

Exact treatment of dispersion relations in pp and $p\bar{p}$ elastic scattering

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Proofs are given of exact analytical solutions for general principal value integrations of the dispersion relations of pp and $p\bar{p}$ scattering amplitudes. The proofs are based on recent developments in the study of properties of the Lerch's transcendent of the mathematical literature. Dispersion relations for the slopes of the amplitudes are mathematically defined and also solved exactly. The results are explicitly expanded, providing an important basis for improvement in the phenomenology of the hadronic interactions.

DOI: [10.1103/PhysRevC.97.014003](https://doi.org/10.1103/PhysRevC.97.014003)**I. INTRODUCTION**

Elastic scattering in the pp and $p\bar{p}$ systems is analytically very simple: spin effects neglected, it is described by a single complex function of two variables (s, t). This formal simplicity hides the fact that elastic scattering is a coherent nonperturbative process involving complicated dynamics. After four decades of studies in the framework of the modern theory of the strong interactions (QCD), still no fundamental microscopic model is successful in the description of the imaginary and real parts of the complex amplitude. Despite the essentially complex fundamental dynamics, the experimental data and the observables on the differential elastic cross sections show simple and regular dependencies on the energy s and on the transferred momentum t . It seems that the global simple behavior is actually a consequence of the extreme internal complexity.

To build a bridge between data and microscopic models, the differential cross sections in the forward range can be represented in terms of a few parameters, with precision and coherence. We present in this paper a formulation, based on principles of dispersion relations (DR), that is appropriate for the analysis of $d\sigma/dt$ data. Our treatment uses two important developments. One is the discovery of the exact solution of the principal value (PV) integrals that occur in dispersion relations, based on our recent work on properties of the mathematical function called Lerch's transcendent [1]. The second is the application of the dispersion relations for slopes (DRS) [2], based on the knowledge of the s dependence of the slopes of the imaginary parts of the amplitudes. The real and imaginary

parts of the amplitudes in the forward direction are treated as independent functions, as required by quantum mechanics, with connections determined by the causality and analyticity foundations of the dispersion relation. The usual assumption of equal real and imaginary slopes is here forbidden.

In the best explored forward region the analysis of pp and $p\bar{p}$ scattering is affected by the comparatively small magnitude of the real amplitude (usually expressed through the parameter ρ that gives the ratio of the real to the imaginary part at $t = 0$), whose sign and strength must be studied through interference with the Coulomb interaction. The extraction of precise information on the real part is very difficult, with consequences on the determination of the optical point in the imaginary part that leads to the total cross sections. Under these conditions, it is essential to use the full potentiality of theoretical controls, such as DRs and DRS.

The original forms of Cauchy PV integrals occurring in DRs are not very practical in calculations. Local forms, called derivative dispersion relations (DDR), are more comfortable and have been used in the analysis of the data. After a period in which the knowledge of DDRs was limited to approximations not valid for low energies, the connection between integral and local forms has been put in exact terms [3–5], giving mathematically correct relations between real and imaginary parts of the complex amplitude. These local forms consist of double [3] and single [4,5] infinite series of fast convergence in the applications.

In the present work we introduce new results [1] for the exact DR forms, written in terms of the function called Lerch's transcendent. From now on, terms of the input imaginary amplitudes used in pp and $p\bar{p}$ phenomenology have their real counterparts written in compact analytical expressions. The new expressions for the exact forms of DRs have their properties discussed and are used to draw consequences of the input form of the imaginary amplitude (namely of the total cross section) proposed by the Particle Data Group/Compete Collaboration [6] (PDG). Because we write exact forms, we call attention to the importance of the influence of the subtraction constant K , which cannot be ignored at low energies.

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We go one step further and explore, again in exact terms, the idea of the DRS [2] that was introduced in the year 2007 and shown to be important for the analysis of pp and $p\bar{p}$ scattering data. It is understood that the imaginary amplitudes for small $|t|$ have the exponential forms $\exp(B_I(s)t/2)$ (with different slopes for pp and $p\bar{p}$). Derivation of the original DR forms with respect to t leads to new relations. With given energy dependence assumed for the B_I slopes, explicit PV calculations can be performed, leading to predictions for the derivatives of the real parts at $t = 0$. If the energy dependence of the imaginary slopes is constructed with a combination of power and logarithm terms, similar to the PDG forms for the total cross sections, the PV integrals are also solved in terms of the Lerch's transcendent. We thus arrive at analytical exact forms for the derivatives of the real amplitudes in the forward direction. This has enormous importance for the phenomenological treatment of pp and $p\bar{p}$ scattering.

DRS can be used to investigate the structure of the real amplitudes in the forward range. In accord with a theorem by Martin [7], at high energies the existence of a zero that approaches the origin as the energy increases is observed [8–10]. This requires more than an exponential slope factor in the real part. For example, assuming the t dependence of the real amplitude with an exponential ($B_R t/2$) times a factor linear in the t variable, we may have the expected zero. DRS predict a relation among the parameters, thus providing an important theoretical control in the analysis of $d\sigma/dt$ data.

We stress that we are here limited to the short-range strong interactions. Coulomb interference is most important in the phenomenology of pp and $p\bar{p}$ scattering and must be properly taken into account, but it is not included in the concerns of the present work.

This paper is organized as follows. In Sec. II we review the connections of imaginary and real parts of the amplitudes as given by general principles and write the forms of the dispersion relations for amplitudes and of the dispersion relations for slopes in terms of principal value integrals of general forms. In the Secs. II A and II B, the expressions for DRs and DRS are fully expanded, using the given inputs. In Sec. III we present the proof of the analytical solution of the principal value integrations in terms of elementary functions and Lerch's transcendents. General properties are described, and explicit forms written for cases of practical occurrence. Cases of apparent singularities are analyzed, and their cancellations explained and explicitly exhibited. In Sec. IV, with subsections for amplitudes and for derivatives, we give explicit expressions for the calculation of the real amplitude and of its derivative at $t = 0$ in terms of the input parameters of the total cross sections and of the imaginary slopes. In Sec. V the connection of the mathematical results and the phenomenology of pp and $p\bar{p}$ scattering is illustrated. In Sec. VI we list the purposes and achievements of the present work.

II. DISPERSION RELATIONS FOR AMPLITUDES AND SLOPES

The well-known DRs for pp and $p\bar{p}$ elastic scattering are written in terms of even and odd dimensionless amplitudes:

$$\text{Re}F_+(E,t) = K + \frac{2E^2}{\pi} \mathbf{P} \int_m^{+\infty} dE' \frac{\text{Im}F_+(E',t)}{E'(E'^2 - E^2)}, \quad (1)$$

$$\text{Re}F_-(E,t) = \frac{2E}{\pi} \mathbf{P} \int_m^{+\infty} dE' \frac{\text{Im}F_-(E',t)}{(E'^2 - E^2)}. \quad (2)$$

Here E is the incident proton energy in laboratory system. The subtraction constant K accounts for the convergence control in the one-subtracted DR.

In high-energy processes the center-of-mass energy \sqrt{s} is most commonly used. For pp and $p\bar{p}$ scattering the connection with the laboratory energy E is

$$s = 2mE + 2m^2, \quad (3)$$

where m is the proton/antiproton mass. To work with the dispersion relations written above, the most useful quantity is the dimensionless ratio

$$x = E/m \quad (4)$$

and then

$$\frac{s}{2m^2} = x + 1. \quad (5)$$

Approximate relations that are often used at high energies are obviously $s = 2mE$ and $x = s/2m^2$.

The optical theorem informs the normalization of the amplitudes by

$$\sigma_{pp} = \frac{\text{Im}F_{pp}(x,t=0)}{2m^2x} \quad (6)$$

and similarly for $p\bar{p}$.

The even and odd combinations of amplitudes are related to the pp and $p\bar{p}$ systems through

$$F_{pp}(x,t) = F_+(x,t) - F_-(x,t), \quad (7)$$

$$F_{p\bar{p}}(x,t) = F_+(x,t) + F_-(x,t). \quad (8)$$

Assuming for small $|t|$ exponential t dependencies for the imaginary parts of the amplitudes, we write

$$\text{Im}F_{pp}(x,t) = 2m^2x\sigma_{pp}(x)\exp(B_I^{pp}(x)t/2), \quad (9)$$

$$\text{Im}F_{p\bar{p}}(x,t) = 2m^2x\sigma_{p\bar{p}}(x)\exp(B_I^{p\bar{p}}(x)t/2), \quad (10)$$

with input functions $\sigma(x)$ and $B_I(x)$. We obtain in this way for the even and odd inputs

$$\text{Im}F_+(x,t) = m^2x[\sigma_{p\bar{p}}(x)e^{B_I^{p\bar{p}}(x)t/2} + \sigma_{pp}(x)e^{B_I^{pp}(x)t/2}], \quad (11)$$

$$\text{Im}F_-(x,t) = m^2x[\sigma_{p\bar{p}}(x)e^{B_I^{p\bar{p}}(x)t/2} - \sigma_{pp}(x)e^{B_I^{pp}(x)t/2}]. \quad (12)$$

Substituting these expressions in Eqs. (1) and (2), written in terms of the dimensionless variable x , we obtain

$$\text{Re}F_+(x,t) = K + \frac{2m^2x^2}{\pi} \mathbf{P} \int_1^{+\infty} \frac{1}{x'^2 - x^2} \{ \sigma_{p\bar{p}}(x') \exp[B_I^{p\bar{p}}(x')t/2] + \sigma_{pp}(x') \exp[B_I^{pp}(x')t/2] \} dx', \quad (13)$$

$$\text{Re}F_-(x,t) = \frac{2m^2x}{\pi} \mathbf{P} \int_1^{+\infty} \frac{x'}{x'^2 - x^2} \{ \sigma_{p\bar{p}}(x') \exp[B_I^{p\bar{p}}(x')t/2] - \sigma_{pp}(x') \exp[B_I^{pp}(x')t/2] \} dx'. \quad (14)$$

The Particle Data Group [6] gives parametrizations for the total cross sections of the pp and $p\bar{p}$ interactions in the well-known forms

$$\sigma^\mp(s) = P' + H' \log^2(s/s_0) + R'_1(s/s_0)^{-\eta'_1} \pm R'_2(s/s_0)^{-\eta'_2}, \quad (15)$$

with parameters P' , H' , R'_1 , and R'_2 in millibarns, s_0 in GeV^2 , and η'_1 and η'_2 dimensionless. The upper and lower indices $-$ and $+$ refer to $p\bar{p}$ and pp scattering, respectively. The parametrization is assumed to be adequate for all energies $s \geq s_0$.

However, dispersion relations are defined with respect to the laboratory system energy, and, for low energies, terms like $\log^2(E+m)$ and $(E+m)^{-\eta}$ appear and spoil the simplicity of DRs preventing one from obtaining closed forms. We then rewrite (reparametrize) the above values for the total cross sections in terms of the dimensionless variables $x = E/m$ and $x_0 = E_0/m$, with $x > 1$, writing

$$\sigma_{pp}(x) = P + H \log^2(x/x_0) + R_1(x/x_0)^{-\eta_1} - R_2(x/x_0)^{-\eta_2}, \quad (16)$$

$$\sigma_{p\bar{p}}(x) = P + H \log^2(x/x_0) + R_1(x/x_0)^{-\eta_1} + R_2(x/x_0)^{-\eta_2}, \quad (17)$$

and we obtain new parameters, with slight changes. Numerical values are given in Sec. V. For mathematical simplicity, from now on we use in this paper the variable x to represent the energy of the collision, with the use of the center-of-mass energy \sqrt{s} in some places.

In terms of the x variable, the slopes $B_I^{pp}(x)$ and $B_I^{p\bar{p}}(x)$ are written in the following Regge-like forms:

$$B_I^{pp}(x) = b_0 + b_1 \log x + b_2 \log^2 x + b_3 x^{-\eta_3} - b_4 x^{-\eta_4}, \quad (18)$$

$$B_I^{p\bar{p}}(x) = b_0 + b_1 \log x + b_2 \log^2 x + b_3 x^{-\eta_3} + b_4 x^{-\eta_4}, \quad (19)$$

with symmetry in the coefficients for pp and $p\bar{p}$. The suggested numerical values are given in Sec. V.

The even and odd inputs are given by Eqs. (11) and (12). Then the DR for the PDG forms, Eqs. (16) and (17), become

$$\begin{aligned} \text{Re}F_+(x,t) = & K + \frac{2m^2x^2}{\pi} \mathbf{P} \int_1^{+\infty} \left\{ \frac{P + H \log^2(x'/x_0) + R_1(x'/x_0)^{-\eta_1}}{x'^2 - x^2} (e^{B_I^{p\bar{p}}(x')t/2} + e^{B_I^{pp}(x')t/2}) \right. \\ & \left. + \frac{R_2(x'/x_0)^{-\eta_2}}{x'^2 - x^2} (e^{B_I^{p\bar{p}}(x')t/2} - e^{B_I^{pp}(x')t/2}) \right\} dx', \end{aligned} \quad (20)$$

$$\begin{aligned} \text{Re}F_-(x,t) = & \frac{2m^2x}{\pi} \mathbf{P} \int_1^{+\infty} x' \left\{ \frac{P + H \log^2(x'/x_0) + R_1(x'/x_0)^{-\eta_1}}{x'^2 - x^2} (e^{B_I^{p\bar{p}}(x')t/2} - e^{B_I^{pp}(x')t/2}) \right. \\ & \left. + \frac{R_2(x'/x_0)^{-\eta_2}}{x'^2 - x^2} (e^{B_I^{p\bar{p}}(x')t/2} + e^{B_I^{pp}(x')t/2}) \right\} dx'. \end{aligned} \quad (21)$$

A. Dispersion relations for amplitudes

Taking $t = 0$ in Eqs. (13) and (14) we have

$$\text{Re}F_+(x,0) = K + \frac{4m^2x^2}{\pi} \mathbf{P} \int_1^{+\infty} \frac{P + H \log^2(x'/x_0) + R_1(x'/x_0)^{-\eta_1}}{x'^2 - x^2} dx', \quad (22)$$

$$\text{Re}F_-(x,0) = \frac{4m^2x}{\pi} \mathbf{P} \int_1^{+\infty} \frac{x' R_2(x'/x_0)^{-\eta_2}}{x'^2 - x^2} dx'. \quad (23)$$

In a glance at the integrands in these equations, we observe that both $\text{Re}F_+(x, 0)$ and $\text{Re}F_-(x, 0)$ result in linear combinations of PV integrals of the form

$$I(n, \lambda, x) = \mathbf{P} \int_1^{+\infty} \frac{x'^{\lambda} \log^n(x')}{x'^2 - x^2} dx'. \quad (24)$$

Exact values of these integrals, based on recent developments in our study of the Lerch's transcendent [1], can be given.

Collecting terms, we have the following for the even and odd parts:

$$\text{Re}F_+(x, 0) = K + \frac{4m^2 x^2}{\pi} \left[I(0, 0, x)(P + H \log^2 x_0) + I(1, 0, x)(-2H \log x_0) + I(2, 0, x)H + I(0, -\eta_1, x)R_1 x_0^{\eta_1} \right], \quad (25)$$

$$\text{Re}F_-(x, 0) = \frac{4m^2 x}{\pi} I(0, 1 - \eta_2, x)R_2 x_0^{\eta_2}. \quad (26)$$

Equations (25) and (26) are known DRs relating imaginary and real parts of the complex amplitude for pp and $p\bar{p}$ elastic scattering.

B. Dispersion relations for slopes

For small $|t|$ we extend the imaginary amplitude of the PDG representation introducing factors $\exp[B_I^{pp}(x)t/2]$ and $\exp[B_I^{p\bar{p}}(x)t/2]$, for all terms in the input form, as written above in Eqs. (9) and (10). The parametrizations of the imaginary slopes as functions of the energy allow us to obtain, from the dispersion relations, information on the derivatives of the real parts at $|t| = 0$. This has essential importance for the construction of the forward real amplitude, with determination of the forward scattering parameters.

Taking derivatives of Eqs. (13) and (14) with respect to t , we obtain

$$\frac{\partial \text{Re}F_+(x, t)}{\partial t} = \frac{m^2 x^2}{\pi} \mathbf{P} \int_1^{+\infty} \frac{1}{x'^2 - x^2} \left\{ \sigma_{p\bar{p}}(x') B_I^{p\bar{p}}(x') \exp[B_I^{p\bar{p}}(x')t/2] + \sigma_{pp}(x') B_I^{pp}(x') \exp[B_I^{pp}(x')t/2] \right\} dx', \quad (27)$$

$$\frac{\partial \text{Re}F_-(x, t)}{\partial t} = \frac{m^2 x}{\pi} \mathbf{P} \int_1^{+\infty} \frac{x'}{x'^2 - x^2} \left\{ \sigma_{p\bar{p}}(x') B_I^{p\bar{p}}(x') \exp[B_I^{p\bar{p}}(x')t/2] - \sigma_{pp}(x') B_I^{pp}(x') \exp[B_I^{pp}(x')t/2] \right\} dx', \quad (28)$$

which, with the parametrizations Eqs. (16) to (19), give the following for the dispersion relations for the derivatives [2] of the real amplitudes at the origin:

$$\left. \frac{\partial \text{Re}F_+(x, t)}{\partial t} \right|_{t=0} = \frac{2m^2 x^2}{\pi} \mathbf{P} \int_1^{+\infty} \left\{ \frac{P + H \log^2(x'/x_0) + R_1(x'/x_0)^{-\eta_1}}{x'^2 - x^2} [b_0 + b_1 \log(x') + b_2 \log^2(x') + b_3 x'^{-\eta_3}] \right. \\ \left. + \frac{R_2(x'/x_0)^{-\eta_2}}{x'^2 - x^2} b_4 x'^{-\eta_4} \right\} dx', \quad (29)$$

$$\left. \frac{\partial \text{Re}F_-(x, t)}{\partial t} \right|_{t=0} = \frac{2m^2 x}{\pi} \mathbf{P} \int_1^{+\infty} x' \left\{ \frac{P + H \log^2(x'/x_0) + R_1(x'/x_0)^{-\eta_1}}{x'^2 - x^2} b_4 x'^{-\eta_4} \right. \\ \left. + \frac{R_2(x'/x_0)^{-\eta_2}}{x'^2 - x^2} [b_0 + b_1 \log(x') + b_2 \log^2(x') + b_3 x'^{-\eta_3}] \right\} dx'. \quad (30)$$

The right-hand sides of these equations are also linear combinations of PV integrals of the form $I(n, \lambda, x)$ defined in Eq. (24). Collecting terms in these equations, we obtain the following for the even and odd parts:

$$\left. \frac{\partial \text{Re}F_+(x, t)}{\partial t} \right|_{t=0} = \frac{2m^2 x^2}{\pi} \left\{ I(0, 0, x)(P + H \log^2 x_0)b_0 + I(1, 0, x)[(-2H \log x_0)b_0 + (P + H \log^2 x_0)b_1] \right. \\ + I(2, 0, x)[Hb_0 - 2H \log x_0 b_1 + (P + H \log^2 x_0)b_2] + I(3, 0, x)[-2H \log x_0 b_2 + Hb_1] \\ + I(4, 0, x)Hb_2 + R_1 x_0^{\eta_1} [I(0, -\eta_1, x)b_0 + I(1, -\eta_1, x)b_1 + I(2, -\eta_1, x)b_2 + I(0, -\eta_1 - \eta_3, x)b_3] \\ + R_2 x_0^{\eta_2} I(0, -\eta_2 - \eta_4, x)b_4 + [(P + H \log^2 x_0)I(0, -\eta_3, x) - 2H \log x_0 I(1, -\eta_3, x) \\ \left. + HI(2, -\eta_3, x)]b_3 \right\}, \quad (31)$$

$$\left. \frac{\partial \text{Re}F_-(x, t)}{\partial t} \right|_{t=0} = \frac{2m^2 x}{\pi} \left\{ R_2 x_0^{\eta_2} [I(0, 1 - \eta_2, x)b_0 + I(1, 1 - \eta_2, x)b_1 + I(2, 1 - \eta_2, x)b_2 + I(0, 1 - \eta_2 - \eta_3, x)b_3] \right. \\ + [(P + H \log^2 x_0)I(0, 1 - \eta_4, x) - 2H \log x_0 I(1, 1 - \eta_4, x) + HI(2, 1 - \eta_4, x) \\ \left. + R_1 x_0^{\eta_1} I(0, 1 - \eta_1 - \eta_4, x)]b_4 \right\}. \quad (32)$$

These equations, here called dispersion relations for slopes, control quantities observed in forward scattering and should be used as basic information for phenomenological and theoretical description of forward pp and $p\bar{p}$ scattering. In their introduction [2], they were shown to be important for the analysis of forward scattering, determining structure and parameters of the real amplitude.

The next section shows our procedure for calculating the PV integrals of the type $I(n, \lambda, x)$ entering Eqs. (25), (26), (31), and (32).

III. PRINCIPAL VALUE INTEGRALS AND LERCH'S TRANSCENDENT

The PV integrals have always had a protagonist role in the application of the principles of DRs to pp and $p\bar{p}$ scattering, requiring much work and concern. Integrands with singularities were not easy to deal with, especially before powerful numerical computation methods became accessible. Dynamical details, such as resonances and thresholds, were mixed up in the efforts to obtain the real part of the amplitude. It was not easy to separate the physical details from the mathematical difficulties.

Nowadays we can show that in high-energy pp and $p\bar{p}$ scattering, analytically very simple inputs, treated with direct exact mathematics, can account for all observation, with high precision. The simplified mathematics helps us to show that the phenomenology of the physical structure is also very simple. Existing physical complications are not visible in the analysis of the data, within the existing experimental precision.

First, the input form of the imaginary amplitude, containing only powers and logarithms, was treated by the DDR, avoiding the need for numerical integrations through singularities. These relations allowed the identification of some global properties of the observable quantities. The limitations of the first DDR forms, restricted to high energies, were solved with exact expressions [3], using double series. The difficulties with proofs on convergence of double infinite series required numerical proofs comparing principal value integrations with series summations. Further progress came with the reduction of the representations to single infinite series, of proved convergence [4]. The convergence of the series is both mathematically sure, and fast and comfortable in practice. Later, another construction of the representation of the PV integrals was introduced [5], identifying the treatment and cure of singularities that occur for certain values of parameters, showing clearly their cancellation. Very recently, further formal progress came with the study of properties of the Lerch's transcendent. The proof given in Ref. [1] for a new representation of the so-called Lerch's transcendent allows one to express the PV integrals of hadronic phenomenology in compact and closed form, in terms of these well-studied functions of the mathematical literature. The theorem is reproduced below, together with the proof that it contains the method to write the solutions of the PV that appear in DRs and DRS.

The Lerch's transcendent $\Phi(z, s, a)$ [11, Chap. 25, Sec. 25.14], also called the Hurwitz-Lerch ζ function, is

defined by its series representation

$$\Phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(a+m)^s}, \quad (33)$$

with

$$a \neq 0, -1, -2, \dots; \quad |z| < 1; \quad |z| = 1; \quad \text{Re}(s) > 1.$$

The restriction on the values of a guarantees that all terms of the series in the right-hand side are finite. Obviously, the series is convergent if $|z| < 1$, independently of the value of s , or if $|z| = 1$ and $\text{Re}(s) > 1$. For other values of its arguments, $\Phi(z, s, a)$ is defined by analytic continuation, which is achieved by means of integral representations. Characteristics of the Φ function are the identities

$$\Phi(z, s, a+1) = \frac{1}{z} \left(\Phi(z, s, a) - \frac{1}{a^s} \right), \quad (34)$$

$$\Phi(z, s-1, a) = \left(a + z \frac{\partial}{\partial z} \right) \Phi(z, s, a), \quad (35)$$

$$\Phi(z, s+1, a) = -\frac{1}{s} \frac{\partial}{\partial a} \Phi(z, s, a), \quad (36)$$

stemming from the series representation in Eq. (33).

A new representation of $\Phi(z, n, a)$ that establishes its connection with the PV integrals of the theory of DRs was recently proved [1] with the following theorem.

Theorem. Let z be a complex number belonging to the open disk of radius 1, excluding its center at the origin, and cut along the negative real semiaxis, that is,

$$z \in \mathcal{C}, \quad 0 < |z| < 1, \quad -\pi < \arg(z) < \pi. \quad (37)$$

Let us denote

$$\varphi = \arg(-\log z). \quad (38)$$

Then, for positive integer values of $n = 1, 2, \dots$ and for complex a such that $\text{Re}[(a-1)e^{i\varphi}] < 0$, the Lerch's transcendent admits the representation

$$\begin{aligned} \Phi(z, n, a) = & \frac{(-1)^{n-1}}{(n-1)!} \left\{ \mathbf{P} \int_0^{\infty e^{i\varphi}} \frac{t^{n-1} e^{at}}{z e^t - 1} dt \right. \\ & \left. + \pi \frac{\partial^{n-1}}{\partial a^{n-1}} [z^{-a} \cot(\pi a)] \right\}, \quad (39) \end{aligned}$$

where the symbol \mathbf{P} stands for the Cauchy principal value of the path integral along the ray $\arg(t) = \varphi$.

With specifications for particular cases, the theorem is applied to obtain the expressions for the general PV integrals in Eq. (24) that we need. Assuming that z and a are real, one has $\varphi = 0$. Changing the integration variable with $t = 2 \log(x')$, we may write

$$\begin{aligned} \Phi(z, n+1, a) = & \frac{(-1)^n}{(n)!} \left[\mathbf{P} \int_1^{+\infty} x'^{2(a-1)} \frac{2^{(n+1)} \log^n(x')}{z x'^2 - 1} dx' \right. \\ & \left. + \pi \frac{\partial^n}{\partial a^n} [z^{-a} \cot(\pi a)] \right]. \quad (40) \end{aligned}$$

Putting $z = 1/x^2$ and $a = (1 + \lambda)/2$ we finally obtain the general form for the PV integrals, defined by Eq. (24),

$$I(n, \lambda, x) = \frac{\pi}{2x} \frac{\partial^n}{\partial \lambda^n} \left[x^\lambda \tan \left(\frac{\pi \lambda}{2} \right) \right] + \frac{(-1)^n n!}{2^{n+1} x^2} \Phi \left(\frac{1}{x^2}, n+1, \frac{1+\lambda}{2} \right). \quad (41)$$

Thus, using the recent developments [1], we have obtained general exact forms of the PV integrals $I(n, \lambda, x)$ with non-negative integer n , complex λ such that $\text{Re } \lambda < 1$, and real $x > 1$. Under these conditions, the function Φ provides elegant exact representations for the PV integrals of the DRs. Equation (33) allows us to write for the function Φ in the right-hand side of Eq. (41) the expansion

$$\Phi \left(\frac{1}{x^2}, n+1, \frac{1+\lambda}{2} \right) = 2^{n+1} \sum_{j=0}^{\infty} \frac{x^{-2j}}{(2j+1+\lambda)^{n+1}}. \quad (42)$$

For its derivative with respect to λ , use can be made of Eq. (36) to write

$$\frac{\partial}{\partial \lambda} \Phi \left(\frac{1}{x^2}, n+1, \frac{1+\lambda}{2} \right) = -\frac{n+1}{2} \Phi \left(\frac{1}{x^2}, n+2, \frac{1+\lambda}{2} \right). \quad (43)$$

In the applications to pp and $p\bar{p}$ scattering, the sums converge rapidly and are easily included in practical computations, requiring only one or a few terms of the series.

We remark that $I(n=0, \lambda=0, x)$ can be written in terms of elementary functions:

$$I(0, 0, x) = \mathbf{P} \int_1^{+\infty} \frac{1}{x'^2 - x^2} dx' = \frac{1}{2x^2} \Phi \left(\frac{1}{x^2}, 1, \frac{1}{2} \right) = \frac{1}{2x} \log \frac{x+1}{x-1}.$$

We may write a simple combination that eliminates the denominator in the PV integrals to obtain

$$I(n, \lambda, x) - x^2 I(n, \lambda - 2, x) = \int_1^{+\infty} x'^{(\lambda-2)} \log^n(x') dx' = (1-\lambda)^{-1-n} \Gamma(1+n), \quad (44)$$

which does not depend on x . On the other hand, using the property of periodicity of the tangent function in Eq. (41), the combination eliminates the terms with derivatives and can also be written as

$$\begin{aligned} I(n, \lambda, x) - x^2 I(n, \lambda - 2, x) &= \frac{(-1)^n}{x^2} 2^{-(n+1)} \Gamma(1+n) \left[\Phi \left(\frac{1}{x^2}, n+1, \frac{1+\lambda}{2} \right) - x^2 \Phi \left(\frac{1}{x^2}, n+1, \frac{1+\lambda-2}{2} \right) \right] \\ &= (1-\lambda)^{-1-n} \Gamma(1+n), \end{aligned} \quad (45)$$

where in the last step use has been made of the property of the Lerch's transcendent in Eq. (34). This confirms the expression for the general formula in Eq. (41) for the PV integration. The combination free of derivatives can be used for noninteger n and can also be useful for computational purposes, because full calculations of integrals need to be made only for $\lambda \in (-1, 1)$.

Specific values for the PV integrals used in this work (values of $n = 0, 1, 2, 3, 4$) are given as follows:

$$I(0, \lambda, x) = \frac{\pi}{2} x^{\lambda-1} \tan \left(\frac{\pi \lambda}{2} \right) + \frac{1}{2x^2} \Phi \left(\frac{1}{x^2}, 1, \frac{1+\lambda}{2} \right), \quad (46)$$

$$I(1, \lambda, x) = \frac{\pi}{2} x^{\lambda-1} \left\{ \log(x) \tan \left(\frac{\pi \lambda}{2} \right) + \frac{\pi}{2} \sec^2 \left(\frac{\pi \lambda}{2} \right) \right\} - \frac{1}{4x^2} \Phi \left(\frac{1}{x^2}, 2, \frac{1+\lambda}{2} \right), \quad (47)$$

$$I(2, \lambda, x) = \frac{\pi}{2} x^{\lambda-1} \left\{ \log^2(x) \tan \left(\frac{\pi \lambda}{2} \right) + \pi \sec^2 \left(\frac{\pi \lambda}{2} \right) \left[\log(x) + \frac{\pi}{2} \tan \left(\frac{\pi \lambda}{2} \right) \right] \right\} + \frac{1}{4x^2} \Phi \left(\frac{1}{x^2}, 3, \frac{1+\lambda}{2} \right). \quad (48)$$

$$\begin{aligned} I(3, \lambda, x) &= \frac{\pi}{2} x^{\lambda-1} \left(\log^3(x) \tan \left(\frac{\pi \lambda}{2} \right) + \frac{\pi}{2} \sec^2 \left(\frac{\pi \lambda}{2} \right) \left\{ 3 \log^2(x) + 3\pi \log(x) \tan \left(\frac{\pi \lambda}{2} \right) + \frac{\pi^2}{2} \left[1 + 3 \tan^2 \left(\frac{\pi \lambda}{2} \right) \right] \right\} \right) \\ &\quad - \frac{3}{8x^2} \Phi \left(\frac{1}{x^2}, 4, \frac{1+\lambda}{2} \right), \end{aligned} \quad (49)$$

$$\begin{aligned} I(4, \lambda, x) &= \frac{\pi}{2} x^{\lambda-1} \left(\log^4(x) \tan \left(\frac{\pi \lambda}{2} \right) + \pi \sec^2 \left(\frac{\pi \lambda}{2} \right) \left\{ 2 \log^3(x) + 3\pi \log^2(x) \tan \left(\frac{\pi \lambda}{2} \right) + \pi^2 \log(x) \left[1 + 3 \tan^2 \left(\frac{\pi \lambda}{2} \right) \right] \right. \right. \\ &\quad \left. \left. + \frac{\pi^3}{2} \tan \left(\frac{\pi \lambda}{2} \right) \left[2 + 3 \tan^2 \left(\frac{\pi \lambda}{2} \right) \right] \right\} \right) + \frac{3}{4x^2} \Phi \left(\frac{1}{x^2}, 5, \frac{1+\lambda}{2} \right). \end{aligned} \quad (50)$$

The use of Eq. (41) is straightforward, except that care must be taken for odd negative integer values of $\lambda = -(2N+1)$, with N being zero or a positive integer, when singularities occur in both trigonometric and Φ function parts of the expression, with

cancellation in a limit procedure that has been explained before [5]. Examples of the calculation of the limits are given below:

$$I[0, -(2N+1), x] = x^{-2N-2} \left[-\log(x) - \frac{1}{2} \sum_{k=1}^N \frac{x^{2k}}{k} + \frac{1}{2} \text{Li}_1(x^{-2}) \right], \quad (51)$$

$$I[1, -(2N+1), x] = x^{-2N-2} \left[-\frac{1}{2} \log^2(x) + \frac{\pi^2}{12} - \frac{1}{4} \sum_{k=1}^N \frac{x^{2k}}{k^2} - \frac{1}{4} \text{Li}_2(x^{-2}) \right], \quad (52)$$

$$I[2, -(2N+1), x] = x^{-2N-2} \left[-\frac{1}{3} \log^3(x) + \frac{\pi^2}{6} \log(x) - \frac{1}{4} \sum_{k=1}^N \frac{x^{2k}}{k^3} + \frac{1}{4} \text{Li}_3(x^{-2}) \right], \quad (53)$$

$$I[3, -(2N+1), x] = x^{-2N-2} \left[-\frac{1}{4} \log^4(x) + \frac{\pi^2}{4} \log^2(x) + \frac{\pi^4}{120} - \frac{3}{8} \sum_{k=1}^N \frac{x^{2k}}{k^4} - \frac{3}{8} \text{Li}_4(x^{-2}) \right], \quad (54)$$

$$I[4, -(2N+1), x] = x^{-2N-2} \left[-\frac{1}{5} \log^5(x) + \frac{\pi^2}{3} \log^3(x) + \frac{\pi^4}{30} \log(x) - \frac{3}{4} \sum_{k=1}^N \frac{x^{2k}}{k^5} + \frac{3}{4} \text{Li}_5(x^{-2}) \right], \quad (55)$$

where Li_m represents the polylogarithm of order m . We recall that the polylogarithm functions that appear above are defined by the simple series

$$\text{Li}_m(x^{-2}) = \sum_{k=1}^{\infty} \frac{x^{-2k}}{k^m}. \quad (56)$$

IV. REAL PARTS OF AMPLITUDES AND DERIVATIVES FROM DR AND DRS

Based on the inputs of total cross sections $\sigma(x)$ for pp and $p\bar{p}$, DRs determine the real amplitudes at $t = 0$ as functions of the energy. The subtraction constant K is required (obtained from data), with a unique energy independent value. On the basis of the input of the imaginary slope $B_I(x)$ for pp and $p\bar{p}$, DRS determine the derivatives of the real amplitudes at $t = 0$ as functions of the energy.

With terms of the general form $x^\lambda \log^n(x)$ in the inputs, exact solutions are written for all PV integrals that appear in DRs and DRS. Thus exact forms, valid for all energies, are written for the real amplitudes and for their derivatives in the forward direction. Because it is known that the real part has important structure in the forward range, these results give essential contributions to the analysis of the dynamics governing elastic processes. Below we provide practical expressions for the results of DRs and DRS, keeping the dominant terms of the Lerch's transcendents. In Sec. V we illustrate the use of these results in the analysis of scattering data.

A. Real amplitudes at $t = 0$

For practical use, taking the low-energy corrections to first order, we write below the expressions for the $\rho\sigma$ products obtained with the PDG (imaginary amplitude) input. We have for the even part

$$\frac{1}{2}[(\sigma\rho)(p\bar{p}) + (\sigma\rho)(pp)] = \frac{1}{2m^2x} \text{Re}F_+(x, 0) = T_1(x) + T_2(x) + T_3(x), \quad (57)$$

with

$$T_1(x) = H\pi \log\left(\frac{x}{x_0}\right), \quad (58)$$

$$T_2(x) = \frac{K}{2m^2x} + \frac{2}{\pi x} (P + H[\log^2(x_0) + 2\log(x_0) + 2]), \quad (59)$$

$$T_3(x) = R_1 x_0^{\eta_1} \left[-x^{-\eta_1} \tan\left(\frac{\pi\eta_1}{2}\right) + \frac{2}{\pi x} \frac{1}{1-\eta_1} \right], \quad (60)$$

and for the odd part

$$\frac{1}{2}[(\sigma\rho)(p\bar{p}) - (\sigma\rho)(pp)] = \frac{1}{2m^2x} \text{Re}F_-(x, 0) = R_2 x_0^{\eta_2} \left[x^{-\eta_2} \cot\left(\frac{\pi\eta_2}{2}\right) + \frac{2}{\pi x} \frac{1}{2-\eta_2} \right]. \quad (61)$$

Additional terms are of the order $O(x^{-4})$.

B. Derivatives of real amplitudes at $t = 0$

We can learn more about the $|t|$ dependence of the amplitudes through the investigation of the DRs for slopes, DRS. For practical purposes we give below explicit expressions for the DRS including only the first term of the expansion of the transcendents. We have

$$\frac{1}{2m^2x} \left. \frac{\partial \text{Re} F_+(x,t)}{\partial t} \right|_{t=0} = \frac{1}{\pi} [(P + H \log^2 x_0) G_1(x) + H G_2(x) + R_1 G_3(x) + R_2 G_4(x)], \quad (62)$$

where

$$G_1(x) \equiv \frac{b_0 - b_1 + 2b_2}{x} + \frac{b_1 \pi^2}{4} + \frac{b_2 \pi^2}{2} \log x + b_3 \left[-\frac{\pi}{2} x^{-\eta_3} \tan\left(\frac{\pi \eta_3}{2}\right) + \frac{1}{x} \frac{1}{1 - \eta_3} \right], \quad (63)$$

$$\begin{aligned} G_2(x) \equiv & \left[\frac{\pi^2}{4} \left(3 \log^2 x + \frac{\pi^2}{2} \right) - \frac{6}{x} \right] (b_1 - 2b_2 \log x_0) - 2b_0 \log x_0 \left(\frac{\pi^2}{4} - \frac{1}{x} \right) + (b_0 - 2b_1 \log x_0) \left(\frac{\pi^2}{2} \log x + \frac{2}{x} \right) \\ & + b_2 \left[\pi^2 \log x \left(\log^2 x + \frac{\pi^2}{2} \right) + \frac{24}{x} \right] - \pi b_3 x^{-\eta_3} \left\{ \log x \tan\left(\frac{\pi \eta_3}{2}\right) \left(-\log x_0 + \frac{1}{2} \log x \right) \right. \\ & \left. - \frac{\pi}{2} \sec^2\left(\frac{\pi \eta_3}{2}\right) \left[\log\left(\frac{x}{x_0}\right) - \frac{\pi}{2} \tan\left(\frac{\pi \eta_3}{2}\right) \right] \right\} + \frac{2b_3}{x(1 - \eta_3)^2} \left(\log x_0 + \frac{1}{1 - \eta_3} \right), \end{aligned} \quad (64)$$

$$\begin{aligned} G_3(x) \equiv & x_0^{\eta_1} \left(b_0 \left[\frac{\pi}{2} x^{-\eta_1} \tan\left(-\frac{\pi \eta_1}{2}\right) + \frac{1}{x} \frac{1}{1 - \eta_1} \right] + b_1 \frac{\pi}{2} x^{-\eta_1} \left[\frac{\pi}{2} \sec^2\left(\frac{\pi \eta_1}{2}\right) - \tan\left(\frac{\pi \eta_1}{2}\right) \log x \right] \right. \\ & \left. - b_2 \frac{\pi^2}{2} x^{-\eta_1} \left\{ \sec^2\left(\frac{\pi \eta_1}{2}\right) \left[\frac{\pi}{2} \tan\left(\frac{\pi \eta_1}{2}\right) - \log x \right] + \frac{1}{\pi} \tan\left(\frac{\pi \eta_1}{2}\right) \log^2 x \right\} \right. \\ & \left. + b_3 \left[-\frac{\pi}{2} x^{-\eta_1 - \eta_3} \tan\left(\frac{\pi(\eta_1 + \eta_3)}{2}\right) + \frac{1}{x} \frac{1}{1 - \eta_1 - \eta_3} \right] + \frac{1}{x} \frac{1}{(1 - \eta_1)^2} \left(-b_1 + \frac{2b_2}{(1 - \eta_1)} \right) \right), \end{aligned} \quad (65)$$

$$G_4(x) \equiv x_0^{\eta_2} b_4 \left[-\frac{\pi}{2} x^{-\eta_2 - \eta_4} \tan\left(\frac{\pi(\eta_2 + \eta_4)}{2}\right) + \frac{1}{x} \frac{1}{1 - \eta_2 - \eta_4} \right]. \quad (66)$$

For the odd combination we have

$$\frac{1}{2m^2x} \left. \frac{\partial \text{Re} F_-(x,t)}{\partial t} \right|_{t=0} = \frac{1}{\pi} [(P + H \log^2 x_0) F_1(x) + H F_2(x) + R_1 F_3(x) + R_2 F_4(x)], \quad (67)$$

where

$$F_1(x) \equiv b_4 \left[\frac{\pi}{2} x^{-\eta_4} \cot\left(\frac{\pi \eta_2}{2}\right) + \frac{1}{x^2} \frac{1}{2 - \eta_4} \right], \quad (68)$$

$$\begin{aligned} F_2(x) \equiv & b_4 \left(\frac{\pi}{2} x^{-\eta_4} \left\{ \pi \csc^2\left(\frac{\pi \eta_4}{2}\right) \left[\log\left(\frac{x}{x_0}\right) + \frac{\pi}{2} \cot\left(\frac{\pi \eta_4}{2}\right) \right] \right. \right. \\ & \left. \left. + \cot\left(\frac{\pi \eta_4}{2}\right) \log x (-2 \log x_0 + \log x) \right\} + \frac{2}{x^2} \frac{1}{(2 - \eta_4)^2} \left(\log x_0 + \frac{1}{2 - \eta_4} \right) \right), \end{aligned} \quad (69)$$

$$F_3(x) \equiv b_4 x_0^{\eta_1} \left[\frac{\pi}{2} x^{-\eta_1 - \eta_4} \cot\left(\frac{\pi}{2}(\eta_1 + \eta_4)\right) + \frac{1}{x^2} \frac{1}{2 - \eta_1 - \eta_3} \right], \quad (70)$$

$$\begin{aligned} F_4(x) \equiv & x_0^{\eta_2} \left(\frac{\pi}{2} x^{-\eta_2} \left\{ \cot\left(\frac{\pi \eta_2}{2}\right) (b_0 + b_1 \log x + b_2 \log^2 x) + \frac{\pi}{2} \csc^2\left(\frac{\pi \eta_2}{2}\right) \left[b_1 + \pi \cot\left(\frac{\pi \eta_2}{2}\right) b_2 + 2 \log x b_2 \right] \right. \right. \\ & \left. \left. + x^{-\eta_3} \cot\left(\frac{\pi}{2}(\eta_2 + \eta_3)\right) b_3 \right\} + \frac{1}{x^2} \frac{1}{2 - \eta_3} \left[b_0 - \frac{b_1}{2 - \eta_2} + \frac{2b_2}{(2 - \eta_2)^2} + \frac{(2 - \eta_2)b_3}{2 - \eta_2 - \eta_3} \right] \right). \end{aligned} \quad (71)$$

V. RELATION WITH PHENOMENOLOGY

We are here concerned with the strong interaction part of pp and $p\bar{p}$ scattering, with no treatment of the Coulomb interaction, but stressing that the proper treatment of the real

part is essential for a correct account of the interference. The determinations of the total cross section, through its connection with the optical point of the imaginary amplitude, and of the ρ parameter of the ratio of real to imaginary amplitudes at $t = 0$, through analysis of $d\sigma/dt$ data, are affected by the structure

(namely the t dependence) of the real part and its interference with the Coulomb amplitude. DRs and DRS determine values of the real amplitude and its derivative at the origin, controlling parameters of the real and imaginary parts.

In this paper we prove that, in DRs and DRS, terms of the general form $x^\lambda \log^n(x)$ with integer n in amplitudes and slopes of the imaginary part can receive exact treatment, through analytic solution of the intervening PV integrals. Luckily, the well-known and well-accepted parametrization of the pp and $p\bar{p}$ total cross sections made by the PDG/Compete Collaboration [6] is a linear combination of such terms. With parametrization of the imaginary slopes B_I also made with such terms, the DRS also come with the exactly calculable forms. Thus, we obtain the new results presented in this work. With knowledge of exact explicit forms for DRs and DRS, we have powerful support for the analysis of forward $d\sigma/dt$ data.

The parametrization for the total cross section for pp and $p\bar{p}$ interaction in the form of Eq. (15) with functions of s has parameter values P' , H' , R'_1 , and R'_2 in millibarns, s_0 in GeV^2 , and η'_1 and η'_2 dimensionless. The parametrization is considered adequate for all energies $s \geq s_0$. Because dispersion relations are written with the laboratory system energy, we reparametrize the above form, writing similar Eqs. (16) and (17) in terms of the variable x and find parameter values $P = 34.37$ mb, $H = 0.2704$ mb, $R_1 = 12.46$ mb, $R_2 = 7.30$ mb, $\eta_1 = 0.4258$, $\eta_2 = 0.5458$, and $x_0 = 8.94$.

Similarly, the experimental B_I slopes are represented by the forms of Eqs. (18) and (19), with suggested parameter values $b_0 = 13.03 \text{ GeV}^{-2}$, $b_1 = -0.3346 \text{ GeV}^{-2}$, $b_2 = 0.04255 \text{ GeV}^{-2}$, $b_3 = -6.94 \text{ GeV}^{-2}$, $b_4 = 17.31 \text{ GeV}^{-2}$, $\eta_3 = 0.5154$, and $\eta_4 = 0.960$. The symmetries in the expressions for pp and $p\bar{p}$ simplify the algebra, without loss in the quality of the representations of the data.

A. The subtraction constant and the parameter ρ

The determination of the dimensionless subtraction constant K , which is particularly important at low energies, uses experimental information on the real part of both pp and $p\bar{p}$ systems. In Fig. 1, values $K = 0$ and $K = -300$ enter as examples to show the influence in the predictions of the real amplitudes for $t = 0$ for low energies.

B. Structure of the real part

We can learn more about the $|t|$ dependence of the real amplitudes through the investigation of the dispersion relations for slopes. From studies at high energies [8–10] it is known that the real part has a nontrivial structure in the forward range, presenting a zero that approaches $|t| = 0$ with increasing energy, in agreement with a theorem by Martin [7]. Because the pure exponential form cannot have a zero, the real part must be described by a more sophisticated structure. Thus we may assume for the real parts the forms

$$(1/2m^2x)\text{Re}F_{pp}(x,t) = \sigma_{pp}[\rho_{pp} - \mu_{pp}t]e^{-B_R^{pp}|t|/2}, \quad (72)$$

$$(1/2m^2x)\text{Re}F_{p\bar{p}}(x,t) = \sigma_{p\bar{p}}[\rho_{p\bar{p}} - \mu_{p\bar{p}}t]e^{-B_R^{p\bar{p}}|t|/2}. \quad (73)$$

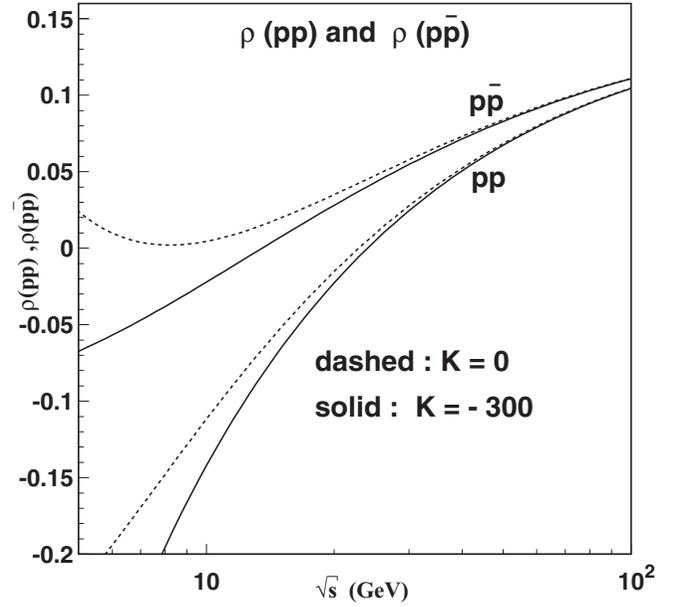


FIG. 1. Energy dependence of $\rho(pp) = \text{Re}F_{pp}(x,0)/\text{Im}F_{pp}(x,0)$ and $\rho(p\bar{p}) = \text{Re}F_{p\bar{p}}(x,0)/\text{Im}F_{p\bar{p}}(x,0)$ predicted by DRs, with illustrative values 0 and -300 for the subtraction constant K .

The derivatives with respect to t at the origin give

$$(1/2m^2x) \frac{\partial \text{Re}F_{pp}(x,t)}{\partial t} \Big|_{|t|=0} = \sigma_{pp} \left[\frac{\rho_{pp}}{2} B_R^{pp} - \mu_{pp} \right], \quad (74)$$

$$(1/2m^2x) \frac{\partial \text{Re}F_{p\bar{p}}(x,t)}{\partial t} \Big|_{|t|=0} = \sigma_{p\bar{p}} \left[\frac{\rho_{p\bar{p}}}{2} B_R^{p\bar{p}} - \mu_{p\bar{p}} \right]. \quad (75)$$

Terms with higher powers do not contribute to the derivative at $t = 0$.

We may observe particularly the combination of parameters

$$\rho B_R/2 - \mu, \quad (76)$$

which together with ρ for the pp and $p\bar{p}$ systems, are the predictions of DRs and DRS for the description of $d\sigma/dt$ data in the forward range. These quantities are shown in Fig. 2. The asymptotic (high energy) values are $\rho \approx \pi/\log(x)$ and $\rho B_R/2 - \mu \approx \pi b_2 \log(x)$. The quantities ρ , B_R , and μ must be extracted from the analysis of the $d\sigma/dt$ data and compared with these predictions.

VI. FINAL REMARKS

The paper presents advances in the formulation and use of dispersion relations in the treatment of pp and $p\bar{p}$ scattering. The main points of our results are reviewed below.

First, we give the proof of the solution of the general form of principal value integrals, arising from terms of the form $x^\lambda \log^n x$ in the imaginary amplitudes. The sum of such terms form the established representation for the energy dependence of total cross sections [6], so that the proof is of fundamental importance for the area, closing a long period in which the solution of the singular integrals of the dispersion relations was the main technical difficulty.

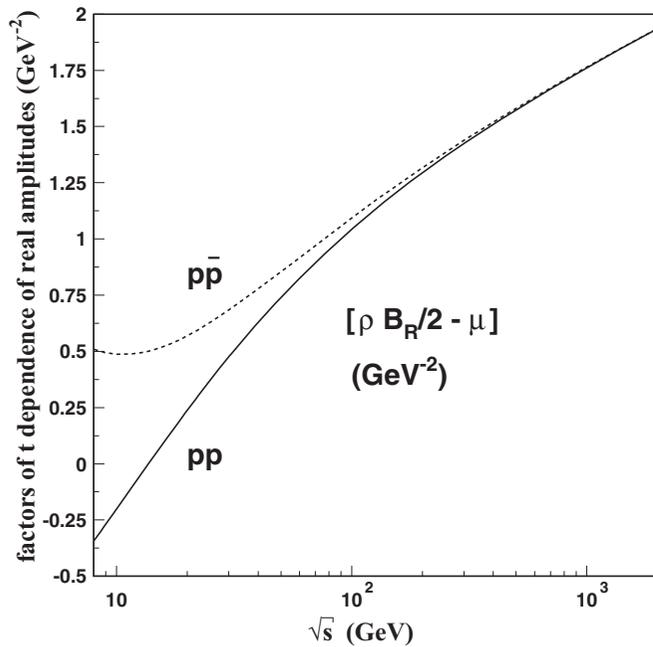


FIG. 2. With σ and B_I (for pp and $p\bar{p}$ systems) given as inputs, the combinations of parameters $\rho B_R/2 - \mu$ (for pp and $p\bar{p}$) are predicted by the DRS, offering efficient control in the analysis of $d\sigma/dt$ data.

The solution is obtained from the study of a new property found by the present authors [1] for the Lerch's transcendent, which is a well-defined function of mathematical analysis. In Sec. III we start from the theorem proving the new integral representation of Lerch functions and construct the analytic solution for the required integrations.

To investigate new consequences of the DR principles, t -dependent extensions are written for the imaginary amplitudes,

with exponential factors of energy-dependent slopes. These extensions are obvious and universally adopted in the description of the forward peak of the elastic differential cross section (the famous diffraction peak). Their use in the framework of dispersion relations was first introduced [2] in 2007 and shown to be an important tool in the control of parameters of forward elastic scattering. Parametrizations are introduced for the energy dependence of the imaginary slopes B_I^{pp} and $B_I^{p\bar{p}}$, describing well the known data, with the analytical structure formed with the combination of terms of the same basic form $x^\lambda \log^n x$.

Derivatives with respect to t of the DR expressions with the t dependencies give origin to new connections between real and imaginary amplitudes. Taking $t = 0$ in these expressions we obtain the derivatives of the real parts in terms of PV integrals that we know how to solve. The novel expressions are here called dispersion relations for slopes. Full expressions are written for these relations in terms of the given inputs of the imaginary parts, using the exact PV solutions.

It is stressed that DRs and DRS together form an important frame for the analysis of elastic scattering, totally based on consequences of the principles of analyticity and causality that are the basis of the theory of dispersion relations.

The relation of these results to the phenomenology of forward elastic scattering is explored, exhibiting the evaluation of observable quantities. Particularly interesting is the study of the behavior of the real amplitudes in the forward range, with identification of constraints that are determined by DRs and DRS and point to controls of the scattering parameters.

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