Since the rotations around the two coordinates ϕ_1 and ϕ_2 are equivalent, we focus on the case in which the black hole rotates either of the two axes ($a \neq 0$, $b = 0$). Then, the metric simplifies to

$$ds_{\text{KA5}}^2 = -\frac{\Delta_{ra}}{\rho_a^2} \left(dt - \frac{a \sin^2 \vartheta}{\Xi_a} d\phi_1 \right)^2 + \frac{\Delta_{\vartheta a} \sin^2 \vartheta}{\rho_a^2} \left(a dt - \frac{r^2 + a^2}{\Xi_a} d\phi_1 \right)^2 + \frac{\Delta_{\vartheta a} \cos^2 \vartheta}{\rho_a^2} r^4 d\phi_2^2 \\ + \frac{\rho_a^2}{\Delta_{ra}} dr^2 + \frac{\rho_a^2}{\Delta_{\vartheta a}} d\vartheta^2 + \frac{1+r^2}{\rho_a^2} a^2 r^2 \cos^4 \vartheta d\phi_2^2 \\ = -\frac{\Delta_{ra} - \Delta_{\vartheta a} a^2 \sin^2 \vartheta}{\rho_a^2} dt^2 + \frac{-\Delta_{ra} a^2 \sin^2 \vartheta + \Delta_{\vartheta a} (r^2 + a^2)^2}{\rho_a^2 \Xi_a^2} \sin^2 \vartheta d\phi_1^2 \\ + 2 \frac{\Delta_{ra} - (r^2 + a^2) \Delta_{\vartheta a}}{\rho_a^2 \Xi_a} a \sin^2 \vartheta dt d\phi_1 + r^2 \cos^2 \vartheta d\phi_2^2 + \frac{\rho_a^2}{\Delta_{ra}} dr^2 + \frac{\rho_a^2}{\Delta_{\vartheta a}} d\vartheta^2, \quad (27)$$

where

$$\rho_a^2 = r^2 + a^2 \cos^2 \vartheta, \quad \Delta_{ra} = (r^2 + a^2)(r^2 + 1) - 2m, \quad \Delta_{\vartheta a} = 1 - a^2 \cos^2 \vartheta, \quad \Xi_a = 1 - a^2. \quad (28)$$

The horizon is located at, by coordinate $z := 1/r$,

$$\Delta_{ra} = 0 \Rightarrow (2m - a^2)z^4 - (1 + a^2)z^2 - 1 = 0, \\ z_h^2 = \frac{1 + a^2 + \sqrt{(1 + a^2)^2 + 8m - 4a^2}}{4m - 2a^2}. \quad (29)$$

The Ramond-Ramond 4-form is determined by $4\text{dvol}[\text{AdS}_5] = dC_4$. Let us find the volume form of the AdS part. The above metric (27) can be written as

$$ds_{\text{KA5}}^2 = -\frac{\Delta_{ra}}{\rho_a^2} dt'^2 + \frac{\Delta_{\vartheta a} \sin^2 \vartheta}{\rho_a^2} d\phi_1'^2 + \text{diag}(r, \vartheta, \phi_2), \quad \begin{bmatrix} dt' \\ d\phi_1' \end{bmatrix} = M \begin{bmatrix} dt \\ d\phi_1 \end{bmatrix}, \quad (30)$$

where

$$M := \begin{bmatrix} 1 & -a \sin^2 \vartheta / \Xi_a \\ a & -(r^2 + a^2) / \Xi_a \end{bmatrix}, \\ \det M = -\frac{r^2 + a^2(1 - \sin^2 \vartheta)}{\Xi_a} = -\frac{r^2 + a^2 \cos^2 \vartheta}{\Xi_a} = -\frac{\rho_a^2}{\Xi_a}.$$

Then, the determinant of the metric for the t and ϕ_1 parts is

$$\det G = \det(M^T G' M) = (\det M)^2 \det G' = -\frac{\Delta_{ra} \Delta_{\vartheta a} \sin^2 \vartheta \rho_a^4}{\rho_a^4 \Xi_a^2} = -\frac{\Delta_{ra} \Delta_{\vartheta a} \sin^2 \vartheta}{\Xi_a^2}, \quad (31)$$

and the total factor in AdS is

$$-\frac{\Delta_{ra} \Delta_{\vartheta a}}{\Xi_a^2} r^2 \sin^2 \vartheta \cos^2 \vartheta \frac{\rho_a^2}{\Delta_{ra}} \frac{\rho_a^2}{\Delta_{\vartheta a}} = -\frac{\rho_a^4}{\Xi_a^2} r^2 \sin^2 \vartheta \cos^2 \vartheta. \quad (32)$$

The volume form of the AdS part is

$$d\text{vol}_{\text{AdS}} = \frac{\rho_a^2 r \sin \vartheta \cos \vartheta}{\Xi_a} dt dr d\vartheta d\phi_1 d\phi_2. \quad (33)$$

Then, the Ramond-Ramond 4-form is

$$C_4 = -\frac{\rho_a^4}{\Xi_a} \sin \vartheta \cos \vartheta dt dr d\phi_1 d\phi_2. \quad (34)$$

From the assumption for the embedding (24) and the gauge flux (25), the sum of the induced metric and the gauge flux is

$$G_{\text{ind}} + \mathcal{F} = \begin{bmatrix} -\frac{\Delta_{ra} - \Delta_{\vartheta a} a^2 \sin^2 \vartheta}{\rho_a^2} & \frac{\Delta_{ra} - \Delta_{\vartheta a} (r^2 + a^2)}{\rho_a^2 \Xi_a} a \sin^2 \vartheta & & & & \\ & \frac{\rho_a^2}{\Delta_{ra}} + \frac{\rho_a^2}{\Delta_{\vartheta a}} \vartheta'^2 & & & & \\ \frac{\Delta_{ra} - \Delta_{\vartheta a} (r^2 + a^2)}{\rho_a^2 \Xi_a} a \sin^2 \vartheta & & -\frac{\Delta_{ra} a^2 \sin^2 \vartheta + \Delta_{\vartheta a} (r^2 + a^2)^2}{\rho_a^2 \Xi_a^2} \sin^2 \vartheta & & & \\ & & & r^2 \cos^2 \vartheta & & \\ & & & & 1 & -\kappa \sin \theta \\ & & & & \kappa \sin \theta & \sin^2 \theta \end{bmatrix}.$$

Its determinant is

$$\sqrt{-\det(G_{\text{ind}} + \mathcal{F})} = \sqrt{1 + \kappa^2 \cos \vartheta \sin \vartheta} \frac{r \rho_a}{\Xi_a} \sqrt{\Delta_{ra} \vartheta'^2 + \Delta_{\vartheta a}}.$$

In the above, since $\vartheta \in [0, \pi/2]$, then $\cos \vartheta \geq 0$. The D5-brane action consists of the following:

$$\begin{aligned} S_{\text{D5}} &= S_{\text{DBI}} + S_{\text{WZ}}, \\ S_{\text{DBI}} &= -T_5 \int \sqrt{-\det(G_{\text{ind}} + \mathcal{F})}, \\ S_{\text{WZ}} &= T_5 \int \mathcal{F} \wedge C_4. \end{aligned} \quad (35)$$

We change the variables $z = 1/r$, $\varphi = \pi/4 - \vartheta$. Introducing a parameter τ , $(z(\tau), \varphi(\tau))$,

$$\sqrt{-\det(G_{\text{ind}} + \mathcal{F})} = \sqrt{1 + \kappa^2 d\tau \cos(2\varphi)} \frac{r \rho_a}{2\Xi_a} \sqrt{\Delta_{ra} \dot{\varphi}^2 + \Delta_{\varphi a} \dot{\tau}^2}, \quad (36a)$$

$$\mathcal{F} \wedge C_4 = \frac{\rho_a^4}{2\Xi_a} d\varphi \cos(2\varphi). \quad (36b)$$

The action is

$$S_{D5} = -T_5(2\pi)^2 \text{vol}[S^2] T \int d\tau \mathcal{L}, \quad (37)$$

where the Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{\sqrt{1+\kappa^2}}{2\Xi_a} \cos(2\varphi) \left(\frac{\rho_a}{z^3} \sqrt{\Delta_{\varphi a} \dot{z}^2 + z^4 \Delta_{ra} \dot{\varphi}^2} - K \rho_a^4 \dot{\varphi} \right) \\ &= \frac{\sqrt{1+\kappa^2} \cos(2\varphi)}{2\Xi_a} \frac{1}{z^4} \left(p_a \sqrt{\Delta_{\varphi a} \dot{z}^2 + \Delta_{za} \dot{\varphi}^2} - K p_a^4 \dot{\varphi} \right). \end{aligned} \quad (38)$$

In the second line of the above, we defined

$$\begin{aligned} \rho_a^2 &= \frac{1}{z^2} + \frac{a^2}{2} (1 + \sin(2\varphi)) =: \frac{p_a^2}{z^2}, \\ \Delta_{ra} &= \left(\frac{1}{z^2} + a^2 \right) \left(\frac{1}{z^2} + 1 \right) - 2m =: \frac{\Delta_{za}}{z^4}, \end{aligned} \quad (39a)$$

$$\Delta_{\varphi a} := 1 - \frac{a^2}{2} (1 + \sin(2\varphi)). \quad (39b)$$

We choose the gauge in which $\Delta_{\varphi a} \dot{z}^2 + \Delta_{za} \dot{\varphi}^2 = 1$ to fix the gauge degrees of freedom.

The equations of motion are

$$z: \frac{d}{d\tau} \left(\frac{\cos(2\varphi)}{z^4} p_a \Delta_{\varphi a} \dot{z} \right) + \frac{4 \cos(2\varphi)}{z^5} (p_a - K p_a^4 \dot{\varphi}) - \frac{\cos(2\varphi)}{z^4} \left(\partial_z p_a + \frac{1}{2} p_a \partial_z \Delta_{za} \dot{\varphi}^2 - 4K p_a^3 \partial_z p_a \dot{\varphi} \right) = 0, \quad (40a)$$

$$\begin{aligned} \varphi: \frac{d}{d\tau} \left(\frac{\cos(2\varphi)}{z^4} p_a \Delta_{za} \dot{\varphi} \right) - \frac{d}{d\tau} \left(\frac{\cos(2\varphi)}{z^4} K p_a^4 \right) + \frac{2 \sin(2\varphi)}{z^4} (p_a - K p_a^4 \dot{\varphi}) \\ - \frac{\cos(2\varphi)}{z^4} \left(\partial_\varphi p_a + \frac{1}{2} p_a \partial_\varphi \Delta_{\varphi a} \dot{z}^2 - 4K p_a^3 \partial_\varphi p_a \dot{\varphi} \right) = 0. \end{aligned} \quad (40b)$$

Let us define variables $u_z := dz/d\tau$ and $u_\varphi := d\varphi/d\tau$. Then, the equations are

$$\frac{dz}{d\tau} = u_z, \quad (41a)$$

$$\frac{d\varphi}{d\tau} = u_\varphi, \quad (41b)$$

$$\frac{du_z}{d\tau} = u_z \left(\frac{4u_z}{z} - \frac{\dot{p}_a}{p_a} - \frac{\dot{\Delta}_{\varphi a}}{\Delta_{\varphi a}} + 2u_\varphi \tan(2\varphi) \right) + \Delta_{\varphi a}^{-1} \left(\frac{\partial_z p_a}{p_a} (1 - 4K p_a^3 u_\varphi) + \frac{1}{2} \partial_z \Delta_{za} u_\varphi^2 \right) - \frac{4}{z} \Delta_{\varphi a}^{-1} (1 - K p_a^3 u_\varphi), \quad (41c)$$

$$\begin{aligned} \frac{du_\varphi}{d\tau} &= u_\varphi \left(\frac{4u_z}{z} - \frac{\dot{p}_a}{p_a} - \frac{\dot{\Delta}_{za}}{\Delta_{za}} + 2u_\varphi \tan(2\varphi) \right) + \left(\frac{\cos(2\varphi)}{z^4} p_a \Delta_{za} \right)^{-1} \frac{d}{d\tau} \left(\frac{\cos(2\varphi)}{z^4} K p_a^4 \right) \\ &\quad + \Delta_{za}^{-1} \left(\frac{\partial_\varphi p_a}{p_a} (1 - 4K p_a^3 u_\varphi) + \frac{1}{2} \partial_\varphi \Delta_{\varphi a} u_z^2 \right) - 2 \tan(2\varphi) \Delta_{za}^{-1} (1 - K p_a^3 u_\varphi). \end{aligned} \quad (41d)$$

Let us confirm the validity of the constraint. By substituting these equations of motion,

$$\begin{aligned} \frac{d}{d\tau} (\Delta_{\varphi a} u_z^2 + \Delta_{za} u_\varphi^2) &= 2\Delta_{\varphi a} u_z \dot{u}_z + 2\Delta_{za} u_\varphi \dot{u}_\varphi + \Delta'_{\varphi a}(\varphi) u_z^2 u_\varphi + \Delta'_{za}(z) u_z u_\varphi^2 \\ &= -2\Delta_{\varphi a} u_z^2 \frac{d}{d\tau} \log \left(\frac{\cos(2\varphi)}{z^4} p_a \Delta_{\varphi a} \right) - \frac{8u_z}{z} (1 - K p_a^3 u_\varphi) + 2u_z \left(\frac{\partial_z p_a}{p_a} + \frac{1}{2} \partial_z \Delta_{za} u_\varphi^2 - 4K p_a^2 \partial_z p_a u_\varphi \right) \\ &\quad - 2\Delta_{za} u_\varphi^2 \frac{d}{d\tau} \log \left(\frac{\cos(2\varphi)}{z^4} p_a \Delta_{za} \right) - 4u_\varphi \tan(2\varphi) (1 - K p_a^2 u_\varphi) \\ &\quad + 2u_\varphi \left(\frac{\partial_\varphi p_a}{p_a} + \frac{1}{2} \partial_\varphi \Delta_{\varphi a} u_z^2 - 4K p_a^2 \partial_\varphi p_a u_\varphi \right) + 2K p_a^3 u_\varphi \frac{d}{d\tau} \log \left(\frac{\cos(2\varphi)}{z^4} p_a^4 \right) + \Delta'_{\varphi a} u_z^2 u_\varphi + \Delta'_{za} u_z u_\varphi^2 \\ &= -2(1 - K p_a^3 u_\varphi) \left(-2 \tan(2\varphi) - \frac{4u_z}{z} \right) - \frac{8u_z}{z} (1 - K p_a^3 u_\varphi) - 4u_\varphi \tan(2\varphi) (1 - K p_a^2 u_\varphi) = 0. \end{aligned}$$

Then, this constraint is valid.

Initial condition As in the previous section, we use the approximation $\varphi = \arcsin(\kappa z)$ near the boundary. Taking account of the constraint, $\Delta_{\varphi a} u_z^2 + \Delta_{za} u_\varphi^2 = 1$, for small z_0 ,

$$\varphi_0 = \arcsin(\kappa z_0), \quad (42a)$$

$$u_z(\tau_0) = \sqrt{\frac{1 - \kappa^2 z^2}{\Delta_{\varphi a}(1 - \kappa^2 z^2) + \Delta_{za} \kappa^2}}, \quad (42b)$$

$$u_\varphi(\tau_0) = \frac{\kappa}{\sqrt{\Delta_{\varphi a}(1 - \kappa^2 z^2) + \Delta_{za} \kappa^2}}. \quad (42c)$$

B. Interface solution

We perform the numerical calculation to solve the equations of motion (41) under the initial condition (42). The whole structure of bulk spacetime is the same as in Fig. 3. The result is shown in the following two figures.

Figure 4 shows the angular momentum dependence. This figure tells us that the D5-brane does not penetrate the

FIG. 4. Angular momentum a dependence (1): $m = 10^5$, $K = 0.01$.

FIG. 5. Angular momentum a dependence (3): $m = 10^5$, $K = 0$.

horizon if the D5-brane has the flux for any angular momentums. The black hole mass is fixed to $m = 10^5$, and the strength of the flux is fixed to $K = 0.01$. The location of the horizon is $z_h \approx 0.047$ for $a = 0$.

Figure 5 shows the a dependence for $K = 0$. We see that the D5-brane can enter the horizon only for $a = 0$. Even if the gauge flux is zero $K = 0$, the D5-brane avoids the horizon for the black holes which have the nonzero angular momentum. For example, the horizon is located at $z_h \approx 0.047$ for $a = 0.05$.

IV. CONCLUSIONS

In this paper, we treat the interface which is a codimension-1 object realized by the D5-brane in IIB string theory. We obtained the interface solution in the AdS black hole spacetime by the numerical calculation. The boundary condition is given in the same way for the flat $\text{AdS}_5 \times S^5$ case imposed in Ref. [1]; that is, the degree of the slope at the boundary changes according to the gauge flux. The D5-brane is bent inside of spacetime by the effect of the black hole. The D5-brane penetrates the horizon only for the case $K = 0$ and $a = 0$. We found in Ref. [21] that this probe brane can enter the horizon only for the case $K = 0$ and in this paper that it cannot enter the horizon if the black holes have the angular momentum, either.

For boundary theories, we can also have an interesting conclusion. By the gauge/gravity correspondence, the bulk has super-Yang-Mills theories with gauge groups $\text{SU}(N)$ and $\text{SU}(N - k)$ on the boundary. Here, the difference k is related to the parameter κ as we can check by integrating the gauge flux on the D5-brane [1]

$$k = -\frac{T_5}{T_3} \int \mathcal{F} = \frac{\kappa}{\pi \alpha'}. \quad (43)$$

The edge of the probe D5-brane touching the boundary corresponds to the interface which separates these two gauge theories. As we see in this paper, the static probe D5-brane without the gauge flux is the only solution which penetrates the horizon. Then, in this case, there is the interface located at $\varphi = 0$, and there are two gauge theories over $-\pi/4 \leq \varphi \leq 0$ and $0 \leq \varphi \leq \pi/4$ (see Fig. 3). If the gauge flux is nonzero, the probe brane bends and touches the boundary at two positions. Then, in this case, there are two interfaces, and there are three gauge theories at the boundary. The difference of the gauge groups between both sides of the interface is given by k . We see the transition from small k to large k (see Fig. 6). Suppose the gauge group of the bottom side of the interface at $\varphi = 0$ is lower than the upper side (left of Fig. 6). There are two interfaces in this case, and they separate the boundary into three gauge theories with gauge groups $\text{SU}(N)$, $\text{SU}(N - k)$, and $\text{SU}(N)$. As the strength of the gauge flux increases, the D5-brane bends more tightly. At the critical point, the two interfaces coincide and degenerate to an interface with the

FIG. 6. A sketch of the transition according to the gauge flux.

same gauge group $SU(N)$ on the both sides (middle of Fig. 6). If the strength of the gauge flux exceeds the critical point, then a self-intersection occurs (right of Fig. 6). For the interface at $\varphi = 0$, the gauge group is lower in the bottom side, as we can see in the left figure. Then, a gauge theory with $SU(N+k)$ occurs on the upper side of this interface. Then, in the resulting situation, there are three gauge theories with gauge groups $SU(N)$, $SU(N+k)$, and $SU(N)$.

In this paper, we consider the case in which the black holes have the angular momentum but the probe brane is static. Considering a moving defect is important to test the complexity-action (CA) conjecture [5,23]. This conjecture

states that the holographic complexity is equal to the bulk action [24]. The test of this conjecture is important for prove the holographic relation. Since the growth of the action is calculated in the inside of the horizon, it is important to study nonlocal operators which affect this region. It will be an important theme to study the CA conjecture and black hole complexity.

ACKNOWLEDGMENTS

I would like to thank Satoshi Yamaguchi, UESTC members, and people who had conversations at the Physical Society of Japan.

-
- [1] K. Nagasaki, H. Tanida, and S. Yamaguchi, Holographic interface-particle potential, *J. High Energy Phys.* **01** (2012) 139.
 - [2] S. R. Das, T. Nishioka, and T. Takayanagi, Probe branes, time-dependent couplings and thermalization in AdS/CFT, *J. High Energy Phys.* **07** (2010) 071.
 - [3] P. Koerber and L. Martucci, D-branes on AdS flux compactifications, *J. High Energy Phys.* **01** (2008) 047.
 - [4] L. Susskind, Three lectures on complexity and black holes, [arXiv:1810.11563](https://arxiv.org/abs/1810.11563).
 - [5] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Holographic Complexity Equals Bulk Action? *Phys. Rev. Lett.* **116**, 191301 (2016).
 - [6] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Complexity, action, and black holes, *Phys. Rev. D* **93**, 086006 (2016).
 - [7] A. R. Brown, H. Gharibyan, H. W. Lin, L. Susskind, L. Thorlacius, and Y. Zhao, The case of the missing gates: Complexity of Jackiw-Teitelboim gravity, *Phys. Rev. D* **99**, 046016 (2019).
 - [8] W. C. Gan and F. W. Shu, Holographic complexity: A tool to probe the property of reduced fidelity susceptibility, *Phys. Rev. D* **96**, 026008 (2017).
 - [9] R. Jefferson and R. C. Myers, Circuit complexity in quantum field theory, *J. High Energy Phys.* **10** (2017) 107.
 - [10] S. Chapman, H. Marrochio, and R. C. Myers, Holographic complexity in Vaidya spacetimes. Part I, *J. High Energy Phys.* **06** (2018) 046.
 - [11] S. Chapman, J. Eisert, L. Hackl, M. P. Heller, R. Jefferson, H. Marrochio, and R. C. Myers, Complexity and entanglement for thermofield double states, *SciPost Phys.* **6**, 034 (2019).
 - [12] M. Guo, J. Hernandez, R. C. Myers, and S.-M. Ruan, Circuit complexity for coherent states, *J. High Energy Phys.* **10** (2018) 011.

- [13] S. Chapman, H. Marrochio, and R. C. Myers, Holographic complexity in Vaidya spacetimes II, *J. High Energy Phys.* **06** (2018) 114.
- [14] R. Khan, C. Krishnan, and S. Sharma, Circuit complexity in Fermionic field theory, *Phys. Rev. D* **98**, 126001 (2018).
- [15] A. R. Brown and L. Susskind, The complexity geometry of a single Qubit, *Phys. Rev. D* **100**, 046020 (2019).
- [16] Z. Fu, A. Maloney, D. Marolf, H. Maxfield, and Z. Wang, Holographic complexity is nonlocal, *J. High Energy Phys.* **02** (2018) 072.
- [17] K. Nagasaki, Complexity of AdS_5 black holes with a rotating string, *Phys. Rev. D* **96**, 126018 (2017).
- [18] K. Nagasaki, Complexity growth of rotating black holes with a probe string, *Phys. Rev. D* **98**, 126014 (2018).
- [19] M. Gutperle and J. D. Miller, Topological interfaces in Chern-Simons theory and $\text{AdS}_3/\text{CFT}_2$, *Phys. Rev. D* **99**, 026014 (2019).
- [20] A. Ovgun and K. Jusufi, Complexity growth rates for AdS black holes with dyonic/ nonlinear charge/ stringy hair/ topological defects, [arXiv:1801.09615](https://arxiv.org/abs/1801.09615).
- [21] K. Nagasaki, Interface in AdS black hole spacetime, [arXiv:1812.07162](https://arxiv.org/abs/1812.07162).
- [22] R. G. Cai, S. M. Ruan, S. J. Wang, R. Q. Yang, and R. H. Peng, Action growth for AdS black holes, *J. High Energy Phys.* **09** (2016) 161.
- [23] S. Janiszewski and A. Karch, Moving defects in AdS/CFT, *J. High Energy Phys.* **11** (2011) 044.
- [24] V. E. Hubeny, Extremal surfaces as bulk probes in AdS/CFT, *J. High Energy Phys.* **07** (2012) 093.