Charged pion condensation and duality in dense and hot chirally and isospin asymmetric quark matter in the framework of the NJL₂ model

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In this paper we investigate in the large- N_c limit (N_c is the number of colored quarks) the phase structure of a massless $(1 + 1)$ -dimensional quark model with four-quark interaction and in the presence of baryon (μ_B), isospin (μ_I) and chiral isospin (μ_{IS}) chemical potentials as well as at nonzero temperature. It is established that the chiral isospin potential leads to the generation of charged pion condensation (PC) in dense (nonzero baryon density) and chiral asymmetric quark matter for a wide range of isospin densities. It is shown that there exists a duality correspondence between the chiral symmetry breaking and the charged PC phenomena at any values of temperature even for very hot quark gluon plasma. Moreover, it is shown that a charged PC phase with nonzero baryon density can be induced in the model at comparatively high temperatures. This opens up new possible physical systems, where it can be of importance, such as heavy-ion collisions, new-born neutron stars (proto-neutron stars), supernovas as well as neutron star mergers.

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I. INTRODUCTION

Recently, much attention has been paid to the study of dense baryon (quark) media with isotopic (isospin) asymmetry (different densities of u and d quarks). Such matter can exist inside compact stars, it can appear in heavy-ion collision experiments, etc., and it is usually described in terms of different nonperturbative QCD methods or effective QCD-like theories such as chiral effective Lagrangians or, especially, Nambu–Jona-Lasinio (NJL) type models [1] with nonzero baryon μ_B and isospin μ_I chemical potentials. It turns out that the (μ_I, μ_B) phase diagram of these models gives us the opportunity to better understand phenomena such as chiral symmetry restoration [2–6], color superconductivity [7–9], and the charged pion condensation (PC) effect [10–22] (let us also recall ideas about pion stars [23]) that can be observed in dense quark matter with isotopic asymmetry.

While the reality of the first two of the above phenomena is beyond doubt, the possibility of condensation of charged pions in a dense quark medium has not been reliably

established (at least in the framework of the NJL model considerations). Indeed, for some values of the model parameters (the coupling constant G , the cutoff parameter Λ , etc.), the phase of quark matter with *nonzero baryon density*, in which the charged pions are condensed, is allowed by the NJL models. However, this effect is prohibited by the NJL models for other physically interesting G and Λ values [12]. In addition, if the condition of electric neutrality is imposed on quark matter, then the charged PC phenomenon strongly depends on the size of the bare quark mass m_0 . In particular, it turns out that in the framework of the NJL models the condensation of charged pions is forbidden in a medium with *nonzero baryon density* if m_0 reaches physically acceptable values of 5–10 MeV (see in Ref. [15]). However, on the basis of the toy $(1 + 1)$ -dimensional model with baryon and isospin chemical potentials, it was shown in Refs. [16,17] that there are factors (the spatial sizes of the system or the spatial inhomogeneity of its condensates) that can stimulate the appearance of the charged PC phenomenon in dense quark matter.

Quite recently, it was found that in dense quark medium in the presence of an external magnetic field two other effects, chiral magnetic and chiral separation effects [24,25], can be observed. Thus they can be realized in compact stars and heavy-ion collision experiments, etc. Usually, these phenomena are inherent in such environments in which there is a

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chiral imbalance (or asymmetry), i.e. when in dense quark matter there is a difference between the density n_R of all right-handed quarks and the density n_L of all left-handed quarks. The quantity $n_5 \equiv n_R - n_L$ is called the chiral density of the system. It has been argued that the chiral density n_5 can be generated dynamically at high temperatures, for example, in the fireball after heavy-ion collisions, by virtue of the Adler-Bell-Jackiw anomaly and quarks interacting with gauge (gluon) field configurations with nontrivial topology, named sphalerons. In the presence of an external strong magnetic field, which can be produced in heavy-ion collisions as well, this can lead to the so-called chiral magnetic effect [24]. Moreover, in the presence of an external magnetic field the chiral density n_5 can be produced (even at a rather low temperature) in dense quark matter due to the so-called chiral separation effect [25] (it can also be produced under fast rotations of the system due to the so-called chiral vortical effect). Now let us notice that usually when one talks about the chiral density n_5 one implies that the chiral density n_{u5} of u quarks and chiral density n_{d5} of d quarks are equal to each other (it is evident that $n_5 = n_{5u} + n_{5d}$). Indeed, that is the case when one has in mind the mechanism of generation of chiral imbalance at high temperatures due to the nontrivial topology of the gauge-field configuration. In this case it is quite plausible that $n_{u5} = n_{d5}$ due to the fact that the gluon field interacts with different quark flavors in exactly the same way and does not feel the difference between flavors. But another mechanism, the chiral separation effect, is sensitive to the flavor of quarks (as it was shown in Appendix A in Refs. [26,27]). So in dense quark matter ($\mu_B \neq 0$) a strong magnetic field separates u and d quarks in different ways. As a result, we see that, e.g., in such astrophysical objects as magnetars there might exist areas, in which the quantity $n_{I5} = n_{5u} - n_{5d}$, called the chiral isospin density, is not zero. Moreover, it has been argued that chiral imbalance is generated by parallel magnetic and electric fields [28], and one can generalize these arguments to chiral isospin imbalance as well.

So in the most general case, chiral imbalance is described by two chemical potentials—the chiral μ_5 and chiral isospin (or isotopic) μ_{I5} chemical potentials—which are thermodynamically conjugated to n_5 and n_{I5} , respectively. The first, μ_5 , is usually used when isotopic asymmetry of quark matter is absent, i.e. in the case $\mu_I = 0$ [29,30]. The second, μ_{I5} , might be taken into account when, in addition to chiral, there is also isotopic asymmetry of matter, in which the charged PC phenomenon may occur, etc. [31]. In particular, it was established in the framework of NJL models that μ_5 catalyzes the chiral symmetry breaking (CSB) [30,31], whereas μ_{I5} promotes the charged PC in dense quark matter [31].

It was also shown in Ref. [31] that in the leading order of the $1/N_c$ expansion the most general $(\mu_B, \mu_I, \mu_5, \mu_{I5})$ -phase portrait of the massless NJL model is symmetric with

respect to a so-called dual transformation between charged PC and CSB phases. If the bare quark mass m_0 is not zero, then this duality is only an approximate, although quite accurate, symmetry of the NJL model phase portrait (see Ref. [26]). In addition, it was established in this paper that the chiral isospin chemical potential μ_{I5} promotes the charged PC phenomenon in dense quark matter even at physically acceptable values of m_0 .

One of the drawbacks of the $(3+1)$ -dimensional NJL (NJL₄) model is its nonrenormalizability. Therefore, the predictions of this effective model depend on the cutoff parameter, which is typically chosen to be of the order of 1 GeV. So, the results when using the NJL₄ model are valid only at *comparatively low energies, temperatures and densities* (chemical potentials). However, there also exists a class of renormalizable theories—the $(1+1)$ -dimensional chiral Gross-Neveu (GN)-type models [32,33]¹—that can be used as a laboratory for the qualitative simulation of specific properties of QCD at *arbitrary energies*. Renormalizability, asymptotic freedom, as well as the spontaneous chiral symmetry breaking (in vacuum) are the most fundamental inherent features both for QCD and all NJL₂-type models. In addition, the $\mu_B - T$ phase diagram (where T denotes temperature) is qualitatively the same for both the QCD and NJL₂ models [34–37]. Let us further mention that $(1+1)$ -dimensional Gross-Neveu-type models are also suitable for the description of physics in quasi-one-dimensional condensed matter systems like polyacetylene [38]. It is currently well understood (see, e.g., the discussion in Refs. [36,37,39]) that the usual *no-go* theorem [40], which generally forbids the spontaneous breaking of any continuous symmetry in two-dimensional spacetime, does not work in the limit $N_c \rightarrow \infty$, where N_c is the number of colored quarks. This follows directly from the fact that in the limit of large N_c the quantum fluctuations, which would otherwise destroy the long-range order corresponding to a spontaneous symmetry breaking, are suppressed by $1/N_c$ factors. Thus, the effects inherent in real dense quark matter, such as the CSB phenomenon [spontaneous breaking of the continuous axial $U(1)$ symmetry] or charged pion condensation (spontaneous breaking of the continuous isospin symmetry) might be simulated in terms of simpler NJL₂-type models, though only in the leading order of the large- N_c approximation (see, e.g., Refs. [39,41–45], respectively).

This paper is devoted to the investigation of such phenomena of dense quark matter as CSB and charged PC, as well as their mutual influence on each other, in the framework of an extended toy NJL₂ model with two massless quark flavors and in the presence of the baryon

¹Below we shall use the notation “NJL₂ model” instead of “chiral GN model” for $(1+1)$ -dimensional models with *continuous chiral and/or isotopic, etc., symmetries*, since the chiral structure of their Lagrangians is the same as that of the corresponding $(3+1)$ -dimensional NJL models.

μ_B , isospin μ_I as well as chiral isospin μ_{I5} chemical potentials (for simplicity, we only consider the case $\mu_5 = 0$). Of course, in this case, in two dimensions, a significant portion of the physical processes that can occur in a real dense quark medium in $(3 + 1)$ dimensions, is removed from our consideration. However, as discussed, e.g., in Ref. [46], the dynamics of the pairing of fermions in strong magnetic fields is $(1 + 1)$ dimensional. Therefore, phenomena of real dense quark matter such as the spontaneous breaking of chiral symmetry, the condensation of charged pions and other pairing phenomena in strong magnetic fields can be investigated effectively in the framework of two-dimensional models. The chemical potential μ_{I5} is also included in our consideration, because the regions with $\mu_{I5} \neq 0$ can appear (due to the chiral separation effect) in neutron stars or arise in quark matter formed by the collision of heavy ions just under the influence of a strong magnetic field (this fact was discussed in Appendix A of Refs. [26,27]). Moreover, in order to avoid the no-go theorem [40], we perform all calculations in the leading order of the large- N_c technique. To clarify the true role of the chiral isospin μ_{I5} chemical potential in the creation of the CSB and charged PC in dense quark matter, it is supposed throughout the paper that all condensates are spatially homogeneous.² Under this constraint the model was already investigated in Refs. [41,42] at $\mu_{I5} = 0$, where it was shown that the charged PC phase with nonzero baryon density is forbidden at arbitrary values of μ_B and μ_I (at zero chiral imbalance). In contrast, we show that at $\mu_{I5} \neq 0$, i.e. when there is an isotopic chiral imbalance of the system, the charged PC phase with nonzero baryon density is allowed to exist for a wide range of isospin densities.

In addition, we show that in the leading order of the large- N_c approximation there arises a duality between CSB and charged PC phenomena in the framework of the NJL₂ model under consideration for the case of cold quark matter as well as for hot media. It means that if at $\mu_I = A$ and $\mu_{I5} = B$ (at arbitrary fixed chemical potential μ_B), e.g., the CSB (or the charged PC) phase is realized in the model, then at the permuted values of these chemical potentials, i.e. at $\mu_I = B$ and $\mu_{I5} = A$, the charged PC (or the CSB) phase is arranged. So, it is enough to know the phase structure of the model at $\mu_I < \mu_{I5}$, in order to establish the phase structure at $\mu_I > \mu_{I5}$ and vice versa. Knowing condensates and other dynamical and thermodynamical quantities of the system, e.g. in the CSB phase, one can then obtain the corresponding quantities in the dually conjugated charged PC phase of the model, by simply performing there the duality transformation, $\mu_I \leftrightarrow \mu_{I5}$.³ Earlier, similar dualities between the charged PC and

chiral symmetry breaking phenomena were observed in the framework of the orbifold equivalence formalism in the limit of large N_c [49]. Namely, it was shown that the whole phase diagram of QCD at the chiral chemical potential must be identical to that of QCD at the isotopic chemical potential in the chiral limit, where the charged pion condensation is replaced by the chiral condensation.

Note that the phase structure of the NJL₂ model under consideration with nonzero values of μ_B , μ_I , and μ_{I5} has already been studied in Ref. [50] at $T = 0$, but an elusive error was made there. Formally, in Ref. [50] all calculations, both numerical and analytical, are correct but one of the statements in the Appendix, which only at first glance may seem to be correct, turned out to be untrue, and this led to an incorrect expression for the thermodynamic potential of the model (see below). However, it is interesting that although the basic qualitative conclusions are still true, some important details of the phase portrait have changed. The predictions became stronger and more interesting for the application to real physical scenarios. This paper is devoted to the investigation and the analysis of these changes, and we consider the influence of the temperature on the phase diagram and the duality of the model in this case.

The paper is organized as follows. In Sec. II a toy $(1 + 1)$ -dimensional massless NJL-type model with two quark flavors (u and d quarks) and with three kinds of chemical potentials, μ_B , μ_I , μ_{I5} , is presented. Next, we discuss the symmetries of the model under consideration and the unrenormalized expression for the thermodynamic potential (TDP) for the cases of both zero and nonzero temperature is obtained in the leading order of the large- N_c expansion. Here the dual symmetry of the model TDP is established at $T \geq 0$. Section III is devoted to the calculation of the TDP at $T = 0$ at both zero and nonzero values of the chemical potentials. Then, in Sec. IV, the phase structure of the model is investigated at $T = 0$. In this section, on the basis of several phase diagrams, we demonstrate the existence of a duality between CSB and charged PC phenomena as well the fact that μ_{I5} promotes the realization of the charged PC phase in dense quark matter. It means that the charged PC phase with nonzero baryon density can be realized in this model only at $\mu_{I5} > 0$. Similar results are obtained in Sec. V, where the case of nonzero temperature is considered. In Sec. VI the main results and conclusions of the paper are formulated, and we emphasize the similarity of the phase properties of the NJL model in two and four spacetime dimensions with isotopic chiral asymmetry. Some technical details are relegated to Appendices A and B.

²As it was noted above, the spatial inhomogeneity of condensates by itself can cause charged PC in dense baryon matter [17].

³Note that another kind of duality correspondence, the duality between CSB and superconductivity, was demonstrated both in $(1 + 1)$ - and $(2 + 1)$ -dimensional NJL models [47,48].

II. THE MODEL AND ITS THERMODYNAMIC POTENTIAL

A. The zero-temperature case

We consider in the chiral limit a $(1+1)$ -dimensional NJL model in order to mimic the phase structure of real dense quark matter with two massless quark flavors (u and d quarks). Its Lagrangian, which is symmetrical under the global color $SU(N_c)$ group, has the form

$$L = \bar{q} \left[\gamma^\nu i \partial_\nu + \frac{\mu_B}{3} \gamma^0 + \frac{\mu_I}{2} \tau_3 \gamma^0 + \frac{\mu_{I5}}{2} \tau_3 \gamma^0 \gamma^5 \right] q + \frac{G}{N_c} [(\bar{q}q)^2 + (\bar{q}i\gamma^5 \vec{\tau}q)^2], \quad (1)$$

where the quark field $q(x) \equiv q_{i\alpha}(x)$ is a flavor doublet ($i = 1, 2$ or $i = u, d$) and color N_c -plet ($\alpha = 1, \dots, N_c$) as well as a two-component Dirac spinor [the summation in Eq. (1) over flavor, color, and spinor indices is implied]; τ_k ($k = 1, 2, 3$) are Pauli matrices in two-dimensional flavor space. The Dirac γ^ν matrices ($\nu = 0, 1$) and γ^5 in Eq. (1) are matrices in two-dimensional spinor space,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

Note that at $\mu_{I5} = 0$ the model was already investigated in detail, e.g., in Refs [41–43,45]. It is evident that the model (1) is a generalization of the two-dimensional GN model [32] with a single massless quark color N_c -plet to the case of two quark flavors and additional baryon μ_B , isospin μ_I and axial isospin μ_{I5} chemical potentials. These parameters are introduced in order to describe in the framework of the model (1) quark matter with nonzero baryon n_B , isospin n_I and axial isospin n_{I5} densities, respectively. It is evident that the Lagrangian (1), both at $\mu_{I5} = 0$ and $\mu_{I5} \neq 0$, is invariant with respect to the Abelian $U_B(1)$, $U_{I3}(1)$ and $U_{AI3}(1)$ groups, where⁴

$$U_B(1): q \rightarrow \exp(i\alpha/3)q; \quad U_{I3}(1): q \rightarrow \exp(i\beta\tau_3/2)q; \\ U_{AI3}(1): q \rightarrow \exp(i\omega\gamma^5\tau_3/2)q. \quad (3)$$

[In Eq. (3) the real parameters α, β, ω specify an arbitrary element of the $U_B(1)$, $U_{I3}(1)$ and $U_{AI3}(1)$ groups, respectively.] So the quark bilinears $\frac{1}{3}\bar{q}\gamma^0 q$, $\frac{1}{2}\bar{q}\gamma^0 \tau^3 q$ and $\frac{1}{2}\bar{q}\gamma^0 \gamma^5 \tau^3 q$ are the zero components of corresponding conserved currents. Their ground-state expectation values are just the baryon n_B , isospin n_I and chiral (axial) isospin

n_{I5} densities of quark matter, i.e. $n_B = \frac{1}{3}\langle \bar{q}\gamma^0 q \rangle$, $n_I = \frac{1}{2}\langle \bar{q}\gamma^0 \tau^3 q \rangle$ and $n_{I5} = \frac{1}{2}\langle \bar{q}\gamma^0 \gamma^5 \tau^3 q \rangle$. As usual, the quantities n_B , n_I and n_{I5} can also be found by differentiating the thermodynamic potential of the system with respect to the corresponding chemical potentials. The goal of the present paper is to investigate the ground-state properties and phase structure of the system (1) and its dependence on the chemical potentials μ_B , μ_I and μ_{I5} .

To find the thermodynamic potential of the system, we use a semibosonized version of the Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$) (in what follows, we use the notations $\mu \equiv \mu_B/3$, $\nu \equiv \mu_I/2$ and $\nu_5 \equiv \mu_{I5}/2$):

$$\tilde{L} = \bar{q}[\gamma^\rho i \partial_\rho + \mu \gamma^0 + \nu \tau_3 \gamma^0 + \nu_5 \tau_3 \gamma^0 \gamma^5 - \sigma - i\gamma^5 \pi_a \tau_a] q - \frac{N_c}{4G} [\sigma^2 + \pi_a^2]. \quad (4)$$

In Eq. (4) the summation over repeated indices is implied. From the Lagrangian (4) one gets the Euler-Lagrange equations for the bosonic fields

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q), \quad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}i\gamma^5 \tau_a q). \quad (5)$$

Note that the composite bosonic field $\pi_3(x)$ can be identified with the physical π_0 meson, whereas the physical $\pi^\pm(x)$ -meson fields are the following combinations of the composite fields: $\pi^\pm(x) = (\pi_1(x) \mp i\pi_2(x))/\sqrt{2}$. Obviously, the semibosonized Lagrangian $\tilde{L}$ is equivalent to the initial Lagrangian (1) when using the equations (5). Furthermore, it is clear from Eqs. (3) and (5) and footnote 4 that the bosonic fields transform under the isospin $U_{I3}(1)$ and axial isospin $U_{AI3}(1)$ groups in the following manner:

$$U_{I3}(1): \sigma \rightarrow \sigma; \pi_3 \rightarrow \pi_3; \pi_1 \rightarrow \cos(\beta)\pi_1 + \sin(\beta)\pi_2; \pi_2 \rightarrow \cos(\beta)\pi_2 - \sin(\beta)\pi_1, \\ U_{AI3}(1): \pi_1 \rightarrow \pi_1; \pi_2 \rightarrow \pi_2; \sigma \rightarrow \cos(\omega)\sigma + \sin(\omega)\pi_3; \pi_3 \rightarrow \cos(\omega)\pi_3 - \sin(\omega)\sigma. \quad (6)$$

In general the phase structure of a given model is characterized by the behavior of some quantities, called order parameters (or condensates), vs external conditions (temperature, chemical potentials, etc.). In the case of model (1), such order parameters are the ground-state expectation values of the composite fields, i.e. the quantities $\langle \sigma(x) \rangle$ and $\langle \pi_a(x) \rangle$ ($a = 1, 2, 3$). It is clear from Eq. (6) that if $\langle \sigma(x) \rangle \neq 0$ and/or $\langle \pi_3(x) \rangle \neq 0$, then the axial isospin $U_{AI3}(1)$ symmetry of the model is spontaneously broken, whereas at $\langle \pi_1(x) \rangle \neq 0$ and/or $\langle \pi_2(x) \rangle \neq 0$ we have a spontaneous breaking of the isospin $U_{I3}(1)$ symmetry. Since in the last case the ground-state expectation values

⁴Recall for the following that $\exp(i\beta\tau_3/2) = \cos(\beta/2) + i\tau_3 \sin(\beta/2)$, $\exp(i\omega\gamma^5\tau_3/2) = \cos(\omega/2) + i\gamma^5\tau_3 \sin(\omega/2)$.

(condensates) of both fields $\pi^+(x)$ and $\pi^-(x)$ are not zero, this phase is usually called the charged pion condensation (PC) phase. The ground-state expectation values $\langle\sigma(x)\rangle$ and $\langle\pi_a(x)\rangle$ are the coordinates of the global minimum point of the thermodynamic potential $\Omega(\sigma, \pi_a)$ of the system.

Starting from the linearized semibosonized model Lagrangian (4), one obtains in the leading order of the large- N_c expansion (i.e. in the one-fermion-loop approximation) the following path integral expression for the effective action $\mathcal{S}_{\text{eff}}(\sigma, \pi_a)$ of the bosonic $\sigma(x)$ and $\pi_a(x)$ fields:

$$\exp(i\mathcal{S}_{\text{eff}}(\sigma, \pi_a)) = N' \int [d\bar{q}][dq] \exp\left(i \int \tilde{L} d^2x\right),$$

where

$$\mathcal{S}_{\text{eff}}(\sigma, \pi_a) = -N_c \int d^2x \left[\frac{\sigma^2 + \pi_a^2}{4G} \right] + \tilde{\mathcal{S}}_{\text{eff}}, \quad (7)$$

and N' is a normalization constant. The quark contribution to the effective action, i.e. the term $\tilde{\mathcal{S}}_{\text{eff}}$ in Eq. (7), is given by

$$\begin{aligned} \exp(i\tilde{\mathcal{S}}_{\text{eff}}) &= N' \int [d\bar{q}][dq] \exp\left(i \int [\bar{q} D q] d^4x\right) \\ &= [\text{Det } D]^{N_c}. \end{aligned} \quad (8)$$

In Eq. (8) we have used the notation $D \equiv D \times I_c$, where I_c is the unit operator in the N_c -dimensional color space and

$$D \equiv \gamma^0 i \partial_\rho + \mu \gamma^0 + \nu \tau_3 \gamma^0 + \nu_5 \tau_3 \gamma^0 \gamma^5 - \sigma - i \gamma^5 \pi_a \tau_a \quad (9)$$

is the Dirac operator, which acts in the flavor, spinor as well as coordinate spaces only. Using the general formula $\text{Det } D = \exp \text{Tr} \ln D$, one obtains for the effective action the following expression:

$$\mathcal{S}_{\text{eff}}(\sigma, \pi_a) = -N_c \int d^2x \left[\frac{\sigma^2 + \pi_a^2}{4G} \right] - i N_c \text{Tr}_{sf} \ln D, \quad (10)$$

where the Tr operation stands for the trace in spinor (s), flavor (f) as well as two-dimensional coordinate (x) spaces, respectively. Using Eq. (10), we obtain the TDP $\Omega(\sigma, \pi_a)$ of the system:

$$\begin{aligned} \Omega(\sigma, \pi_a) &\equiv - \frac{\mathcal{S}_{\text{eff}}(\sigma, \pi_a)}{N_c \int d^2x} \Big|_{\sigma, \pi_a = \text{const}} = \frac{\sigma^2 + \pi_a^2}{4G} + i \frac{\text{Tr}_{sf} \ln D}{\int d^2x} \\ &= \frac{\sigma^2 + \pi_a^2}{4G} + i \text{Tr}_{sf} \int \frac{d^2p}{(2\pi)^2} \ln \bar{D}(p), \end{aligned} \quad (11)$$

where the σ and π_a fields are now x -independent quantities, and

$$\bar{D}(p) = \not{p} + \mu \gamma^0 + \nu \tau_3 \gamma^0 + \nu_5 \tau_3 \gamma^0 \gamma^5 - \sigma - i \gamma^5 \pi_a \tau_a \quad (12)$$

is the momentum-space representation of the Dirac operator D [Eq. (9)]. In what follows we are going to investigate the μ, ν, ν_5 dependence of the global minimum point of the function $\Omega(\sigma, \pi_a)$ vs σ, π_a . To simplify the task, let us note that due to the $U_{I_3}(1) \times U_{A_{I_3}}(1)$ invariance of the model, the TDP (11) depends effectively only on the two combinations $\sigma^2 + \pi_3^2$ and $\pi_1^2 + \pi_2^2$ of the bosonic fields, as is easily seen from Eq. (6). In this case, without loss of generality, one can put $\pi_2 = \pi_3 = 0$ in Eq. (11), and study the TDP (11) as a function of only two variables: $M \equiv \sigma$ and $\Delta \equiv \pi_1$. Taking into account this constraint in Eqs. (12) and (11) as well as the general relation

$$\text{Tr}_{sf} \ln \bar{D}(p) = \ln \text{Det } \bar{D}(p) = \sum_i \ln \epsilon_i, \quad (13)$$

where the summation over all four eigenvalues ϵ_i of the 4×4 matrix $\bar{D}(p)$ is implied and

$$\epsilon_{1,2,3,4} = -M \pm \sqrt{(p_0 + \mu)^2 - p_1^2 - \Delta^2 + \nu^2 - \nu_5^2} \pm 2 \sqrt{[(p_0 + \mu)\nu + p_1 \nu_5]^2 - \Delta^2(\nu^2 - \nu_5^2)}, \quad (14)$$

we have from Eq. (11)

$$\Omega(M, \Delta) = \frac{M^2 + \Delta^2}{4G} + i \int \frac{d^2p}{(2\pi)^2} \ln P_4(p_0). \quad (15)$$

In Eq. (15) we use the notation

$$P_4(p_0) = \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = \eta^4 - 2a\eta^2 - b\eta + c, \quad (16)$$

where $\eta = p_0 + \mu$ and

$$\begin{aligned} a &= M^2 + \Delta^2 + p_1^2 + \nu^2 + \nu_5^2, & b &= 8p_1 \nu \nu_5, \\ c &= a^2 - 4p_1^2(\nu^2 + \nu_5^2) - 4M^2 \nu^2 - 4\Delta^2 \nu_5^2 - 4\nu^2 \nu_5^2. \end{aligned} \quad (17)$$

Thus, it follows from Eqs. (16) and (17) that the TDP (15) is invariant with respect to the so-called duality transformation (for an analogous case of duality between chiral and superconducting condensates, see Refs. [47,48]),

$$\mathcal{D}: M \leftrightarrow \Delta, \nu \leftrightarrow \nu_5. \quad (18)$$

In powers of Δ and M the fourth-degree polynomial $P_4(p_0)$ has the following forms:

$$P_4(p_0) \equiv \Delta^4 - 2\Delta^2(\eta^2 - p_1^2 - M^2 + \nu_5^2 - \nu^2) + [M^2 + (p_1 - \nu_5)^2 - (\eta + \nu)^2] \times [M^2 + (p_1 + \nu_5)^2 - (\eta - \nu)^2] \quad (19)$$

$$\equiv M^4 - 2M^2(\eta^2 - p_1^2 - \Delta^2 + \nu^2 - \nu_5^2) + [\Delta^2 + (p_1 - \nu)^2 - (\eta + \nu_5)^2][\Delta^2 + (p_1 + \nu)^2 - (\eta - \nu_5)^2]. \quad (20)$$

Note that according to the general theorem of algebra, the polynomial $P_4(p_0)$ can also be presented in the form

$$P_4(p_0) \equiv (p_0 - p_{01})(p_0 - p_{02})(p_0 - p_{03})(p_0 - p_{04}), \quad (21)$$

where the roots p_{01}, p_{02}, p_{03} and p_{04} of this polynomial are the energies of quasiparticle or quasiantiparticle excitations of the system. In particular, it follows from Eq. (19) that at $\Delta = 0$ the set of roots p_{0i} looks like

$$\{p_{01}, p_{02}, p_{03}, p_{04}\}|_{\Delta=0} = \{-\mu - \nu \pm \sqrt{M^2 + (p_1 - \nu_5)^2}, -\mu + \nu \pm \sqrt{M^2 + (p_1 + \nu_5)^2}\}, \quad (22)$$

whereas it is clear from Eq. (20) that at $M = 0$ it has the form

$$\{p_{01}, p_{02}, p_{03}, p_{04}\}|_{M=0} = \{-\mu - \nu_5 \pm \sqrt{\Delta^2 + (p_1 - \nu)^2}, -\mu + \nu_5 \pm \sqrt{\Delta^2 + (p_1 + \nu)^2}\}. \quad (23)$$

Taking into account the relation (21) as well as the formula (it was proved, e.g., in Ref. [17])

$$\int_{-\infty}^{\infty} dp_0 \ln(p_0 - K) = i\pi|K| \quad (24)$$

(being true up to an infinite term independent on the real quantity K), it is possible to integrate in Eq. (15) over p_0 . Then, the *unrenormalized* TDP (15) can be presented in the following form:

$$\Omega(M, \Delta) \equiv \Omega^{un}(M, \Delta) = \frac{M^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} (|p_{01}| + |p_{02}| + |p_{03}| + |p_{04}|). \quad (25)$$

It is clear directly from the relations (16) and (17) that the TDP (15) is an even function over each of the variables M and Δ . In addition, it is invariant under each of the transformations $\mu \rightarrow -\mu, \nu \rightarrow -\nu, \nu_5 \rightarrow -\nu_5$.⁵ Hence, without loss of generality in the following we only consider the values $\mu \geq 0, \nu \geq 0, \nu_5 \geq 0, M \geq 0$, and $\Delta \geq 0$. Finally, note that the integrand in Eq. (25) is not an even function of p_1 (see also Appendix A).

B. The nonzero-temperature case

Although the effect of nonzero temperatures is quite predictable (one can expect that the temperature just restores all the broken symmetries of the model), here we consider nonzero temperatures because it is instructive to know how robust the charged PC phase is under temperature.

To introduce a finite temperature T , it is very convenient to use the zero-temperature expression (15) for the TDP,

$$\Omega(M, \Delta) = \frac{M^2 + \Delta^2}{4G} + i \int \frac{d^2 p}{(2\pi)^2} \ln(p_0 - p_{01}) \times (p_0 - p_{02})(p_0 - p_{03})(p_0 - p_{04}), \quad (26)$$

where we took into account Eq. (21). Then, to find the temperature-dependent TDP $\Omega_T(M, \Delta)$ one should replace in Eq. (26) the integration over p_0 in favor of the summation over Matsubara frequencies ω_n by the rule

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} (\dots) \rightarrow iT \sum_{n=-\infty}^{\infty} (\dots),$$

$$p_0 \rightarrow p_{0n} \equiv i\omega_n \equiv i\pi T(2n + 1),$$

$$n = 0, \pm 1, \pm 2, \dots \quad (27)$$

In the expression obtained, it is possible to sum over Matsubara frequencies using the general formula (the corresponding technique was presented, e.g., in Ref. [51])

$$\sum_{n=-\infty}^{\infty} \ln(i\omega_n - a) = \ln[\exp(\beta|a|/2) + \exp(-\beta|a|/2)]$$

$$= \frac{\beta|a|}{2} + \ln[1 + \exp(-\beta|a|)], \quad (28)$$

where $\beta = 1/T$. As a result, one can obtain the following expression for the TDP $\Omega_T(M, \Delta)$:

⁵Indeed, if we perform simultaneously with $\mu \rightarrow -\mu$ the change of variables $p_0 \rightarrow -p_0$ and $p_1 \rightarrow -p_1$ in the integral (15), then one can easily see that the expression (15) remains intact. Finally, if only ν (only ν_5) is replaced by $-\nu$ (by $-\nu_5$), we should transform $p_1 \rightarrow -p_1$ in the integral (15) in order to see that the TDP remains unchanged.

$$\Omega_T(M, \Delta) = \Omega(M, \Delta) - T \sum_{i=1}^4 \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \ln(1 + e^{-\beta|p_{0i}|}), \quad (29)$$

where $\Omega(M, \Delta)$ is the TDP (25) of the system at zero temperature. Since each root p_{0i} in Eq. (29) is a dually $\mathcal{D}$ -invariant quantity (see Appendix A), it is clear that the temperature-dependent TDP (29) is also symmetric with respect to the duality transformation $\mathcal{D}$ (18).

III. CALCULATION OF THE TDP AT $T=0$

A. Thermodynamic potential in vacuum:

The case of $\mu=0, \nu=0, \nu_5=0$

First of all, let us obtain a finite, i.e. renormalized, expression for the TDP (25) at $\mu=0, \nu=0$ and $\nu_5=0$, i.e. in vacuum, and at zero temperature. Since in this case a thermodynamic potential is usually called an effective potential, we use for it the notation $V^{un}(M, \Delta)$. As a consequence of Eqs. (15)–(17) and using Eq. (24), it is clear that at $\mu=\nu=\nu_5=0$ the effective potential $V^{un}(M, \Delta)$ looks like

$$V^{un}(M, \Delta) = \frac{M^2 + \Delta^2}{4G} + 2i \int \frac{d^2 p}{(2\pi)^2} \ln[p_0^2 - p_1^2 - M^2 - \Delta^2] \\ = \frac{M^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{\pi} \sqrt{p_1^2 + M^2 + \Delta^2}. \quad (30)$$

It is evident that the effective potential (30) is an ultraviolet (UV)-divergent quantity. So, we need to renormalize it. This procedure consists of two steps. (i) First of all we need to regularize the divergent integral in Eq. (30), i.e. we suppose there that $|p_1| < \Lambda$. (ii) Second, we must suppose also that the bare coupling constant G depends on the cutoff parameter Λ in such a way that in the limit $\Lambda \rightarrow \infty$ one obtains a finite expression for the effective potential.

Following step (i) of this procedure, we have

$$V^{\text{reg}}(M, \Delta) = \frac{M^2 + \Delta^2}{4G} - \frac{2}{\pi} \int_0^\Lambda dp_1 \sqrt{p_1^2 + M^2 + \Delta^2} \\ = \frac{M^2 + \Delta^2}{4G} - \frac{1}{\pi} \left\{ \Lambda \sqrt{\Lambda^2 + M^2 + \Delta^2} \right. \\ \left. + (M^2 + \Delta^2) \ln \frac{\Lambda + \sqrt{\Lambda^2 + M^2 + \Delta^2}}{\sqrt{M^2 + \Delta^2}} \right\}. \quad (31)$$

Further, according to step (ii) we suppose that in Eq. (31) the bare coupling $G \equiv G(\Lambda)$ has the following Λ dependence:

$$\frac{1}{4G(\Lambda)} = \frac{1}{\pi} \ln \frac{2\Lambda}{m}, \quad (32)$$

where m is a new mass scale of the model, and m is a free model parameter. It appears instead of the dimensionless bare coupling constant G (dimensional transmutation) and, evidently, does not depend on a normalization point, i.e. it is a renormalization-invariant quantity. Substituting Eq. (32) into Eq. (31) and ignoring there an unessential term $(-\Lambda^2/\pi)$, we obtain in the limit $\Lambda \rightarrow \infty$ the finite and renormalization-invariant expression for the effective potential,

$$V_0(M, \Delta) = \frac{M^2 + \Delta^2}{2\pi} \left[\ln \left(\frac{M^2 + \Delta^2}{m^2} \right) - 1 \right]. \quad (33)$$

B. Calculation of the TDP (25) in the general case:

$\mu > 0, \nu > 0, \nu_5 > 0$

In Appendix A the properties of the quasiparticle energies p_{0i} , where $i = 1, \dots, 4$, are investigated. In particular, it is clear from the asymptotic expansion (A12) that the integral over p_1 in Eq. (25) is ultraviolet divergent. It is possible to transform the expression (25) in the following way:

$$\Omega^{un}(M, \Delta) = \frac{M^2 + \Delta^2}{4G} \\ - \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} (|p_{01}| + |p_{02}| + |p_{03}| + |p_{04}|) \Big|_{\mu=\nu=\nu_5=0} \\ - \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} \left[\sum_{i=1}^4 |p_{0i}| - \left(\sum_{i=1}^4 |p_{0i}| \right) \Big|_{\mu=\nu=\nu_5=0} \right]. \quad (34)$$

Since the asymptotic expansion (A12) does not depend on the chemical potentials μ, ν and ν_5 , it is evident that the last integral in Eq. (34) is convergent and all UV divergences of the TDP are located in the first integral of Eq. (34). Moreover, it is clear due to the relation (A13) that the first two terms on the right-hand side of Eq. (34) are just the unrenormalized effective potential in vacuum (30). So to obtain a finite expression for the TDP (34), it is enough to proceed as in the previous subsection, where just these two terms, i.e. the vacuum effective potential, were renormalized. As a result, we have

$$\Omega^{\text{ren}}(M, \Delta) = V_0(M, \Delta) \\ - \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} \left\{ |p_{01}| + |p_{02}| + |p_{03}| + |p_{04}| \right. \\ \left. - 4 \sqrt{p_1^2 + M^2 + \Delta^2} \right\}, \quad (35)$$

where $V_0(M, \Delta)$ is the renormalized TDP (effective potential) (33) of the model at $\mu=\nu=\mu_5=0$. Moreover, we have used in Eq. (35) the relation (A13) for the sum of quasiparticle energies in vacuum. Note also that (as it

follows from the considerations of Appendix A) the quasiparticle energies p_{0i} , where $i = 1, \dots, 4$, are invariant with respect to the duality transformation (18). So the renormalized TDP (35) is also symmetric under the duality transformation $\mathcal{D}$.

In Ref. [50] the erroneous assumption was used that the quantity $\sum_{i=1}^4 |p_{0i}|$ is an even function of p_1 . This further led to an incorrect expression for the TDP (35), in which the integration over $p_1 \in (0, +\infty)$ was used, and to erroneous phase diagrams. Here we correct this mistake. Therefore, in Eq. (35), the integration over $p_1 \in (-\infty, +\infty)$ is carried out. However, the main qualitative conclusions about the properties of the chiral isospin chemical potential ν_5 and the duality of the model under consideration do not change.

Let us denote by (M_0, Δ_0) the global minimum point (GMP) of the TDP (35). Then, by investigating the behavior of this point vs μ , ν and ν_5 it is possible to construct the (μ, ν, ν_5) -phase portrait (diagram) of the model. A numerical algorithm for finding the quasi(anti) particle energies p_{01} , p_{02} , p_{03} , and p_{04} is elaborated in Appendix A. Based on this algorithm, it is possible to study the TDP of the model in the region of sufficiently small values of chemical potentials, e.g., at $\mu, \nu, \nu_5 \leq 2m$, where we did not find any GMP of the form $(M_0 \neq 0, \Delta_0 \neq 0)$

(this conclusion probably will not change at larger values of chemical potentials). Strictly speaking, this property of the TDP is typical for the NJL models only in the chiral limit. At the physical point, i.e. when quarks have a nonzero bare mass, in the charged PC phase both the Δ_0 and M_0 condensates are nonzero (see, e.g., Refs. [11,26,44]). But these condensates are of the same order of magnitude only in the transitory region (in the region between the CSB and charged PC phases), whereas deeper in the phases one of the condensates dominates. For example, in the charged PC phase, first (at ν slightly larger than half of the pion mass) there is a nonzero chiral condensate, or dynamical quark mass, $M_0 \gg \Delta_0$ but it decreases very fast and at larger ν the chiral condensate M_0 is much smaller than the charged pion one, Δ_0 . Let us also recall that the chiral limit overall is a good approximation as it has been shown in Ref. [26]. Hence, in order to establish the phase portrait of the massless model (1), e.g., in the region $\mu, \nu, \nu_5 \leq 2m$, it is enough to study the projections $F_1(M) \equiv \Omega^{\text{ren}}(M, \Delta = 0)$ and $F_2(\Delta) \equiv \Omega^{\text{ren}}(M = 0, \Delta)$ of the TDP (35) onto the M and Δ axes, correspondingly. Taking into account the relations (22) for the quasiparticle energies p_{0i} at $\Delta = 0$, it is possible to obtain the following expression for the projection $F_1(M)$:

$$F_1(M) = \frac{M^2}{2\pi} \ln \left(\frac{M^2}{m^2} \right) - \frac{M^2}{2\pi} - \frac{\nu_5^2}{\pi} - \frac{\theta(|\mu - \nu| - M)}{2\pi} \left(|\mu - \nu| \sqrt{(\mu - \nu)^2 - M^2} - M^2 \ln \frac{|\mu - \nu| + \sqrt{|\mu - \nu|^2 - M^2}}{M} \right) - \frac{\theta(\mu + \nu - M)}{2\pi} \left((\mu + \nu) \sqrt{(\mu + \nu)^2 - M^2} - M^2 \ln \frac{\mu + \nu + \sqrt{(\mu + \nu)^2 - M^2}}{M} \right). \quad (36)$$

Then, the projection $F_2(\Delta)$ of the TDP onto the Δ axis can be obtained directly from Eq. (23) or, alternatively, using the dual symmetry $\mathcal{D}$ of the TDP,

$$F_2(\Delta) = F_1(\Delta)|_{\nu \leftrightarrow \nu_5}. \quad (37)$$

(Details of the derivation of these expressions are given in Appendix B.) After simple transformations, one can see that $F_1(M)$ and $F_2(\Delta)$ coincide at $\nu_5 = 0$ with the corresponding TDPs (12) and (13) of Ref. [43]. Moreover, it is obvious that the GMP of the TDP (35) is defined by a comparison between the minimum values of the functions $F_1(M)$ and $F_2(\Delta)$.

C. Quark number density

As it is clear from the above consideration, there are three phases in the model (1). The first one is the symmetric phase, which corresponds to the global minimum point

(M_0, Δ_0) of the TDP (35) of the form $(M_0 = 0, \Delta_0 = 0)$. In the CSB phase the TDP reaches its minimum value at the point $(M_0 \neq 0, \Delta_0 = 0)$. Finally, in the charged PC phase the GMP lies at the point $(M_0 = 0, \Delta_0 \neq 0)$. [Notice, that in the most general case the coordinates (condensates) M_0 and Δ_0 of the global minimum point depend on chemical potentials.]

In the present subsection we would like to obtain the expression for the quark number (or particle) density n_q in the ground state of each phase. Recall that in the most general case this quantity is defined by the relation⁶

$$n_q = - \frac{\partial \Omega^{\text{ren}}(M_0, \Delta_0)}{\partial \mu}. \quad (38)$$

⁶The density of baryons n_B and the quark number density n_q are connected by the relation $n_q = 3n_B$.

Hence, in the chiral-symmetry-breaking phase we have

$$\begin{aligned} n_q|_{\text{CSB}} &= -\frac{\partial \Omega^{\text{ren}}(M_0 \neq 0, \Delta_0 = 0)}{\partial \mu} = -\frac{\partial F_1(M_0)}{\partial \mu} \\ &= \frac{\theta(\mu + \nu - M_0)}{\pi} \sqrt{(\mu + \nu)^2 - M_0^2} \\ &\quad + \frac{\text{sign}(\mu - \nu)\theta(|\mu - \nu| - M_0)}{\pi} \sqrt{(\mu - \nu)^2 - M_0^2}, \end{aligned} \quad (39)$$

where $\text{sign}(x)$ denotes the sign function and the quantity $F_1(M)$ is given in Eq. (36). The quark number density in the charged pion condensation phase can be easily obtained from Eq. (39) by the simple replacement,

$$\begin{aligned} n_q|_{\text{PC}} &= -\frac{\partial \Omega^{\text{ren}}(M_0 = 0, \Delta_0 \neq 0)}{\partial \mu} = -\frac{\partial F_2(\Delta_0)}{\partial \mu} \\ &= \{n_q|_{\text{CSB}}\}|_{M_0 \rightarrow \Delta_0; \nu \leftrightarrow \nu_5}, \end{aligned} \quad (40)$$

which is due to the relation (37). Supposing in Eq. (39) that $M_0 = 0$, one can find the following expression for the particle density in the symmetric phase (of course, we take into account the constraints $\mu \geq 0$, $\nu \geq 0$ and $\nu_5 \geq 0$):

$$n_q|_{\text{SYM}} = \frac{\mu + \nu}{\pi} + \frac{\text{sign}(\mu - \nu)}{\pi} |\mu - \nu| = \frac{2\mu}{\pi}. \quad (41)$$

IV. PHASE STRUCTURE AT ZERO TEMPERATURE

A. The role of the duality symmetry $\mathcal{D}$ [Eq. (18)] of the TDP

Suppose now that at some fixed particular values of the chemical potentials μ , $\nu = A$ and $\nu_5 = B$ the global minimum of the TDP (35) lies at the point, e.g., $(M = M_0 \neq 0, \Delta = 0)$. It means that for such fixed values of the chemical potentials the CSB phase is realized in the model. Then it follows from the duality invariance of the TDP (15) [or Eq. (35)] with respect to the transformation $\mathcal{D}$ [Eq. (18)] that the permutation of the chemical potential values (i.e. $\nu = B$ and $\nu_5 = A$ and the intact value of μ) moves the global minimum of the TDP $\Omega^{\text{ren}}(M, \Delta)$ to the point $(M = 0, \Delta = M_0)$, which corresponds to the charged PC phase (and vice versa). This is the so-called duality correspondence between CSB and charged PC phases in the framework of the model under consideration.⁷

Hence, the knowledge of a phase of the model (1) at some fixed values of the external free model parameters μ, ν, ν_5 is sufficient to understand what phase (we call it a

dually conjugated phase) is realized at rearranged values of the external parameters, $\nu \leftrightarrow \nu_5$, at fixed μ . Moreover, different physical parameters such as condensates, densities, etc., which characterize both the initial phase and the dually conjugated phase, are connected by the duality transformation $\mathcal{D}$. For example, the chiral condensate of the initial CSB phase at some fixed μ, ν, ν_5 is equal to the charged-pion condensate of the dually conjugated charged PC phase, in which one should perform the replacement $\nu \leftrightarrow \nu_5$. Knowing the particle density n_q of the initial CSB phase [see Eq. (39)] as a function of the external chemical potentials μ, ν and ν_5 , one can find the particle density (40) in the dually conjugated charged PC phase by interchanging ν and ν_5 in the initial expression n_q for the particle density in the CSB phase (see also Sec. III C), etc.

The duality transformation $\mathcal{D}$ of the TDP can also be applied to an arbitrary phase portrait of the model (see below). In particular, it is clear that if we have a most general phase portrait, i.e. the correspondence between any point (ν, ν_5, μ) of the three-dimensional space of external parameters and possible model phases (CSB, charged PC and symmetric phases), then under the duality transformation ($\nu \leftrightarrow \nu_5$, CSB $\leftrightarrow$ charged PC and the symmetrical phase does not change) this phase portrait is mapped to itself, i.e. the most general (ν, ν_5, μ) -phase portrait is self-dual. The self-duality of the (ν, ν_5, μ) -phase portrait means that the regions of the CSB and charged PC phases in the three-dimensional (ν, ν_5, μ) space are arranged mirror symmetrically with respect to the plane $\nu = \nu_5$ of this space. In Sec. IV B, we will present a few sections of this three-dimensional (ν, ν_5, μ) -phase portrait of the model by the planes of the form $\mu = \text{const}$, $\nu = \text{const}$ and $\nu_5 = \text{const}$, respectively.

B. Promotion of the dense charged PC phase by $\nu_5 \neq 0$

First of all, we will study the phase structure of the model (1) at different fixed values of the chiral isospin chemical potential ν_5 . To this end, we determine numerically the global minimum points of the TDPs $F_1(M)$ [Eq. (36)] and $F_2(\Delta)$ [Eq. (37)] and then compare the minimum values of these functions with the external parameters μ, ν, ν_5 . Moreover, using the expressions (39) and (40), it is possible to find the quark number density n_q or baryon density n_B (note that $n_q = 3n_B$) inside each phase. As a result, in Fig. 1 we show several (ν, μ) -phase portraits, corresponding to $\nu_5 = 0$, $\nu_5 = 0.75m$ and $\nu_5 = 1.3m$, respectively. Recall that m is a free renormalization-invariant mass scale parameter, which appears in the vacuum case of the model after renormalization [see Eqs. (32) and (33)].

The phase portrait of the model in Fig. 1 with $\nu_5 = 0$ was obtained earlier (see e.g. Refs. [16,43]). It is clear from it that at $\nu_5 = 0$ the charged PC phase with *nonzero baryon density* n_B is not realized in the model under consideration. Only the charged PC phase with *zero baryon density* can be observed at rather small values of μ . (Physically, it means

⁷It is worth noting that in some $(1+1)$ - and $(2+1)$ -dimensional models there is a duality between CSB and superconductivity [47,48].

FIG. 1. The (ν, μ) -phase portraits of the model for different values of the chiral chemical potential ν_5 : the case $\nu_5 = 0$ (left panel), the case $\nu_5 = 0.75m$ (middle panel) and the case $\nu_5 = 1.3m$ (right panel). The notations PC and PC_d mean the charged pion condensation phase with zero and nonzero baryon density, respectively. Analogously, the notations CSB and CSB_d mean the chiral-symmetry-breaking phase with zero and nonzero baryon density, respectively, and SYM denotes the symmetrical phase. The parameter m is introduced in Eq. (32). All the lines are solid and denote phase transitions of the first order.

that at $\nu_5 = 0$ the model predicts the charged PC phenomenon in the medium with $n_B = 0$ only. For example, it might consist of charged pions, etc. But in quark matter with nonzero baryon density the charged PC is forbidden.) Instead, at large values of μ there exist two phases—the chiral-symmetry-breaking phase and the symmetrical phase—both with nonzero baryon density, i.e. the model predicts only the CSB or symmetrical phases of dense quark matter at $\nu_5 = 0$. However, as we can see from the other phase diagrams in Fig. 1, at rather high values of ν_5 , e.g., at $\nu_5 = 0.75m$ or $\nu_5 = 1.3m$, there might appear in the phase portrait a charged PC phase with *nonzero baryon*

density (in Fig. 1 and other figures it is denoted as PC_d). Hence, in chirally asymmetric (i.e. at $\nu_5 > 0$) and dense quark matter the charged PC phenomenon is allowed to exist in the framework of the toy model (1). Thus, we see that $\nu_5 \neq 0$ is a factor which promotes the charged PC phenomenon in dense quark matter.

Now, suppose that we want to obtain a (ν_5, μ) -phase portrait of the model at some fixed value $\nu = \text{const}$. In this case there is no need to perform the direct numerical investigations of the TDP (35). In contrast [due to the dual invariance (18) of the model TDP], one can simply make the dual transformation of the (ν, μ) -phase diagram at the

FIG. 2. The (ν_5, μ) -phase portrait of the model for different values of the isospin chemical potential ν : the case $\nu = 0$ (left panel) and case $\nu = 1.3m$ (right panel). All the lines are the first-order phase transition curves. The notations are the same as in Fig. 1.

FIG. 3. The (ν, ν_5) -phase portraits of the model for different values of the quark number chemical potential μ : the case $\mu = 0.65m$ (left panel) and the case $\mu = m$ (right panel). All the lines are the first-order phase transition curves. The notations are the same as in Fig. 1.

corresponding fixed value $\nu_5 = \text{const}$. For example, to find the (ν_5, μ) -phase diagram at $\nu = 0$ we should start from the (ν, μ) diagram at fixed $\nu_5 = 0$ of Fig. 1 and make the simplest replacement in the notations of this figure: $\nu \rightarrow \nu_5$, $\text{PC} \leftrightarrow \text{CSB}$, $\text{PC}_d \leftrightarrow \text{CSB}_d$ and the notation “sym” does not change. As a result of this mapping, we obtain the phase diagram of Fig. 2 (left panel) with the PC_d phase. From this phase diagram, we can also draw the interesting conclusion that the PC_d phase can occur in a dense, *isotopically symmetric* baryonic medium, $\nu = 0$, but at the same time ν_5 should be nonzero.

In a similar way, to obtain the (ν_5, μ) -phase diagram, e.g., at $\nu = 1.3m$, it is sufficient to apply the duality transformation to the (ν, μ) -phase portrait of the model at $\nu_5 = 1.3m$ (see the right panel of Fig. 1). The resulting mapping is the right panel of Fig. 2, etc. It thus supports the above conclusion: the charged PC phenomenon can be realized in chirally asymmetric quark matter with nonzero baryon density.

Finally, let us consider the (ν, ν_5) -phase diagrams of the model at different fixed values of μ . It is clear from the previous discussions that each of these diagrams is a self-dual one, i.e. the CSB and charged PC phases are arranged symmetrically with respect to the line $\nu = \nu_5$ of the (ν, ν_5) plane. This fact is confirmed by the (ν, ν_5) -phase portraits of Fig. 3, obtained by direct numerical analysis of the TDPs $F_1(M)$ [Eq. (36)] and $F_2(\Delta)$ [Eq. (37)]. Moreover, the phase diagrams of Fig. 3 support once again the main conclusion of our paper: the charged PC phase with nonzero baryon density, i.e. the phase denoted in Figs. 1–3 as PC_d , might be realized in the framework of the model (1) only at $\nu_5 \neq 0$.

We have already noted that the phase structure of this NJL₂ model was previously studied in Ref. [50]. And there the conclusion was drawn that the chiral isospin chemical potential $\mu_{I5} \equiv 2\nu_5$ generates the charged PC phase in dense quark matter, but only for sufficiently small values of the isospin chemical potential $\mu_I \equiv 2\nu$. However, this result was significantly different from the predictions of the NJL₄ model, which were obtained later in Refs. [26,31], where it was stated that the charged PC phase can exist in a chirally asymmetric medium only at sufficiently high values of the isospin chemical potential μ_I . Based on this latter result, it is possible to predict that the charged PC phase can be realized in the cores of such astrophysical objects as neutron stars, in which both the isotopic μ_I and chiral isotopic μ_{I5} chemical potential can reach large values. (The possibility of the appearance of regions with $\mu_{I5} \neq 0$ inside neutron stars was discussed in Appendix A of Ref. [26]). In this case, the charged PC phenomenon can have a significant influence on the physical processes in neutron stars and lead to observable effects. However, as noted in the Introduction, the results of Ref. [50] were based on an erroneous expression for the TDP. The correct phase diagrams of the NJL₂ model (1) are presented just in the present work, from which it can be seen (see Figs. 1 and 3) that the charged PC_d phase occupies an infinitely long strip along the ν axis (for some fixed values of ν_5), which is largely consistent with the predictions of the NJL₄ model (see Refs. [26,31]). Thus, we see that two generally different effective models, NJL₂ and NJL₄, predict qualitatively identical properties of quark matter with chiral isotopic asymmetry, which, in our opinion, enhances the realism of their predictions.

FIG. 4. The (T, μ) -phase portraits of the model for different values of the chemical potentials ν and ν_5 : the case $\nu = m$ and $\nu_5 = 0$ (left panel), the case $\nu = 0.9m$ and $\nu_5 = 0.75m$ (middle panel) and the case $\nu = 1.3m$ and $\nu_5 = 0.6m$ (right panel). Dashed lines are the second-order phase transition curves, while solid lines are the first-order phase transition curves. All the notations are the same as in Fig. 1. The dots are the tricritical points where phase transitions of two types meet.

V. PHASE STRUCTURE AT $T \neq 0$

First of all note that the temperature-dependent TDP $\Omega_T(M, \Delta)$ in Eq. (29) is a UV-divergent quantity. While the second term on the right-hand side of this relation is a convergent improper integral that does not contain UV divergences, the first term $\Omega(M, \Delta)$ is a UV-divergent TDP of the model at $T = 0$. So both the TDP $\Omega(M, \Delta)$ at $T = 0$ and the TDP $\Omega_T(M, \Delta)$ at nonzero temperature can be renormalized using the technique presented in the Sec. III B. As a result, one can obtain the following expression for the renormalized TDP of the model at $T \neq 0$:

$$\Omega_T^{\text{ren}}(M, \Delta) = \Omega^{\text{ren}}(M, \Delta) - T \sum_{i=1}^4 \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} \ln(1 + e^{-\beta|p_{0i}|}), \quad (42)$$

where $\Omega^{\text{ren}}(M, \Delta)$ is presented in Eq. (35). A numerical algorithm for finding the quasi(anti)particle energies p_{01} , p_{02} , p_{03} , and p_{04} is elaborated in Appendix A. Based on this, it can be shown numerically that the GMP of the TDP (42) can never be of the form ($M_0 \neq 0$, $\Delta_0 \neq 0$). Hence, in order to establish the phase portrait of the model at $T \neq 0$, it is enough to study the projections $F_1^T(M) \equiv \Omega_T^{\text{ren}}(M, \Delta = 0)$ and $F_2^T(\Delta) \equiv \Omega_T^{\text{ren}}(M = 0, \Delta)$ of the TDP (42) onto the M and Δ axes, correspondingly. Taking into account the relations (22) and (23) for the quasiparticle energies p_{0i} at $\Delta = 0$ or $M = 0$, we have

$$F_1^T(M) = F_1(M) - T \sum_{\kappa=\pm} \int_0^{\infty} \frac{dq}{\pi} \ln(1 + e^{-\beta|\mu+\nu+\kappa\sqrt{M^2+q^2}|}) - T \sum_{\kappa=\pm} \int_0^{\infty} \frac{dq}{\pi} \ln(1 + e^{-\beta|\mu-\nu+\kappa\sqrt{M^2+q^2}|}), \quad (43)$$

where $F_1(M)$ is given in Eq. (36). Then, the projection $F_2^T(\Delta)$ of the TDP $\Omega_T^{\text{ren}}(M = 0, \Delta)$ on the Δ axis reads

$$F_2^T(\Delta) = F_1^T(\Delta)|_{\nu \leftrightarrow \nu_5} = F_2(\Delta) - T \sum_{\kappa=\pm} \int_0^{\infty} \frac{dq}{\pi} \ln(1 + e^{-\beta|\mu+\nu_5+\kappa\sqrt{\Delta^2+q^2}|}) - T \sum_{\kappa=\pm} \int_0^{\infty} \frac{dq}{\pi} \ln(1 + e^{-\beta|\mu-\nu_5+\kappa\sqrt{\Delta^2+q^2}|}), \quad (44)$$

where $F_2(\Delta)$ is given in Eq. (37). By comparing the smallest values of the functions $F_1^T(M)$ and $F_2^T(\Delta)$, one can construct the simplest temperature phase diagrams of the model.

First, let us notice that the duality is exact in the chiral limit at any temperature in the framework of the NJL₂ model.

The phase diagrams of Fig. 4 are the extensions of some phase portraits at $T = 0$ shown in Figs. 1 and 2 to nonzero values of temperature. For example, Fig. 4 (left panel) is the temperature extension of the section $\nu = m$ of the left panel diagram of Fig. 1 and shows at what temperatures the phase structure remains nontrivial. The middle diagram of Fig. 4 is the temperature extension of the section $\nu/m = 0.9$ of the middle panel at Fig. 1. Finally, in the right panel of Fig. 4 one can see the temperature extension of the section $\nu_5/m = 0.6$ of the right panel of Fig. 2. The dashed lines denote second-order phase transitions, but one can note that there are regions of first-order phase transitions at not very large temperatures denoted by the solid lines. The points, where two types of phase transitions meet, are the so-called tricritical points (at the physical point, i.e. at nonzero bare quark mass, they correspond to the critical end points of a phase diagram). Recall that at zero temperature all phase transitions were of first order. So, in Fig. 4 we show three (μ, T) -phase diagrams of the model. Each of them corresponds to a rather high fixed value of the chemical potential $\nu \gtrsim m$, but to substantially not very large fixed values of $\nu_5 = 0, 0.6m$ and $0.75m$, respectively. It is clear that the temperature phase diagrams of Fig. 4 also support the fact that the chiral isospin chemical potential μ_{I5} promotes the

creation of a phase with condensation of charged pions in dense quark matter (it is the PC_d phase in Fig. 4). In addition, it is clear from Fig. 4 that the PC_d phase is quite stable with respect to temperature effects and can be realized in dense quark matter up to rather high temperatures. In Fig. 4 (middle and right panels) one can see that the PC_d phase persists up to $T = 0.2m - 0.35m$ respectively, which are comparatively high temperatures. (If one considers m to be the constituent quark mass value in vacuum around 300 MeV, then the PC_d phase corresponds to temperatures as high as 100 MeV.) This part of the phase diagram is rather interesting because these conditions are realized in various physical scenarios such as in new-born neutron stars (proto-neutron stars) [52], supernovae [53] and neutron star mergers [54] as well as in heavy-ion collisions [55].

Let us recall that, according to the previous study of quark matter in the framework of the NJL₂ model (1) (see Ref. [50]) at sufficiently high baryon densities, the PC_d and CSB_d phases could be realized there only in narrow regions of the (μ_I, μ_{I5}) -phase diagram, namely (i) at rather large μ_{I5} and small μ_I and (ii) at rather large μ_I and rather small μ_{I5} , respectively. Other regions of the phase diagram were occupied by either the phases without baryon density or a symmetric phase. According to this analysis, in the cores of neutron stars, where there is a large isospin imbalance μ_I , only the CSB_d phase can be realized at a not very high chiral imbalance μ_{I5} , and the PC_d phase was not likely to be present (because it required a rather low isospin imbalance, which is not the case for the neutron stars). In a similar way, the NJL₂ analysis forbade the realization of the PC_d phase in heavy-ion collisions where, if one assumes that the chiral imbalance is generated due to the chiral separation effect and is rather large, one needs a large baryon chemical potential μ_B that can be attained only at comparatively low energy and in that regime the isospin imbalance μ_I can not be small enough for the PC_d phase generation. Or if it is more energetic collisions and due to smaller μ_B probably (let us put aside the magnitude of magnetic field) chiral imbalance is small, then isospin imbalance could be not large enough (due to larger energy). (Or conversely for the CSB_d phase, which is probably less interesting but still, if the isospin imbalance is large, then the chiral imbalance should be necessarily small.) Of course, these conditions can easily be attained but still they put some restrictions on the generation of PC_d in heavy-ion collisions. In this paper we show that in the framework of NJL₂ at large baryon density (i.e. at large values of μ_B) the regions occupied by the PC_d and CSB_d phases are rather large (two bands) and they are not confined to the small areas near the μ_{I5} and μ_I axes of the (μ_I, μ_{I5}) -phase diagram. This leads to the possible generation of charged pion condensates in dense quark matter with large isospin imbalance, for example in the cores of neutron stars. The possibility of generating the PC_d phase in heavy-ion collisions is less bounded by

various constraints and the values of different imbalances. So one can see that the possibility of charged pion condensation in dense quark matter was underestimated in Ref. [50] and is actually a more considerable result of the NJL₂ model predictions.

Finally, recall that in the model (1) there is a duality between the CSB and charged PC phenomena. In this case it is possible, relying on the phase portraits of Fig. 4, to get an idea of the (μ, T) -phase structure of the model, when ν and ν_5 are fixed in a qualitatively different way than in Fig. 4 area of the chemical potentials. Namely, when ν_5 is fixed at rather high, but ν is fixed at not very high values, respectively. For example, if we apply the duality transformation in which $\nu \leftrightarrow \nu_5$, $CSB \leftrightarrow PC$ and $CSB_d \leftrightarrow PC_d$ to Fig. 4 (left panel), we can obtain the (μ, T) -phase portrait of the model at $\nu_5 = m$ and $\nu = 0$. In this new (μ, T) -phase diagram, one can find in the vicinity of the point $\mu = m$ a charged PC phase with nonzero baryon density, which in itself is already an interesting fact since in this case there is no isotopic asymmetry ($\nu = 0$). In a similar way, one can construct the duality mapping of other phase portraits in Fig. 4.

VI. SUMMARY AND CONCLUSIONS

In this paper the phase structure of the massless NJL₂ model (1) with two quark flavors was investigated in the large- N_c limit in the presence of baryon μ_B , isospin μ_I and chiral isospin μ_{I5} chemical potentials. The particular case with $\mu_{I5} = 0$ was solved earlier in Refs. [16,42,43], where it was shown that the toy model (1) does not predict a charged PC phase of dense and isotopically asymmetric quark matter. The case $\mu_{I5} \neq 0$ was first investigated in Ref. [50], where the phase structure of this NJL₂ model was studied at $T = 0$. It was shown there that the charged PC phase with nonzero baryon density (this phase is denoted by the symbol PC_d), prohibited at $\mu_{I5} = 0$, appears at rather large values of $\mu_{I5} \neq 0$, and hence, it was concluded that chiral isospin asymmetry (i.e. $\mu_{I5} \neq 0$) can generate a charged PC phenomenon in dense quark matter. (Note that two other known factors promoting a charged PC phase in dense quark matter are a finite volume [16] and the possible spatial inhomogeneity of condensates [17].)

After this prediction was made in the framework of the NJL₂ model (1), the phase structure of a more realistic $(3 + 1)$ -dimensional NJL model with nonzero μ_B , μ_I and μ_{I5} was investigated in Refs. [26,31] and similar results were obtained, i.e., it was shown in these papers that the chiral isotopic chemical potential μ_{I5} (as well as the chiral chemical potential μ_5 [31]) generates charged PC phenomenon in dense quark matter. The results were qualitatively similar, but one feature was quite different, namely it was shown that the PC_d phase can be generated only at rather small isospin chemical potential μ_I values in the NJL₂ model and for a wide range of μ_I values in the NJL₄ model.

In the present paper we noted that in Ref. [50] a miscalculation was performed in the calculation of the expression for the TDP of the NJL₂ model, and here the corrected and enlarged consideration was presented. In addition, we included in the consideration the case of nonzero temperature. It is shown that although the right expression for the TDP is very different from the original one of Ref. [50] (the right one is much simpler), qualitatively the basic features and the main predictions of the NJL₂ model analysis turned out to be similar.

(i) Namely, the main conclusion that chiral isospin imbalance generates charged pion condensation in dense and cold ($T = 0$) quark matter remains valid. Moreover, it is quite interesting that the PC_d phase can occur in a dense, *isotopically symmetric* baryonic medium, $\nu = 0$, but at the same time ν_5 should be nonzero (see the left panel of Fig. 2).

(ii) This generation of the PC_d phase by ν_5 not only remains valid, but is also greatly enhanced and the PC_d phase occupies a long strip along the μ_I axis of different phase portraits (see Figs. 1–3) of the NJL₂ model, instead of a small and compact part of the phase diagram at not too large values of the isospin chemical potential as in Ref. [50].

(iii) The possibility for the generation of charged pion condensation in dense quark matter was underestimated in Ref. [50] and is actually more considerable. It can be realized in real physical situations such as heavy-ion collisions or compact stars (see the discussion in Sec. V).

(iv) Moreover, it was also shown in the present paper that in the leading order of the large- N_c expansion there is a duality correspondence between CSB and charged PC phenomena. The simplest and most obvious manifestation of this phenomenon is that in each (μ_I, μ_{I5}) -phase diagram of the model (see, e.g., Fig. 3), each CSB phase is located mirror-symmetrically to some charged PC phase with respect to the line $\mu_I = \mu_{I5}$ (and vice versa).

(v) It has been shown that in the chiral limit duality stays the exact symmetry of the phase diagram of dense quark matter even at very high temperatures as long as the low-energy effective NJL model description is valid, and can be used for phase structure investigations.

(vi) It has also been demonstrated that the predicted generation of charged pion condensation by a chiral isospin imbalance in dense and cold ($T = 0$) quark matter remains valid even at temperatures as high as several dozens or even a hundred MeV. These conditions are rather pertinent in proto-neutron stars [52], supernovae [53] and neutron star mergers [54] as well as in heavy-ion collisions [55].

So this consideration shows that the possibility of charged pion condensation in dense quark matter at zero temperature was previously underestimated and it is more likely to happen in compact stars. Additionally, it was shown that this phenomenon can take place even at rather high temperatures, which makes it feasible in various

physical situations such as heavy-ion collisions, new-born neutron stars (proto-neutron stars), supernovae as well as neutron star mergers.

Taking into account earlier investigations [31], we see that within the framework of the two different effective models—the NJL₄ and NJL₂ models—quite similar properties (i.e., similar phase portraits) of dense and chirally asymmetric quark matter are predicted. This fact increases our confidence that in the cores of neutron stars, where μ_B and μ_I can reach significant values, the effect of charged PC can be realized. Then, to explain the similarity between the phase diagrams of these two effective models, two circumstances can be taken into account. (i) Under the influence of a strong magnetic field in the depths of neutron stars, regions with a nonzero chiral isotopic density (with $\mu_{I5} \neq 0$) can appear (for more details, see Appendix A of Refs. [26,27]). (ii) Just in the presence of a strong magnetic field, the dynamics of fermion pairing is essentially $(1+1)$ dimensional [46]. Therefore, the phase portraits of dense quark matter at $\mu_{I5} \neq 0$ obtained in framework of the NJL₄ and NJL₂ models are qualitatively the same. However, it is worth noting that $(1+1)$ -dimensional models are not comprehensive. Thus, phase transformations in real dense media with a strong magnetic field are still available for description in a two-dimensional framework, which cannot be said of many other physical properties. For example, the influence of the charged PC phase on various processes in neutron stars, such as transport processes etc., cannot be considered within the NJL₂ model. It is not the subject of the present paper, and will be investigated in the framework of a more realistic model elsewhere.

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APPENDIX A: EVALUATION OF THE ROOTS OF THE POLYNOMIAL $P_4(p_0)$ (16)

It is very convenient to present the fourth-order polynomial (16) of the variable $\eta \equiv p_0 + \mu$ as a product of two second-order polynomials (this method was proposed in Ref. [56]), i.e. we assume that

$$\eta^4 - 2a\eta^2 - b\eta + c = (\eta^2 + r\eta + q)(\eta^2 - r\eta + s) \quad (A1)$$

$$\begin{aligned} &= \left[\left(\eta + \frac{r}{2} \right)^2 + q - \frac{r^2}{4} \right] \left[\left(\eta - \frac{r}{2} \right)^2 + s - \frac{r^2}{4} \right] \\ &\equiv (\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)(\eta - \eta_4), \end{aligned} \quad (A2)$$

where r , q and s are some real-valued quantities, such that [see the relations (17)]

$$\begin{aligned}
-2a &\equiv -2(M^2 + \Delta^2 + p_1^2 + \nu^2 + \nu_5^2) = s + q - r^2, \\
-b &\equiv -8p_1\nu\nu_5 = rs - qr, \\
c &\equiv a^2 - 4p_1^2(\nu^2 + \nu_5^2) - 4M^2\nu^2 - 4\Delta^2\nu_5^2 - 4\nu^2\nu_5^2 = sq.
\end{aligned} \tag{A3}$$

In the most general case, i.e. at $M \geq 0$, $\Delta \geq 0$, $\nu \geq 0$, $\nu_5 \geq 0$ and for arbitrary values of p_1 , one can solve the system of equations (A3) with respect to q , s , r and find

$$\begin{aligned}
q &= \frac{1}{2} \left(-2a + R + \frac{b}{\sqrt{R}} \right), & s &= \frac{1}{2} \left(-2a + R - \frac{b}{\sqrt{R}} \right), \\
r &= \sqrt{R},
\end{aligned} \tag{A4}$$

where R is an arbitrary real positive solution of the equation

$$X^3 + AX = BX^2 + C \tag{A5}$$

with respect to a variable X , and

$$\begin{aligned}
A &= 4a^2 - 4c = 16[\nu_5^2\Delta^2 + M^2\nu^2 + \nu_5^2\nu^2 + p_1^2(\nu^2 + \nu_5^2)], \\
B &= 4a = 4(M^2 + \Delta^2 + \nu^2 + \nu_5^2 + p_1^2), \\
C &= b^2 = (8\nu_5\nu p_1)^2.
\end{aligned} \tag{A6}$$

Finding (numerically) the quantities q , s and r , it is possible to obtain from Eq. (A2) the roots η_i :

$$\begin{aligned}
\eta_1 &= -\frac{r}{2} + \sqrt{\frac{r^2}{4} - q}, & \eta_2 &= \frac{r}{2} + \sqrt{\frac{r^2}{4} - s}, \\
\eta_3 &= -\frac{r}{2} - \sqrt{\frac{r^2}{4} - q}, & \eta_4 &= \frac{r}{2} - \sqrt{\frac{r^2}{4} - s}.
\end{aligned} \tag{A7}$$

A numerical investigation shows that in the most general case the discriminant of the third-order algebraic equation (A5), i.e. the quantity $18ABC - 4B^3C + A^2B^2 - 4A^3 - 27C^2$, is always non-negative. So Eq. (A5) vs X has three real solutions R_1 , R_2 and R_3 (this fact was presented in Ref. [56]). Moreover, since the coefficients A , B and C [Eq. (A6)] are non-negative, it is clear that, due to the form of Eq. (A5), all its roots R_1 , R_2 and R_3 are also non-negative quantities (usually, they are positive and different). So we are free to choose the quantity R from Eq. (A4) as one of the positive solutions R_1 , R_2 or R_3 . In each case, i.e. for $R = R_1$, $R = R_2$, or $R = R_3$, we will obtain the same set of roots (A7) (possibly rearranged), which depends only on ν , ν_5 , M , Δ and p_1 , and does not depend on the choice of R . Due to the relations (A2)–(A7), one can find numerically (at fixed values of μ , ν , ν_5 , M , Δ and p_1) the roots $\eta_i = p_{0i} + \mu$ [Eq. (A7)] and, as a result, numerically investigate the TDP (25).

Analyzing the relations (A1)–(A7), we can draw another important conclusion concerning the properties of the roots η_i of the polynomial (A1) with respect to the duality

transformations $\mathcal{D}$ [Eq. (18)]. It is clear that the coefficients of this polynomial are invariant under the action of $\mathcal{D}$, so each root η_i [Eq. (A7)] is transformed under the dual mapping of $\mathcal{D}$ to itself, if the quantity R from Eq. (A4) remains intact under the action of $\mathcal{D}$ [see Eq. (A11)]. Note that in our work the quantity R is chosen with exactly this property. However, in the most general case (when R is selected to not be invariant under the $\mathcal{D}$ transformation) the set of roots η_i also remains unchanged (with the cost of a possible rearrangement of these roots) under the action of $\mathcal{D}$. Hence, the TDPs of Eqs. (29), (35) etc. are invariant with respect to the duality transformation (18).

On the basis of the relations (A1)–(A7) let us consider the asymptotic behavior of the quasiparticle energies p_{0i} at $p_1 \rightarrow \pm\infty$. First of all, we start from the asymptotic analysis of the roots $R_{1,2,3}$ of Eq. (A5) at $p_1 \rightarrow \pm\infty$,

$$R_1 = 4\nu^2 - \frac{4\Delta^2\nu^2}{p_1^2} + \mathcal{O}(1/p_1^4), \tag{A8}$$

$$R_2 = 4\nu_5^2 - \frac{4M^2\nu_5^2}{p_1^2} + \mathcal{O}(1/p_1^4), \tag{A9}$$

$$R_3 = 4p_1^2 + 4(M^2 + \Delta^2) + \frac{4(\nu_5^2M^2 + \nu^2\Delta^2)}{p_1^2} + \mathcal{O}(1/p_1^4). \tag{A10}$$

It is clear from these relations that R_3 is invariant under the duality transformation (18), whereas $R_1 \leftrightarrow R_2$. Then, using for example R_3 [Eq. (A10)] as the quantity R in Eqs. (A4) and (A7), one can get the asymptotics of the quasiparticle energies $p_{0i} \equiv \eta_i - \mu$ at $p_1 \rightarrow \pm\infty$,

$$\begin{aligned}
p_{01} &= -|p_1| - \mu + |\nu_5 - \nu| - \frac{\Delta^2 + M^2}{2|p_1|} + \mathcal{O}(1/p_1^2), \\
p_{02} &= |p_1| - \mu + \nu_5 + \nu + \frac{\Delta^2 + M^2}{2|p_1|} + \mathcal{O}(1/p_1^2), \\
p_{03} &= -|p_1| - \mu - |\nu_5 - \nu| - \frac{\Delta^2 + M^2}{2|p_1|} + \mathcal{O}(1/p_1^2), \\
p_{04} &= |p_1| - \mu - \nu_5 - \nu + \frac{\Delta^2 + M^2}{2|p_1|} + \mathcal{O}(1/p_1^2).
\end{aligned} \tag{A11}$$

Finally, it follows from Eq. (A11) that at $p_1 \rightarrow \pm\infty$

$$\begin{aligned}
&|p_{01}| + |p_{02}| + |p_{03}| + |p_{04}| \\
&= 4|p_1| + \frac{2(\Delta^2 + M^2)}{|p_1|} + \mathcal{O}(1/p_1^2).
\end{aligned} \tag{A12}$$

For the purposes of the renormalization of the TDP (25), it is very important that the leading terms of this asymptotic behavior do not depend on different chemical potentials, i.e. the quantity $\sum_{i=1}^4 |p_{0i}|$ at $\mu = \nu = \nu_5 = 0$ has the same

asymptotic expansion (A12). Furthermore, there is an exact expression for this sum,

$$(|p_{01}| + |p_{02}| + |p_{03}| + |p_{04}|)|_{\mu=\nu=\nu_5=0} = 4\sqrt{M^2 + \Delta^2 + p_1^2}. \quad (\text{A13})$$

We would like to emphasize once again that the asymptotic behavior (A12) does not depend on which of the roots R_1 , R_2 or R_3 of Eq. (A5) is taken as the quantity R in the relations (A4). Finally, it is important to note that the quantity $\sum_{i=1}^4 |p_{0i}|$ is not an even function with respect to the momentum p_1 . [This statement is supported, in particular, by the relation (B1) of Appendix B.] Indeed, the

polynomial (A1) is not invariant under the transformation $p_1 \rightarrow -p_1$ (since its coefficient b changes). So each root η_i and the sum $\sum_{i=1}^4 |p_{0i}|$ are changed under this transformation. (In contrast, if $\nu = 0$ or $\nu_5 = 0$ then this sum is an even function of p_1 .) Unfortunately, the opposite, incorrect assertion was made in Ref. [50], which led to an incorrect expression for the TDPs $F_1(M)$ and $F_2(\Delta)$.

APPENDIX B: DERIVATION OF THE RELATION (36)

If $\Delta = 0$ and $M \neq 0$, then the quasiparticle energies p_{0i} are presented in Eq. (22). So

$$\begin{aligned} & (|p_{01}| + |p_{02}| + |p_{03}| + |p_{04}|)|_{\Delta=0} \\ &= \sum_{\kappa=\pm} \left(\left| -\mu + \kappa\nu + \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right| + \left| -\mu + \kappa\nu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right| \right) \\ &= 2 \sum_{\kappa=\pm} \left\{ \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} + \left(\mu - \kappa\nu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \theta \left(\mu - \kappa\nu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \right. \\ & \quad \left. + \left(\kappa\nu - \mu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \theta \left(\kappa\nu - \mu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \right\}, \end{aligned} \quad (\text{B1})$$

where we have taken into account the well-known relations $|x| = x\theta(x) - x\theta(-x)$ and $\theta(x) = 1 - \theta(-x)$. Hence, the expression (35) at $\Delta = 0$ and $M \neq 0$ can be presented in the following form:

$$F_1(M) \equiv \Omega^{\text{ren}}(M, \Delta = 0) = -\frac{M^2}{2\pi} + \frac{M^2}{2\pi} \ln \left(\frac{M^2}{m^2} \right) - U - V, \quad (\text{B2})$$

where

$$U = \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left\{ \sqrt{M^2 + (p_1 + \nu_5)^2} + \sqrt{M^2 + (p_1 - \nu_5)^2} - 2\sqrt{M^2 + p_1^2} \right\} = \frac{\nu_5^2}{\pi}, \quad (\text{B3})$$

$$\begin{aligned} V &= \sum_{\kappa=\pm} \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left\{ \left(\mu - \kappa\nu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \theta \left(\mu - \kappa\nu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \right. \\ & \quad \left. + \left(\kappa\nu - \mu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \theta \left(\kappa\nu - \mu - \sqrt{M^2 + (p_1 + \kappa\nu_5)^2} \right) \right\} \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left(\mu - \nu - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \theta \left(\mu - \nu - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \\ & \quad + \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left(\nu - \mu - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \theta \left(\nu - \mu - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \\ & \quad + \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left(\mu + \nu - \sqrt{M^2 + (p_1 - \nu_5)^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + (p_1 - \nu_5)^2} \right). \end{aligned} \quad (\text{B5})$$

Notice that a calculation of the convergent improper integral U [Eq. (B3)] can be found, e.g., in Appendix C of Ref. [16]. Moreover, when summing in Eq. (B4) over $\kappa = \pm$, we have taken into account that $\mu \geq 0$ and $\nu \geq 0$. So there are only three integrals in the expression (B5). Due to the presence of the step function $\theta(x)$, each integral in Eq. (B5) is indeed a proper one. Let us denote the sum of the first two integrals of Eq. (B5) as V_1 and the last integral as V_2 , i.e. $V = V_1 + V_2$. Then, it is evident that

$$V_1 = \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left(|\mu - \nu| - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \theta \left(|\mu - \nu| - \sqrt{M^2 + (p_1 + \nu_5)^2} \right), \quad (\text{B6})$$

$$V_2 = \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left(\mu + \nu - \sqrt{M^2 + (p_1 - \nu_5)^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + (p_1 - \nu_5)^2} \right). \quad (\text{B7})$$

In these relations the region of integration over $p_1 \in (-\infty, +\infty)$ can be transformed to the region $p_1 \in (0, +\infty)$.

$$\begin{aligned} V_1 = & \int_0^{\infty} \frac{dp_1}{2\pi} \left(|\nu - \mu| - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \theta \left(|\nu - \mu| - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \\ & + \int_0^{\infty} \frac{dp_1}{2\pi} \left(|\nu - \mu| - \sqrt{M^2 + (p_1 - \nu_5)^2} \right) \theta \left(|\nu - \mu| - \sqrt{M^2 + (p_1 - \nu_5)^2} \right), \end{aligned} \quad (\text{B8})$$

$$\begin{aligned} V_2 = & \int_0^{\infty} \frac{dp_1}{2\pi} \left(\mu + \nu - \sqrt{M^2 + (p_1 - \nu_5)^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + (p_1 - \nu_5)^2} \right) \\ & + \int_0^{\infty} \frac{dp_1}{2\pi} \left(\mu + \nu - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + (p_1 + \nu_5)^2} \right) \equiv V_{2-} + V_{2+}, \end{aligned} \quad (\text{B9})$$

where V_{2-} and V_{2+} denote the first and the second integral of Eq. (B9), respectively. Carrying out in the integrals of Eq. (B9) the change of variables, $q = p_1 + \nu_5$ for V_{2+} and $q = p_1 - \nu_5$ for V_{2-} , respectively, we have

$$\begin{aligned} V_{2+} = & \int_{\nu_5}^{\infty} \frac{dq}{2\pi} \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \\ = & \left(\int_0^{\infty} - \int_0^{\nu_5} \right) \frac{dq}{2\pi} \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + q^2} \right), \end{aligned} \quad (\text{B10})$$

$$\begin{aligned} V_{2-} = & \int_{-\nu_5}^{\infty} \frac{dq}{2\pi} \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \\ = & \left(\int_0^{\infty} + \int_0^{\nu_5} \right) \frac{dq}{2\pi} \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + q^2} \right). \end{aligned} \quad (\text{B11})$$

Hence,

$$\begin{aligned} V_2 \equiv V_{2+} + V_{2-} = & \int_0^{\infty} \frac{dq}{\pi} \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \theta \left(\mu + \nu - \sqrt{M^2 + q^2} \right) \\ = & \frac{\theta(\mu + \nu - M)}{2\pi} \left((\mu + \nu) \sqrt{(\mu + \nu)^2 - M^2} - M^2 \ln \frac{\mu + \nu + \sqrt{(\mu + \nu)^2 - M^2}}{M} \right). \end{aligned} \quad (\text{B12})$$

In a similar way it is possible to show that

$$\begin{aligned} V_1 = & \int_0^{\infty} \frac{dq}{\pi} \left(|\mu - \nu| - \sqrt{M^2 + q^2} \right) \theta \left(|\mu - \nu| - \sqrt{M^2 + q^2} \right) \\ = & \frac{\theta(|\mu - \nu| - M)}{2\pi} \left(|\mu - \nu| \sqrt{(\mu - \nu)^2 - M^2} - M^2 \ln \frac{|\mu - \nu| + \sqrt{(\mu - \nu)^2 - M^2}}{M} \right). \end{aligned} \quad (\text{B13})$$

Since $V = V_1 + V_2$ and U is given in Eq. (B3), we have for the TDP $F_1(M)$ [Eq. (B2)] the expression (36).

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