Effective alignments and the landscape of S_4 flavor models

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We explore the concept of effective alignments: contractions of multiple flavor symmetry breaking flavon fields. These contractions give rise to directions that are hard or impossible to obtain directly by breaking the flavor symmetry. Within this context, and using S_4 as the flavor symmetry to exemplify, we perform a phenomenological check of lepton flavor models built from pairing any two effective alignments up to order 2 (in flavon contractions). The check is performed for each pair of effective alignments in a framework with models of constrained sequential dominance type, in a basis where the charged leptons are diagonal. We thus obtain an indication of which effective alignments are interesting for model building, within this so-called S_4 landscape. We find three types of viable topologies and provide examples of models realizing this strategy for each topology.

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I. INTRODUCTION

The formulation of the Standard Model (SM) was one of the biggest successes of particle physics, accounting for most interactions of matter known to date. Notwithstanding its success, it is still a theory that leaves unanswered some questions. Why are there three copies of fermions? Why are their masses hierarchical? What gives rise to the specific mixing patterns observed? These questions (among others) are part of what is known as the “flavor problem” (for insight and reviews, see e.g., [1–7]).

The inclusion of a flavor symmetry to the SM became a prolific way to attempt to meaningfully answer some of the questions posed by the flavor problem. These flavor symmetries do not affect the gauge structure of the model; rather they restrict the way the different particles are able to interact with each other in the Yukawa sector. As such, this strategy has been widely used to predict the leptonic mixing structure, for which the flavor symmetries favored are typically non-Abelian discrete symmetries. In these flavor models, we can take the SM to be a low-energy version of a more complete model which includes a flavor symmetry that has been broken by either scalar fields that are singlets with respect to the SM gauge structure (these are called

flavons and it is this paradigm that we will be discussing) or by enlargement of the Higgs sector. Then, the vacuum expectation values (VEVs) of the scalars dictate the Yukawa structures. In leptonic flavor models, the type-I seesaw mechanism is employed and sequential dominance (SD) [8–11] is a useful paradigm for model building, with constrained sequential dominance (CSD) [12] remaining a viable strategy to produce predictive models [13–18].

Since in the indirect approach models the mixing structure is defined by the flavon VEVs, the problem trickles down to choosing appropriate VEVs to match the observed mixing pattern. Nevertheless, this is still a question that must be in accordance with the scalar potential of the chosen flavor model (both the flavor symmetry chosen and the number of flavons will influence the outcome). Moreover, cross terms in the scalar potential that arise from the interaction of the different flavons will, in general, influence the direction of the flavon VEVs [19].

One novel way to explore possible leptonic flavor models is through effective alignments (EAs) [20]. EAs are directions that arise from higher-order operators in which multiple flavons contract, giving rise to a much richer structure. While there are some ambiguities with respect to invariants built from flavon contractions in higher-order operators, it was shown in [20] (see Sec. III therein) that the way flavons contract can be uniquely determined by considering the UV completion, namely, by including heavy vectorlike fermions. Given a renormalizable model at high scale, the decoupling of the vectorlike fermions leaves only unique EAs at low energy. We will systematically do a scan for more EAs. We will focus on the S_4 symmetry group, a group with a long tradition of being used to explain large neutrino mixing angles [21]. To simplify the work, we will

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consider contractions of no more than two S_4 flavons, and do an exploratory survey of the resulting landscape, to ascertain at a primary level if there are any EAs compatible with current experimental data of neutrino oscillations.

After reviewing the list of possible minima of the S_4 scalar potential [22–24], we take these to be the possible VEV alignments our flavons can take. These will be the starting point from which EAs can be constructed, through contractions up to two flavons.

The S_4 symmetry group, this mechanism of flavon contraction, as well as the list of possible VEV alignments, will be the focus of Sec. II A. Afterwards, in Sec. II B, we consider the leptonic sector of CSD models, describing how lepton mixing is formulated in this framework, as well as how pairs of EAs can be used to produce leptonic mixing. In Sec. III, we go through the numerical method used, and present the obtained results, to ascertain which of the possible EAs studied could lead to the observed leptonic mixing. In Sec. IV, we present specific UV complete [25–28] realizations of models with the pairs of EAs that were found to lead to viable mixing. Finally, we present our conclusions in Sec. V.

II. FLAVON VACUUM ALIGNMENTS IN S_4

We briefly review the idea of effective alignments [20] in the flavor space.

The theory is assumed to be invariant under a flavor symmetry. As in [20], we consider as example S_4 , the group of the permutations of four objects; see, e.g., [29–31]. It has 24 distinct elements, which are generated by three generators S , T , and U satisfying the relations $T^3 = S^2 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = 1$, which automatically

leads to $(STU)^4 = 1$. S_4 has five irreducible representations (irreps), **1**, **1'**, **2**, **3** and **3'**. Representation matrices of generators are given in Table IV, Appendix A.

A. Flavon VEV configurations

Following the discussion in [20], we take the S_4 potential of a complex triplet flavon $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$, in the irreducible representation **3** or **3'** of S_4 . By embedding an additional $U(1)$ symmetry, the renormalizable flavon potential takes a simple form as

$$V(\varphi) = \mu_\varphi^2 I_1 + g_1 I_1^2 + g_2 I_2 + \frac{1}{2} g_3 (I_3 + \text{H.c.}) \quad (1)$$

with

$$\begin{aligned} I_1 &= |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2, \\ I_2 &= |\varphi_1 \varphi_2|^2 + |\varphi_2 \varphi_3|^2 + |\varphi_3 \varphi_1|^2, \\ I_3 &= (\varphi_1^* \varphi_2)^2 + (\varphi_2^* \varphi_3)^2 + (\varphi_3^* \varphi_1)^2. \end{aligned} \quad (2)$$

Vacuum stability requires the conditions $\mu_\varphi^2 < 0$, $g_1 > 0$ and $3g_1 + g_2 > |g_3|$ to be satisfied at tree level.

The vacuum is selected by minimizing the potential, in detail, by solving the extremization condition $\partial V(\varphi)/\partial \varphi_i = 0$ and requiring the positivity condition $\partial^2 V(\varphi)/\partial \varphi_i \partial \varphi_j$. The equation $\partial V(\varphi)/\partial \varphi_i = 0$ can be solved explicitly. Up to a normalization factor, there are in total 21 VEV candidates obtained [20]. They are classified into four classes, which are defined by sets of VEVs related by flavor symmetry elements. In the basis of [32,33] these classes are given by

$$\begin{aligned} a &= \frac{1}{\sqrt{3}} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \right\}; \\ b &= \left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{3+\sqrt{3}} \\ \frac{-1}{3-\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{3-\sqrt{3}} \\ \frac{1}{3+\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\omega^2}{3+\sqrt{3}} \\ \frac{-\omega}{3-\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-\omega^2}{3-\sqrt{3}} \\ \frac{\omega}{3+\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\omega}{3+\sqrt{3}} \\ \frac{-\omega^2}{3-\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-\omega}{3-\sqrt{3}} \\ \frac{\omega^2}{3+\sqrt{3}} \end{pmatrix} \right\}; \\ c &= \left\{ \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3}\omega^2 \\ -\frac{2}{3}\omega \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3}\omega \\ -\frac{2}{3}\omega^2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}; \\ d &= \left\{ \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3}\omega \\ -\frac{2}{3}\omega^2 \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3}\omega^2 \\ \frac{1}{3} \\ -\frac{2}{3}\omega \end{pmatrix}, \begin{pmatrix} -\frac{2}{3}\omega^2 \\ -\frac{2}{3}\omega \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3}\omega \\ \frac{1}{3} \\ -\frac{2}{3}\omega^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \end{aligned} \quad (3)$$

where $\omega = e^{i2\pi/3}$. Whether these solutions are global minimums depends on values of g_1 , g_2 and g_3 .

- (i) For $g_2 + g_3 > 0$ and $g_2 - g_3 > 0$, class a is the global minimum.
- (ii) For $g_2 + g_3 > 0$ and $g_2 - g_3 < 0$, class b is the global minimum.
- (iii) For $g_2 + g_3 < 0$ and $g_3 < 0$, class c is the global minimum.
- (iv) For $g_2 + g_3 > 0$ and $g_3 > 0$, class d is the global minimum.

All solutions in each class (e.g., a_1 , a_2 and a_3 in class a) are degenerate. Given that they are related by a flavor symmetry element, in the single flavon case, it is a random choice to select any of the solutions within a class. However, once the model involves several flavons whose VEVs are in the same class, the situation with VEVs taking the same solution is physically different from the case where they take different solutions [34]. Furthermore, there may be some residual symmetry preserved in some of these VEVs, e.g., a_1 preserving $Z_2 = \{1, S\}$, c_4 preserving $Z_3 = \{1, T, T^2\}$. For more discussion on this aspect, please see [20].

We collect all these solutions in the set

$$S_{\text{VEV}}^{(1)} = \{a_1, a_2, a_3, b_1, b_1^*, b_2, b_2^*, b_3, b_3^*, c_1, c_2, c_3, c_4, d_1, d_1^*, d_2, d_2^*, d_3, d_3^*, d_4, d_4^*\}. \quad (4)$$

In the following, we regard the 21 VEV alignments in $S_{\text{VEV}}^{(1)}$ as building blocks of the other EAs. For any $x, y, z, \dots \in S_{\text{VEV}}^{(1)}$, the three-dimensional irreducible representations $(xy)_3$ and $(xy)_{3'}$ form a set $S_{\text{VEV}}^{(2)}$; $((xy)_3 z)_3$, $((xy)_3 z)_{3'}$, $((xy)_{3'} z)_3$ and $((xy)_{3'} z)_{3'}$ form another set $S_{\text{VEV}}^{(3)}$; and so on. Following the procedure, one can in principle get $S_{\text{VEV}}^{(3)}$, $S_{\text{VEV}}^{(4)}$, We define their connection as S_{VEV} ,

$$S_{\text{VEV}} = S_{\text{VEV}}^{(1)} \cup S_{\text{VEV}}^{(2)} \cup S_{\text{VEV}}^{(3)} \cup \dots \quad (5)$$

We have checked that from the 21 EAs in $S_{\text{VEV}}^{(1)}$, 174 additional EAs are generated in $S_{\text{VEV}}^{(2)}$ and another 2988 EAs are generated in $S_{\text{VEV}}^{(3)}$.

We note that unlike the VEVs within $S_{\text{VEV}}^{(1)}$, EAs within $S_{\text{VEV}}^{(2)}$ and beyond ($S_{\text{VEV}}^{(3)}$ and so on) are not actually VEVs of physical fields, and they do not affect the minimization of the potential. We consider the potential to be renormalizable, and it is correct to minimize with the actual scalar fields only. It is only in the terms involving the fermions that we couple EAs, in nonrenormalizable terms which can be made renormalizable in suitable UV completions (see Sec. IV for explicit examples). We do note however that cross terms between multiple flavons (whose VEVs would be within $S_{\text{VEV}}^{(1)}$) can be relevant. Such cross terms can be avoided in supersymmetric flavor models, and the issue of

cross terms in general has been discussed in more detail in [20].

B. Lepton mixing in the CSD framework

We can now use the flavon VEVs and the EAs to build models in the framework of CSD. We write the Lagrangian terms as

$$-\mathcal{L}_l = (Y'_\tau \bar{\ell}_L)_1 H \tau_R + (Y'_\mu \bar{\ell}_L)_1 H \mu_R + (Y'_e \bar{\ell}_L)_1 H e_R + \text{H.c.},$$

$$-\mathcal{L}_\nu = \sum_i (Y_i^{(i)} \bar{\ell}_L)_{1^{(i)}} \tilde{H} N_i + \frac{1}{2} M_i \bar{N}_i N_i + \text{H.c.}, \quad (6)$$

where i labels the different right-handed neutrinos (RHNs). The minimal case corresponding to a massless active neutrino occurs for $i = 1, 2$. We assume for this paper that the charged lepton Yukawa couplings are diagonal up to a row permutation, namely that

$$Y_{e,\mu,\tau} \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (7)$$

This is easily achieved with S_4 in the basis we are working on, e.g., by using the VEV $d_4 = (0, 0, 1)^T$ and EAs built from it (e.g., in [20] the flavon Φ_ℓ had the VEV of this form). Other EAs, e.g., $c_4 = (1, 0, 0)^T$, $d_4^* = (0, 1, 0)^T$ or their contractions can also be used to construct a diagonal charged lepton mass matrix up to a row permutation. With this assumption, the model building basis coincides with the basis where charged leptons are diagonal and mixing comes just from the neutrino sector, where the mass matrix is

$$M_\nu = \sum_i \frac{v^2}{M_{N_i}} Y_i^{(i)} Y_i^{(i)T} = \sum_i \mu_i V_i V_i^T, \quad (8)$$

μ_i being complex mass parameters. Here, V_i and all EAs in Eq. (7) should be included in the set S_{VEV} .

We focus on models with only two RHNs ($i = 1, 2$) which are particularly predictive, where one μ_i vanishes and correspondingly one of the active neutrinos is massless. We can then distinguish two cases:

- (i) V_1 is orthogonal to V_2 , $V_1^\dagger V_2 = 0$. This completely fixes the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix

$$U_{\text{PMNS}} = (U_1, V_1, V_2) \quad \text{or} \quad (U_1, V_2, V_1), \quad (9)$$

up to Majorana phases and a row permutation, and where U_1 denotes the column of the PMNS that is orthogonal to V_1 and V_2 . The mixing matrix is unrelated to neutrino masses (so-called mass-independent mixing schemes or form-diagonalizable

schemes [35]). To fit current data, often small corrections δV_2 or δV_3 are necessary (see e.g., [36,37]).

- (ii) V_1 is not orthogonal to V_2 . The mixing matrix can be parametrized by a rotation between the second and third columns,

$$U_{\text{PMNS}} = (U_1, V_1, V_2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta e^{-i\gamma} \\ 0 & -\sin \theta e^{i\gamma} & \cos \theta \end{pmatrix}, \quad (10)$$

where θ and γ depend on μ_1/μ_2 , i.e., depend on the neutrino mass ratio. A well-known example is TM1 mixing [38–41], when $V_1, V_2 \perp (2, -1, -1)^T$. S_4 can be used to achieve TM1 [42], as can other groups [43]. Most of the models we consider here are examples of this type, as are the CSD2 models with two RHN neutrinos [13,15]. The ambiguity of the permutation of the rows of the PMNS matrix will always be taken into account.

III. THE S_4 LANDSCAPE

We are interested in testing the landscape of S_4 flavor models in the CSD paradigm described in Sec. II B. We thus test each pair of EAs resultant of Eq. (5) up to $S_{\text{VEV}}^{(2)}$. This includes cases with just two flavons (one flavon for each EA), with three flavons (one flavon for one EA and two for the other) and four flavons (two flavons for each EA).

Following the steps in the Appendix of [44], the mixing angles, δ_{CP} , and the two mass-squared differences are extracted from the neutrino mass matrix (we take the ambiguity of the permutation of the rows of the PMNS matrix into account, as discussed in Sec. II B). Then a standard χ^2 analysis is performed to compare these observables and ascertain to which extent each model (corresponding to a pair of EAs) is compatible with experimental data. The χ^2 is computed individually for each experimental value,

$$\chi_i^2 = \frac{(x_i^{\text{model}} - x_i^{\text{bfp}})^2}{\sigma_{x_i}^2}, \quad (11)$$

where x_i is any one of the six test metrics (meaning the six tested observables: three mixing angles, two mass-squared differences, and a phase); x_i^{model} is the value the test metric assumes for that particular model; x_i^{bfp} is the best-fit point; and σ_{x_i} is the corresponding difference between the three (or one) σ values for the test metric. Since the best-fit point is not perfectly centered around the three (one) σ regions, the σ_{x_i} values are distinct whether the test metric for the model is larger, or smaller, than the best-fit point. A model

is then considered compatible with experimental data (taken from [45]) when $\chi_i^2 \leq 1$ for all six test metrics.

In order to achieve the optimal x_i^{model} values for the test metrics for each model, we compute the sum over all χ_i^2 , which acts as the minimizing value that is the target for the MINUIT minimization routine.¹ This routine takes random starting point values for the model's free parameters (the μ_i), and then searches for optimal values for these values which minimize the target variable ($\sum_i \chi_i^2$ in our case). In this way, the MINUIT routine will search for the optimal set of values of the free parameter space which results in the average best fit for the model's predictions (our six test metrics), given the best-fit points of each test metric and the chosen validity interval (three or one σ range). After searching for EA pairs compatible with the 1σ experimental range for all observables and not finding any, we always set σ_{x_i} to the respective 3σ range.

With all σ_{x_i} set to the respective 3σ range we compute the χ_i^2 and their sum, χ^2 , and display the results for each pair of EAs in Fig. 1 for normal ordering (top) and inverted ordering (bottom), where a heat map was constructed according to the χ^2 . In Fig. 1, the axes have EAs with progressively larger entries, e.g., the first EA is (0,0,1) and up to EA 17 the first entry of the EA remains at zero, with EA 18 having a first entry with a small magnitude and so on. Since each EA pair leads to identical results regardless of their ordering (permuting the two EAs is equivalent to permuting μ_1 with μ_2), we omit the redundant results from the upper triangle of each plot in Fig. 1.

To better display the results, we show two different ranges, and additionally mark viable EA pairs (i.e., those for which our six observables fall within the 3σ range) with black crosses. The plots on the left of Fig. 1 have $\chi^2 \leq 10$, whereas those on the right have $\chi^2 \leq 100$. We selected these ranges to enhance the presentation of the relevant results.

Figure 1 shows that the overwhelming majority of the EA pairs cannot achieve results which are compatible with the experimental data, returning $\chi^2 = \sum_i \chi_i^2$ orders of magnitude above the unit value. Furthermore, it is also clear that the normal ordering scenario is generally favored over the inverted ordering scenario. An interesting aspect that can be visualized in Fig. 1 is the presence of some vertical and horizontal features, seen more clearly in the plots on the right. These features indicate that some EAs are preferred by current experimental data, in the sense that those preferred EAs have (relatively) small χ^2 when paired with multiple other EAs. We also see that pairs in the bottom left corner never have $\chi^2 < 100$, which is due to both EAs having the form (0, y, z), which would lead to a vanishing mixing angle.

¹The MINUIT minimization routine is part of the CERN program library [46].

FIG. 1. Heat maps showing the χ^2 for the S_4 flavor model landscape in the CSD paradigm with two RHNs, for each pair of EAs up to $S_{\text{VEV}}^{(2)}$ [Eq. (5)]. The axes label the EAs according to the criteria described in the text. The top row is for normal ordering and the bottom row for inverted ordering. Panels on the left (right) show only pairs of EAs that have χ^2 under 10 (100). Crosses mark the EA pairs that are viable (within the 3σ experimental ranges).

Nevertheless, a small number of cases for each scenario are able to achieve results within the 3σ experimental range (these EA pairs are marked with black crosses). These cases are among the viable pairs (viable meaning here within the 3σ experimental range). Note that since we have allowed a row permutation in Eq. (7), some of these pairs may give the same predictions, e.g.,

$$V_1 = c_1 \propto \begin{pmatrix} 1 \\ -2\omega^2 \\ -2\omega \end{pmatrix}, \quad V_2 = (d_1^* d_4)_{3_s} \propto \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \quad (12)$$

compared with

$$V'_1 = c_2 \propto \begin{pmatrix} 1 \\ -2\omega \\ -2\omega^2 \end{pmatrix}, \quad V'_2 = (d_1 d_4^*)_{3_s} \propto \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}. \quad (13)$$

After extracting out this redundancy, of the 102 pairs of EAs that are viable, we are left with 53 viable pairs, which give different predictions for neutrino masses and flavor mixing. These viable cases for normal ordering are listed in Table V of Appendix B. For the inverted ordering, only 12

pairs of EAs are viable. After extracting out the same type of redundancy, there are 6 pairs left, as shown in Table VI of Appendix B.

We present here some examples of viable EA pairs, showing the respective contractions from VEVs [shown in (3)] that give rise to those EAs:

$$c_1 \propto \begin{pmatrix} 1 \\ -2\omega^2 \\ -2\omega \end{pmatrix}, \quad (d_1^* d_4)_{3_s} \propto \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}; \quad (14)$$

$$(a_1 c_2)_{3_s} \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (b_2 b_3)_{3_s} \propto \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} + \frac{1}{\sqrt{3}} \\ \frac{1}{3} - \frac{1}{\sqrt{3}} \end{pmatrix}; \quad (15)$$

$$(a_2 c_1)_{3_s} \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (c_1 d_2)_{3_s} \propto \begin{pmatrix} 2\omega^2 \\ 3\omega - 1 \\ 2\omega \end{pmatrix}. \quad (16)$$

We note that the pair in Eq. (14) is particularly interesting, since one of the EAs is one of the VEVs (in this case, c_2) in Eq. (3), i.e., part of $S_{\text{VEV}}^{(1)}$. Also noteworthy is the other EA in this pair: up to the minus sign it is a permutation of the (1,4,2) direction, which we obtain here in a much simpler fashion than in the original CSD4 alignment [14]. In addition, Eqs. (14) and (15) correspond to examples with TM1 mixing [38–41].

We note also that there are no viable cases where both EAs are VEVs (part of $S_{\text{VEV}}^{(1)}$). Within the viable cases we have found three distinct topologies, represented by each of the examples in Eqs. (14)–(16):

- (i) $\Phi_a + \Phi_b \Phi_c$, where one EA is a VEV Φ_a (part of $S_{\text{VEV}}^{(1)}$) and the other belongs to $S_{\text{VEV}}^{(2)}$ without involving Φ_a .
- (ii) $\Phi_a \Phi_b + \Phi_c \Phi_d$, where both EAs belong to $S_{\text{VEV}}^{(2)}$ without repeated VEVs.
- (iii) $\Phi_a \Phi_b + \Phi_b \Phi_c$, where both EAs belong to $S_{\text{VEV}}^{(2)}$ with a single repeated VEV appearing in both EAs.

There are also models with repeated flavons such as $\Phi_a + \Phi_a \Phi_b$ or $\Phi_a \Phi_b + \Phi_b \Phi_a$, but in the S_4 landscape none of these options is experimentally viable. Indeed of the cases with repeated flavons, $\Phi_a \Phi_b + \Phi_b \Phi_c$ leads to viable cases for normal ordering only.

IV. EXAMPLES OF VIABLE MODELS

We now produce some example models for each topology found in Sec. III. In the following $\tilde{H}_\alpha$ transforms (like H) as an $SU(2)$ doublet, $\tilde{H}_\alpha = \epsilon_{\alpha\beta} H_\beta$. Further, we do not specify the S_4 assignments here for simplicity (see Appendix A and [20] for more details).

As we are giving specific UV completions, the scalar potential is required to be renormalizable at the UV-energy

scale. Nevertheless, when taken as a low-energy effective theory, there are some corrections to the scalar potential, stemming from the scalars' couplings to the heavy fermion messengers A , B , and C , as well as the Majorana neutrinos N , and even the SM leptons. There are no relevant corrections at tree level, but at loop level, it is possible for some of the topologies presented in the UV-completions to allow for contributions to the quartic scalar cross terms, such as e.g., $(\Phi_B \Phi_C)(\Phi_B^\dagger \Phi_C^\dagger)$. Nevertheless, cross terms of this type are already allowed at the renormalizable level (they are not forbidden by the Z_N symmetry; see the discussion in [20]), and the potential contributions stemming from the presence of the fermion messengers are both loop suppressed and suppressed by the masses of the fermions running in the loop.

With respect to the mass scales of the heavy fermionic messengers, one viable possibility is to have a hierarchy such that their masses are heavier than the masses of the RH neutrinos which in turn are heavier than the Higgs mass $M_A, M_B, M_C \gg M_{1,2} \gg m_H$. Such a hierarchy would further suppress all higher order corrections, both to the scalar potential as well as to e.g., the neutrino masses.

A. $\Phi_a + \Phi_b \Phi_c$

The first topology found in Sec. III is where one of the EAs is a single flavon and the other EA is a two-flavon contraction without repeated flavons. For this topology we write the Lagrangian with renormalizable terms

$$\mathcal{L}_\nu = \bar{\ell}_L \tilde{H} A + \bar{A} \Phi_a N_1 + \bar{A} \Phi_b B + \bar{B} \Phi_c N_2 \quad (17)$$

$$+ M_A \bar{A} A + M_B \bar{B} B + M_1 \bar{N}_1^c N_1 + M_2 \bar{N}_2^c N_2 + \text{H.c.}, \quad (18)$$

which can be implemented for example with an additional Z_8 with charges listed in Table I.

The Lagrangian in Eq. (18) corresponds to the effective diagrams in Figs. 2 and 3.

One specific example of this topology is Eq. (14), i.e., for $\Phi_a = c_2$, and $\Phi_b = d_1^*$ and $\Phi_c = d_4$ contracting as $(d_1^* d_4)_{3_s}$ to make the EA.

B. $\Phi_a \Phi_b + \Phi_c \Phi_d$

The second topology found in Sec. III is where both EAs are two-flavon contractions without repeated flavons. The corresponding renormalizable Lagrangian terms are

$$\mathcal{L}_\nu = \bar{\ell}_L \tilde{H} A + \bar{A} \Phi_a B + \bar{B} \Phi_b N_1 + \bar{A} \Phi_c C + \bar{C} \Phi_d N_2 \quad (19)$$

TABLE I. Fields and Z_8 charges in additive notation (−3 to 4) for models with topology $\Phi_a + \Phi_b \Phi_c$.

Field	ℓ_L	H	N_1	N_2	Φ_a	Φ_b	Φ_c	A	B
Z_8	−1	0	0	4	−1	−3	−2	−1	2

FIG. 2. Effective diagram for EA coming from Φ_a . FIG. 3. Effective diagram for EA coming from Φ_b and Φ_c .

$$+M_A \bar{A}A + M_B \bar{B}B + M_C \bar{C}C + M_1 \bar{N}_1^c N_1 + M_2 \bar{N}_2^c N_2 + \text{H.c.}, \quad (20)$$

which can be implemented for example with an additional Z_{10} with charges listed in Table II.

The Lagrangian in Eq. (22) corresponds to the effective diagrams in Figs. 4 and 5.

Equation (15) exemplifies this topology, with $\Phi_a = a_1$, $\Phi_b = c_2$, $\Phi_c = b_2$, $\Phi_d = b_3$ and the EAs being the contractions $(a_1 c_2)_{3_s}$, $(b_2 b_3)_{3_s}$.

C. $\Phi_a \Phi_b + \Phi_b \Phi_c$

The third topology found in Sec. III is where both EAs are two-flavon contractions with one repeated flavon, $\Phi_a \Phi_b + \Phi_b \Phi_c$. The renormalizable Lagrangian for this topology is

$$\mathcal{L}_\nu = \bar{\ell}_L \tilde{H} A + \bar{A} \Phi_a B + \bar{B} \Phi_b N_1 + \bar{A} \Phi_c C + \bar{C} \Phi_d N_2 \quad (21)$$

$$+M_A \bar{A}A + M_B \bar{B}B + M_C \bar{C}C + M_1 \bar{N}_1^c N_1 + M_2 \bar{N}_2^c N_2 + \text{H.c.}, \quad (22)$$

which can be implemented for example with an additional Z_6 with charges listed in Table III.

TABLE II. Fields and Z_8 charges in additive notation (-3 to 4) for models with topology $\Phi_a \Phi_b + \Phi_c \Phi_d$.

Field	ℓ_L	H	N_1	N_2	Φ_a	Φ_b	Φ_c	Φ_d	A	B	C
Z_8	-1	0	0	4	4	3	1	2	-1	3	-2

FIG. 4. Effective diagram for EA coming from Φ_a and Φ_b . FIG. 5. Effective diagram for EA coming from Φ_c and Φ_d .

The Lagrangian in Eq. (22) corresponds to the effective diagrams in Figs. 4 and 3. The same diagrams already appeared in the other topologies; the fields take different charges according to the respective topology.

Equation (16) exemplifies this topology, with EAs contracted as $\Phi_a = a_1$, $\Phi_b = d_2^*$, $\Phi_c = d_2$ and the EAs being $(a_1 d_2^*)_{3_s}$, $(d_2^* d_1)_{3_s}$.

V. CONCLUSIONS

We have explored the concept of effective alignments introduced in [20], that is of directions in flavor space that can be constructed from contractions of vacuum expectation values.

As in [20], we have considered an S_4 flavor symmetry as an example, and perform an exploratory survey. Here, we considered the 21 VEVs of S_4 , from which a total of 195 distinct EAs are obtained with contractions of up to two flavons.

Assuming the charged lepton Yukawa couplings to be diagonal in a convenient basis (which can be easily obtained in S_4 [20]), we then take the paradigm of constrained sequential dominance leptonic flavor models with two right-handed neutrinos, and consider the set of all such models that can be built with any pair of the 195 EAs of S_4 .

We test all these models to find which pairs of EAs are phenomenologically viable (meaning all observables within the respective 3σ ranges). In total we found 102 cases for normal ordering, corresponding to 53 mixing patterns (listed in Table V) and 12 cases for inverted

TABLE III. Fields and Z_6 charges in additive notation (-2 to 3) for models with topology $\Phi_a \Phi_b + \Phi_b \Phi_c$.

Field	ℓ_L	H	N_1	N_2	Φ_a	Φ_b	Φ_c	A	B	C
Z_6	-1	0	0	3	2	3	1	1	3	2

ordering, corresponding to 6 mixing patterns (listed in Table VI). Analyzing our results, we found no viable models where both EAs are VEVs, but there are cases in a topology with one EA being a VEV (the other being a contraction of two VEVs). Viable cases where both EAs are contractions of two VEVs exist in a topology without repeated VEVs and also in a topology where a given VEV appears in both EAs. Within this framework, we construct example models for each type of viable model, showing the Lagrangian, field content and charge assignments, and effective diagrams for each topology.

This type of study can be further generalized, for example considering models with more flavon contractions or even models based on other flavor symmetries.

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APPENDIX A: GROUP THEORY OF S_4

S_4 is the permutation group of four objects; see e.g., [29–31]. It has 24 distinct elements, which are generated by three generators S , T and U satisfying the relations

$$T^3 = S^2 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = 1. \quad (\text{A1})$$

The Kronecker products between different irreducible representations can be easily obtained:

$$\begin{aligned} \mathbf{1}' \otimes \mathbf{1}' &= \mathbf{1}, \quad \mathbf{1}' \otimes \mathbf{2} = \mathbf{2}, \quad \mathbf{1}' \otimes \mathbf{3} = \mathbf{3}', \\ \mathbf{1}' \otimes \mathbf{3}' &= \mathbf{3}, \quad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}, \\ \mathbf{2} \otimes \mathbf{3} &= \mathbf{2} \otimes \mathbf{3}' = \mathbf{3} \oplus \mathbf{3}', \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}', \\ \mathbf{3} \otimes \mathbf{3}' &= \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'. \end{aligned} \quad (\text{A2})$$

The generators T , S , and U in different irreducible representations are listed in Table IV. This basis is widely used in the literature since the charged lepton mass matrix invariant under T is diagonal in this basis. The products of two three-dimensional irreducible representations $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ can be expressed as

TABLE IV. The representation matrices for the S_4 generators T , S and U , where $\omega = e^{2\pi i/3}$.

	T	S	U
$\mathbf{1}$	1	1	1
$\mathbf{1}'$	1	1	-1
$\mathbf{2}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{3}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\mathbf{3}'$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\begin{aligned} (ab)_{\mathbf{1}_i} &= a_1 b_1 + a_2 b_3 + a_3 b_2, \\ (ab)_{\mathbf{2}} &= (a_2 b_2 + a_1 b_3 + a_3 b_1, a_3 b_3 + a_1 b_2 + a_2 b_1)^T, \\ (ab)_{\mathbf{3}_i} &= \frac{1}{\sqrt{6}} (2a_1 b_1 - a_2 b_3 - a_3 b_2, 2a_3 b_3 - a_1 b_2 \\ &\quad - a_2 b_1, 2a_2 b_2 - a_3 b_1 - a_1 b_3)^T, \\ (ab)_{\mathbf{3}_j} &= \frac{1}{\sqrt{2}} (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_3 b_1 - a_1 b_3)^T, \end{aligned} \quad (\text{A3})$$

where

$$\begin{aligned} \mathbf{1}_i &= \mathbf{1}, \quad \mathbf{3}_i = \mathbf{3}, \quad \mathbf{3}_j = \mathbf{3}' \quad \text{for } a \sim b \sim \mathbf{3}, \mathbf{3}', \\ \mathbf{1}_i &= \mathbf{1}', \quad \mathbf{3}_i = \mathbf{3}', \quad \mathbf{3}_j = \mathbf{3} \quad \text{for } a \sim \mathbf{3}, b \sim \mathbf{3}'. \end{aligned} \quad (\text{A4})$$

The products of two doublets $a = (a_1, a_2)^T$ and $b = (b_1, b_2)^T$ are reduced to

$$\begin{aligned} (ab)_{\mathbf{1}} &= a_1 b_2 + a_2 b_1, \quad (ab)_{\mathbf{1}'} = a_1 b_2 - a_2 b_1, \\ (ab)_{\mathbf{2}} &= (a_2 b_2, a_1 b_1)^T. \end{aligned} \quad (\text{A5})$$

The Kronecker products of multiplets of S_4 require the following properties: if the trilinear combination of three multiplets $a \sim \mathbf{r}$, $b \sim \mathbf{r}'$ and $c \sim \mathbf{r}''$ is an invariance of S_4 , e.g., $((ab)_{\mathbf{r}''} c)_{\mathbf{1}}$, then

$$\begin{aligned} ((ab)_{\mathbf{r}_c} c)_{\mathbf{1}} &= ((bc)_{\mathbf{r}_a} a)_{\mathbf{1}} = ((ca)_{\mathbf{r}_b} b)_{\mathbf{1}} = (a(bc)_{\mathbf{r}_a})_{\mathbf{1}} \\ &= (b(ca)_{\mathbf{r}_b})_{\mathbf{1}} = (c(ab)_{\mathbf{r}_c})_{\mathbf{1}}, \\ ((ab)_{\mathbf{r}_c} c)_{\mathbf{1}'} &= ((bc)_{\mathbf{r}_a} a)_{\mathbf{1}'} = ((ca)_{\mathbf{r}_b} b)_{\mathbf{1}'} = (a(bc)_{\mathbf{r}_a})_{\mathbf{1}'} \\ &= (b(ca)_{\mathbf{r}_b})_{\mathbf{1}'} = (c(ab)_{\mathbf{r}_c})_{\mathbf{1}'} \end{aligned} \quad (\text{A6})$$

holds, where $\mathbf{r}' = \mathbf{1}', \mathbf{1}, \mathbf{2}, \mathbf{3}', \mathbf{3}$ for $\mathbf{r} = \mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{3}, \mathbf{3}'$, respectively [20]. This property is used for flavon contractions in Sec. IV.

APPENDIX B: VIABLE PAIRS OF EAs

In this appendix, we list analytical expressions of EAs which are compatible with oscillation data in 3σ ranges. Results in the normal mass ordering are shown in Table V, and those in the inverted mass ordering are shown in Table VI. All

vectors are normalized to 1. The phase difference between $e^{i\alpha_1} V_1$ and V_1 and that between $e^{i\alpha_2} V_2$ and V_2 (for arbitrary α_1 and α_2) can be absorbed by the redefinition of the parameters μ_1 and μ_2 , respectively, and thus $e^{i\alpha_1} V_1$ and $e^{i\alpha_2} V_2$ do not need to be listed independently from V_1 and V_2 .

TABLE V. The two EAs compatible with data in 3σ ranges in the normal mass ordering. Each EA as a flavon VEV or a contraction of two flavon VEVs (not unique) is also shown.

V_1	V_2
$(0, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{5}})^T = (c_4 d_1)_{3_A}$	$(\frac{1}{\sqrt{3}}, -\frac{(-3+\sqrt{3})\omega^2}{6}, -\frac{(3+\sqrt{3})\omega}{6})^T = b_2$
$(0, \frac{\omega^2}{\sqrt{5}}, \frac{2\omega}{\sqrt{5}})^T = (c_4 d_2)_{3_A}$	$(\frac{1}{\sqrt{3}}, -\frac{(-3+\sqrt{3})\omega}{6}, -\frac{(3+\sqrt{3})\omega^2}{6})^T = b_3$
$(0, \frac{-2\omega^2}{\sqrt{5}}, \frac{-\omega}{\sqrt{5}})^T = (c_4 d_2^*)_{3_A}$	$(\frac{1}{\sqrt{3}}, \frac{-3-\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6})^T = b_1^*$
$(0, \frac{-2\omega}{\sqrt{5}}, \frac{-\omega^2}{\sqrt{5}})^T = (c_4 d_3^*)_{3_A}$	$(\frac{1}{\sqrt{3}}, -\frac{(3+\sqrt{3})\omega^2}{6}, -\frac{(-3+\sqrt{3})\omega}{6})^T = b_2^*$
$(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T = (a_1 c_2)_{3_S}$	$(\frac{1}{3}, \frac{1}{3} + \frac{1}{\sqrt{3}}, \frac{1}{3} - \frac{1}{\sqrt{3}})^T = (b_2 b_3)_{3_S}$
$(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T = (a_1 c_2)_{3_S}$	$(\frac{2\omega^2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, -\frac{3+4\omega}{\sqrt{21}})^T = (c_2 d_1^*)_{3_S}$
$(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}})^T = (a_1 d_1)_{3_S}$	$(\sqrt{\frac{2}{3}}, \frac{(-3+\sqrt{3})\omega^2}{6\sqrt{2}}, \frac{(3+\sqrt{3})\omega}{6\sqrt{2}})^T = (b_2 c_4)_{3_S}$
$(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}})^T = (a_1 d_1)_{3_S}$	$(\sqrt{\frac{2}{3}}, \frac{(-3+\sqrt{3})\omega}{6\sqrt{2}}, \frac{(3+\sqrt{3})\omega^2}{6\sqrt{2}})^T = (b_3 c_4)_{3_S}$
$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = (a_2 c_1)_{3_S}$	$(\frac{1}{3}, \frac{(1+\sqrt{3})\omega^2}{3}, -\frac{(-1+\sqrt{3})\omega}{3})^T = (b_1 b_3)_{3_S}$
$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = (a_2 c_1)_{3_S}$	$(\frac{1}{3}, -\frac{(-1+\sqrt{3})\omega^2}{3}, \frac{(1+\sqrt{3})\omega}{3})^T = (b_1^* b_3^*)_{3_S}$
$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = (a_2 c_1)_{3_S}$	$(\frac{2\omega^2}{\sqrt{21}}, \frac{3\omega-1}{\sqrt{21}}, \frac{2\omega}{\sqrt{21}})^T = (c_1 d_2)_{3_S}$
$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = (a_2 c_1)_{3_S}$	$(\frac{2\omega^2}{\sqrt{21}}, \frac{2\omega^2}{\sqrt{21}}, \frac{3-\omega^2}{\sqrt{21}})^T = (c_3 d_2^*)_{3_S}$
$(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})^T = (a_2 d_1^*)_{3_S}$	$(\frac{3-\omega^2}{\sqrt{21}}, \frac{2\omega^2}{\sqrt{21}}, \frac{2\omega^2}{\sqrt{21}})^T = (d_2 d_3^*)_{3_S}$
$(-\frac{1}{\sqrt{6}}, -\frac{\omega^2}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega)^T = (a_2 d_2)_{3_S}$	$(\sqrt{\frac{2}{3}}, \frac{-3+\sqrt{3}}{6\sqrt{2}}, \frac{3+\sqrt{3}}{6\sqrt{2}})^T = (b_1 c_4)_{3_S}$
$(-\frac{1}{\sqrt{6}}, -\frac{\omega^2}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega)^T = (a_2 d_2)_{3_S}$	$(\sqrt{\frac{2}{3}}, \frac{(-3+\sqrt{3})\omega}{6\sqrt{2}}, \frac{(3+\sqrt{3})\omega^2}{6\sqrt{2}})^T = (b_3 c_4)_{3_S}$
$(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega^2, -\frac{\omega}{\sqrt{6}})^T = (a_2 d_2^*)_{3_S}$	$(\sqrt{\frac{2}{3}}, \frac{3+\sqrt{3}}{6\sqrt{2}}, \frac{-3+\sqrt{3}}{6\sqrt{2}})^T = (b_1^* c_4)_{3_S}$
$(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega^2, -\frac{\omega}{\sqrt{6}})^T = (a_2 d_2^*)_{3_S}$	$(\sqrt{\frac{2}{3}}, \frac{(3+\sqrt{3})\omega}{6\sqrt{2}}, \frac{(-3+\sqrt{3})\omega^2}{6\sqrt{2}})^T = (b_3^* c_4)_{3_S}$
$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = (a_3 c_1)_{3_S}$	$(\frac{1}{3}, \frac{(1+\sqrt{3})\omega}{3}, -\frac{(-1+\sqrt{3})\omega^2}{3})^T = (b_1 b_2)_{3_S}$
$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = (a_3 c_1)_{3_S}$	$(\frac{1}{3}, -\frac{(-1+\sqrt{3})\omega}{3}, \frac{(1+\sqrt{3})\omega^2}{3})^T = (b_1^* b_2^*)_{3_S}$
$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = (a_3 c_1)_{3_S}$	$(\frac{2\omega^2}{\sqrt{21}}, \frac{2\omega}{\sqrt{21}}, \frac{3\omega^2-1}{\sqrt{21}})^T = (c_1 d_3^*)_{3_S}$
$(\frac{1-i}{\sqrt{6}}, -\frac{\sqrt{3}(1+2i)+3}{6\sqrt{2}}, \frac{(3\sqrt{2}-(1+2i)\sqrt{6})}{12})^T = (b_1 c_3)_{3_S}$	$(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)^T = (a_1 d_2)_{3_S}$
$(\frac{(-3+\sqrt{3})}{3\sqrt{2(4+\sqrt{3})}}, -\frac{2(3+\sqrt{3})}{3\sqrt{2(4+\sqrt{3})}}, -\sqrt{\frac{2}{3(4+\sqrt{3})}})^T = (b_1 d_4^*)_{3_S}$	$(\frac{1}{\sqrt{3}}, -\frac{(3+\sqrt{3})\omega}{6}, -\frac{(-3+\sqrt{3})\omega^2}{6})^T = b_3^*$
$(\frac{1-i}{\sqrt{6}}, \frac{(-3+3i)+\sqrt{3}(1+5i)}{12\sqrt{2}}, \frac{\sqrt{3}(1+5i)+(3-3i)}{12\sqrt{2}})^T = (b_2 c_1)_{3_S}$	$(\frac{1}{\sqrt{2}}, -\frac{\omega^2}{\sqrt{2}}, 0)^T = (a_2 d_1)_{3_S}$

(Table continued)

TABLE V. (Continued)

V_1	V_2
$\left(\frac{(-3+\sqrt{3})\omega^2}{3\sqrt{2(4+\sqrt{3})}}, -\frac{2(3+\sqrt{3})\omega}{3\sqrt{2(4+\sqrt{3})}}, -\sqrt{\frac{2}{3(4+\sqrt{3})}}\right)^T = (b_2 d_4^*)_{3_s}$	$\left(\frac{1}{\sqrt{3}}, \frac{(-3-\sqrt{3})}{6}, \frac{(3-\sqrt{3})}{6}\right)^T = b_1^*$
$\left(\frac{1-i}{\sqrt{6}}, \frac{\sqrt{3}(1-i)+(-9+3i)}{12\sqrt{2}}, \frac{(1-i)(\sqrt{3}+(6+3i))}{12\sqrt{2}}\right)^T = (b_2^* c_3)_{3_s}$	$\left(\frac{1}{\sqrt{2}}, 0, -\frac{\omega}{\sqrt{2}}\right)^T = (a_2 d_1^*)_{3_s}$
$\left(\frac{(-3+\sqrt{3})\omega}{3\sqrt{2(4+\sqrt{3})}}, -\sqrt{\frac{2}{3(4+\sqrt{3})}}, -\frac{2(3+\sqrt{3})\omega^2}{3\sqrt{2(4+\sqrt{3})}}\right)^T = (b_2^* d_4)_{3_s}$	$\left(\frac{1}{3}, -\frac{(-3+\sqrt{3})\omega}{6}, -\frac{(3+\sqrt{3})\omega^2}{6}\right)^T = b_3$
$\left(\frac{2\omega^2}{\sqrt{21}}, \frac{3\omega^2-1}{\sqrt{21}}, \frac{2\omega}{\sqrt{21}}\right)^T = (c_1 d_2)_{3_s}$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)^T = (a_1 d_2)_{3_s}$
$\left(\frac{2}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}\right)^T = (c_1 d_4)_{3_s}$	$\left(-\frac{2}{3}, \frac{\omega^2}{3}, -\frac{2}{3}\omega\right)^T = d_2$
$\left(\frac{2}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}\right)^T = (c_1 d_4)_{3_s}$	$\left(-\frac{2}{3}, \frac{\omega}{3}, -\frac{2}{3}\omega^2\right)^T = d_3$
$\left(\frac{2\omega^2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, -\frac{3+4\omega^2}{\sqrt{21}}\right)^T = (c_2 d_1^*)_{3_s}$	$\left(\frac{1}{\sqrt{2}}, 0, -\frac{\omega}{\sqrt{2}}\right)^T = (a_2 d_1^*)_{3_s}$
$\left(\frac{2\omega^2}{\sqrt{21}}, \frac{3-\omega^2}{\sqrt{21}}, \frac{2\omega^2}{\sqrt{21}}\right)^T = (c_2 d_3)_{3_s}$	$\left(\frac{1}{\sqrt{2}}, -\frac{\omega^2}{\sqrt{2}}, 0\right)^T = (a_2 d_1)_{3_s}$
$\left(\frac{2\omega}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, -\frac{4\omega^2}{\sqrt{21}}\right)^T = (c_2 d_4)_{3_s}$	$\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)^T = d_1^*$
$\left(\frac{2\omega}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, -\frac{4\omega^2}{\sqrt{21}}\right)^T = (c_2 d_4)_{3_s}$	$\left(-\frac{2}{3}, \frac{\omega}{3}, -\frac{2}{3}\omega^2\right)^T = d_3$
$\left(\frac{2\omega^2}{\sqrt{21}}, -\frac{4\omega}{\sqrt{21}}, -\frac{1}{\sqrt{21}}\right)^T = (c_2 d_4^*)_{3_s}$	$\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)^T = d_1$
$\left(\frac{2\omega^2}{\sqrt{21}}, -\frac{4\omega}{\sqrt{21}}, -\frac{1}{\sqrt{21}}\right)^T = (c_2 d_4^*)_{3_s}$	$\left(-\frac{2}{3}, -\frac{2\omega}{3}, \frac{\omega^2}{3}\right)^T = d_3^*$
$\left(-\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)^T = (d_1 d_4^*)_{3_s}$	$\left(\frac{1}{3}, -\frac{2}{3}\omega^2, -\frac{2}{3}\omega\right)^T = c_2$
$\left(-\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)^T = (d_1 d_4^*)_{3_s}$	$\left(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2\right)^T = c_3$
$\left(-\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)^T = (d_1 d_4^*)_{3_s}$	$\left(-\frac{1+\sqrt{3}}{\sqrt{2(4+\sqrt{3})}}, -\sqrt{\frac{2}{4+\sqrt{3}}}, 0\right)^T = (b_1^* d_1)_{3_s}$
$\left(-\frac{\omega^2}{\sqrt{21}}, -\frac{4\omega}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)^T = (d_2 d_4^*)_{3_s}$	$\left(\frac{1}{3}, -\frac{2}{3}\omega^2, -\frac{2}{3}\omega\right)^T = c_1$
$\left(-\frac{\omega^2}{\sqrt{21}}, -\frac{4\omega}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)^T = (d_2 d_4^*)_{3_s}$	$\left(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2\right)^T = c_3$
$\left(-\frac{\omega^2}{\sqrt{21}}, -\frac{4\omega}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)^T = (d_2 d_4^*)_{3_s}$	$\left(-\frac{1+\sqrt{3}}{\sqrt{2(4+\sqrt{3})}}, -\sqrt{\frac{2}{4+\sqrt{3}}}\omega^2, 0\right)^T = (b_2^* d_2)_{3_s}$
$\left(-\frac{\omega}{\sqrt{21}}, \frac{2}{\sqrt{21}}, -\frac{4\omega^2}{\sqrt{21}}\right)^T = (d_2^* d_4)_{3_s}$	$\left(\frac{1}{3}, -\frac{2}{3}\omega^2, -\frac{2}{3}\omega\right)^T = c_1$
$\left(-\frac{\omega}{\sqrt{21}}, \frac{2}{\sqrt{21}}, -\frac{4\omega^2}{\sqrt{21}}\right)^T = (d_2^* d_4)_{3_s}$	$\left(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2\right)^T = c_3$
$\left(-\frac{\omega}{\sqrt{21}}, \frac{2}{\sqrt{21}}, -\frac{4\omega^2}{\sqrt{21}}\right)^T = (d_2^* d_4)_{3_s}$	$\left(-\frac{1+\sqrt{3}}{\sqrt{2(4+\sqrt{3})}}, 0, -\sqrt{\frac{2}{4+\sqrt{3}}}\omega\right)^T = (b_2 d_2^*)_{3_s}$
$\left(\frac{1}{\sqrt{3}}, \frac{(3-\sqrt{3})}{6}, \frac{(-3-\sqrt{3})}{6}\right)^T = b_1$	$\left(\frac{2\omega}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)^T = (c_2 d_4)_{3_A}$
$\left(\frac{1}{\sqrt{3}}, -\frac{(-3+\sqrt{3})\omega^2}{6}, -\frac{(3+\sqrt{3})\omega}{6}\right)^T = b_2$	$\left(\frac{2\omega^2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)^T = (c_3 d_4)_{3_A}$
$\left(\frac{1}{\sqrt{3}}, -\frac{(-3+\sqrt{3})\omega}{6}, -\frac{(3+\sqrt{3})\omega^2}{6}\right)^T = b_2^*$	$\left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)^T = (c_1 d_4^*)_{3_A}$
$\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)^T = d_1$	$\left(-\frac{4}{\sqrt{21}}, -\frac{\omega^2}{\sqrt{21}}, \frac{2\omega}{\sqrt{21}}\right)^T = (c_4 d_2)_{3_s}$
$\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)^T = d_1$	$\left(-\frac{4}{\sqrt{21}}, -\frac{\omega}{\sqrt{21}}, \frac{2\omega^2}{\sqrt{21}}\right)^T = (c_4 d_3)_{3_s}$
$\left(-\frac{2}{3}, \frac{\omega^2}{3}, -\frac{2}{3}\omega\right)^T = d_2$	$\left(-\frac{4}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)^T = (c_4 d_1)_{3_s}$
$\left(-\frac{2}{3}, \frac{\omega^2}{3}, -\frac{2}{3}\omega\right)^T = d_2$	$\left(-\frac{4}{\sqrt{21}}, -\frac{\omega}{\sqrt{21}}, \frac{2\omega^2}{\sqrt{21}}\right)^T = (c_4 d_3)_{3_s}$
$\left(-\frac{2}{3}, -\frac{2\omega^2}{3}, \frac{\omega}{3}\right)^T = d_2^*$	$\left(-\frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}}, -\frac{1}{\sqrt{21}}\right)^T = (c_4 d_1^*)_{3_s}$
$\left(-\frac{2}{3}, -\frac{2\omega^2}{3}, \frac{\omega}{3}\right)^T = d_2^*$	$\left(-\frac{4}{\sqrt{21}}, \frac{2\omega}{\sqrt{21}}, -\frac{\omega^2}{\sqrt{21}}\right)^T = (c_4 d_3^*)_{3_s}$

TABLE VI. The two EAs compatible with data in 3σ ranges in the inverted mass ordering. Each EA as a contraction of two flavon VEVs (not unique) is also shown.

V_1	V_2
$(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}})^T = (a_1 d_1)_{3_s}$	$(\frac{1-i}{\sqrt{6}}, \frac{(3\sqrt{2}-(1+2i)\sqrt{6})}{12}, -\frac{\sqrt{3}(1+2i)+3}{6\sqrt{2}})^T = (b_1^* c_2)_{3_s}$
$(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}})^T = (a_1 d_1^*)_{3_s}$	$(\frac{1+i}{\sqrt{6}}, \frac{-\sqrt{3}(1-2i)-3}{6\sqrt{2}}, \frac{3-(1-2i)\sqrt{3}}{6\sqrt{2}})^T = (b_1 c_2)_{3_s}$
$(-\frac{1}{\sqrt{6}}, -\frac{\omega^2}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega)^T = (a_2 d_2)_{3_s}$	$(\frac{1+i}{\sqrt{6}}, \frac{(3+3i)+\sqrt{3}(1-5i)}{12\sqrt{2}}, \frac{(-3-3i)+\sqrt{3}(1-5i)}{12\sqrt{2}})^T = (b_2^* c_1)_{3_s}$
$(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega^2, -\frac{\omega}{\sqrt{6}})^T = (a_2 d_2^*)_{3_s}$	$(\frac{1-i}{\sqrt{6}}, \frac{(-3+3i)+\sqrt{3}(1+5i)}{12\sqrt{2}}, \frac{\sqrt{3}(1+5i)+(3-3i)}{12\sqrt{2}})^T = (b_2 c_1)_{3_s}$
$(-\frac{1}{\sqrt{6}}, -\frac{\omega}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega^2)^T = (a_3 d_3)_{3_s}$	$(\frac{1-i}{\sqrt{6}}, \frac{\sqrt{3}(1+5i)+(3-3i)}{12\sqrt{2}}, \frac{(-3+3i)+\sqrt{3}(1+5i)}{12\sqrt{2}})^T = (b_3^* c_1)_{3_s}$
$(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\omega, -\frac{\omega^2}{\sqrt{6}})^T = (a_3 d_3^*)_{3_s}$	$(\frac{1+i}{\sqrt{6}}, \frac{(-3-3i)+\sqrt{3}(1-5i)}{12\sqrt{2}}, \frac{(3+3i)+\sqrt{3}(1-5i)}{12\sqrt{2}})^T = (b_3 c_1)_{3_s}$

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