Frame-independent spatial coordinate $\tilde{z}$: Implications for light-front wave functions, deep inelastic scattering, light-front holography, and lattice QCD calculations

Gerald A. Miller *Department of Physics, University of Washington, Seattle, Washington 98195-1560, USA*Stanley J. Brodsky *SLAC National Accelerator Laboratory, Stanford University, Stanford, California 94309, USA*

(Received 5 May 2020; accepted 16 July 2020; published 4 August 2020)

A general procedure for obtaining frame-independent three-dimensional light-front coordinate-space wave functions is introduced. The third spatial coordinate $\tilde{z}$ is the boost and Lorentz frame-independent coordinate conjugate to the light-front momentum coordinate $x = \frac{k^+}{p^+}$ which appears in the momentum-space light-front wave functions underlying generalized parton distributions, structure functions, distribution amplitudes, form factors, and other hadronic observables. These causal light-front coordinate-space wave functions are used to derive a general expression for the quark distribution function of hadrons as an integral over the frame-independent longitudinal distance (the Ioffe time) between virtual-photon absorption and emission appearing in the forward virtual photon-hadron Compton scattering amplitude. Specific examples using models derived from light-front holographic QCD show that the spatial extent of the proton eigenfunction in the longitudinal direction can have a very large extent in $\tilde{z}$.

DOI: [10.1103/PhysRevC.102.022201](https://doi.org/10.1103/PhysRevC.102.022201)

Introduction. Much recent effort has been devoted to understanding and measuring the generalized parton distributions [1–3] which encode the fundamental structure of hadrons in terms of the three-dimensional momentum-space coordinates of their quark and gluon constituents. Recent lattice calculations of quasi-parton distribution functions (-pdfs) evaluate a Fourier transform of a matrix element which depends on the spatial separation between the points of virtual-photon absorption and emission that appears in the virtual photon-proton Compton scattering amplitude. It is, therefore, of considerable interest to understand the spatial longitudinal dependence of the virtual Compton amplitude from a causal frame-independent perspective. In this Rapid Communication, we show that the frame-independent eigen-solutions of the Quantum Chromodynamics (QCD) light-front (LF) Hamiltonian that underly hadronic observables can be expressed in terms of a longitudinal spatial coordinate $\tilde{z}$ that is simply related to the spatial separation between a struck quark and the spectators. One, thus, obtains a frame-independent fully three-dimensional spatial description of the hadron structure which complements analyses using the usual transverse spatial variables [4–7].

The light-front Fock representation. The light-front expansion of any hadronic system is constructed by quantizing QCD at fixed light-front time $\tau = t + z/c$ [8–12]. The LF time-evolution operator $P^- = i \frac{d}{d\tau}$ can be derived directly from the QCD Lagrangian. The light-front Lorentz-invariant Hamiltonian for the composite hadrons $H_{\text{LF}}^{\text{QCD}} = P^- P^+ - \mathbf{P}^2$ ($P^\pm = P^0 \pm P^z$ and boldface denotes the two-dimensional transverse vectors) has eigenvalues $\mathcal{M}_h^2$, corresponding to the mass spectrum of the color-singlet states in QCD [10].

In principle, the complete set of bound-state and scattering eigensolutions of $H_{\text{LF}}^{\text{QCD}}$ can be obtained by solving the light-front Heisenberg equation

$H_{\text{LF}}^{\text{QCD}} |\psi_h\rangle = \mathcal{M}_h^2 |\psi_h\rangle$, where $|\psi_h\rangle$ is an expansion in multiparticle Fock eigenstates $\{|n\rangle\}$ of the free light-front Hamiltonian: $|\psi_h\rangle = \sum_n \psi_{n/h} |n\rangle$. The light-front wave-functions $\psi_{n/h}(x_i, \mathbf{k}_i, \lambda_i)$ provide a complete causal frame-independent representation of hadrons, relating the quark and gluon degrees of freedom in each n -particle Fock state to the hadronic eigenstate.

Twist-two operators and the need for a longitudinal spatial coordinate. The quark distributions of a hadron are matrix elements of quark operators at lightlike separation [13–16],

$$q(x) = \int \frac{dx^-}{4\pi} e^{ixP^+x^-} \langle P | \bar{\psi} \left(-\frac{x^-}{2} \right) \gamma^+ \psi \left(\frac{x^-}{2} \right) | P \rangle,$$

$$\bar{q}(x) = -q(-x), \quad (1)$$

where the notation $(x^-/2)$ refers to the four-vector $(x^-/2, x^+ = 0, \mathbf{x} = 0)$; the LF helicity and flavor labels as well as the Q^2 dependence are suppressed. The operator $\gamma^0 \gamma^+$

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

FIG. 1. Forward virtual Compton scattering.

that appears in the matrix element in the $A^+ = 0$ gauge serves to project [9] the field operator ψ onto its independent component ψ_+ so that field operators and their adjoints appear. The variable x ranges between 0 and 1.

The expression for $q(x)$ is the leading-twist approximation to the virtual photon forward-scattering amplitude shown in Fig. 1. Our focus, here, is the distance along the light cone between the emission and absorption of the virtual photon, shown by the thick line. This is needed for a complete understanding of the three-dimensional spatial dependence of the quark distributions.

The matrix element appearing in Eq. (2) is directly relevant to several techniques that seek to obtain quark distributions as functions of x , e.g., Refs. [17–20]. See the extensive reviews [21,22]. These techniques represent significant advances over efforts based on computing moments of distributions. Lattice theorists compute the lattice version of the matrix element appearing in Eq. (2), for example, Ref. [18] as $h_{\gamma^+}(P, x^-)$, and then take a Fourier transform in order to obtain the quasi-pdfs. Therefore, it is useful to obtain physical intuition regarding the matrix element appearing in Eq. (2). This will be performed here by employing recent models derived from holographic light-front QCD.

Of course, we are not the first to study the variable x^- . It has commonly been called the Ioffe time [23–25]. This quantity is known to be large if x is small. The study of the *matrix element* appearing in Eq. (2) as the Fourier transform of quark probability distributions was initiated in Refs. [26,27]. Our procedure elucidates the dependence on x^- that appears in Eq. (2) as derived from light-front wave functions in coordinate space, and it is, thus, not the same as the procedure of Refs. [26,27].

We study hadronic light-front wave functions as a function of the longitudinal spatial coordinate of the quark and gluon constituents. The appearance of wave functions arises by inserting a complete set of states $|n-1\rangle$ in Eq. (2) so that

$$q(x) = 2^{1/2} \int \frac{dx^-}{4\pi} e^{ixP^+x^-} \times \sum_n \langle P | \psi_+^\dagger \left(-\frac{x^-}{2} \right) | n-1 \rangle \langle n-1 | \psi_+ \left(\frac{x^-}{2} \right) | P \rangle. \quad (2)$$

The quantity $\langle n-1 | \psi_+ \left(\frac{x^-}{2} \right) | P \rangle$ is an overlap of amplitudes which projects out the active struck quark integrated over the spectator particles. This is simply the light-front Fock-space wave function of a quark (or antiquark). In the momentum-space representation of the standard light-front Fock-space description [10–12], one has for the quark wave-functions $\langle n-1 | \psi_+(x, \mathbf{k}, \lambda) | P \rangle \equiv \psi_n(x, \mathbf{k}, \lambda) 2^{-1/4}$ in which the indices that refer to specific states have been suppressed to simplify the presentation. The contribution of this component (q_n) $q(x)$ of Eq. (2) is given by

$$q_n(x) = \int \frac{d^2k}{(2\pi)^2} |\psi_n(x, \mathbf{k})|^2. \quad (3)$$

For quarks $|\psi_n(x, \mathbf{k}, \lambda)|^2 \propto |\langle n-1 | b(k^+, \mathbf{k}, \lambda) | P \rangle|^2$, where $b(k^+, \mathbf{k}, \lambda)$ is the destruction operator, and, for antiquarks, $|\psi_n(x, \mathbf{k}, \lambda)|^2 \propto |\langle n-1 | d(k^+, \mathbf{k}, \lambda) | P \rangle|^2$ [28,29].

Converting these momentum-space wave functions to coordinate space is the next step. The transverse momentum coordinate $\mathbf{k}$ is transformed into the canonically conjugate impact parameter $\mathbf{b}$ to obtain $\psi_n(x, \mathbf{b})$ using standard methods [4–7]. The dependence of ψ_n on the frame-independent longitudinal spatial coordinate has not previously appeared.

The frame-independent longitudinal space coordinate $\tilde{z}$. The momentum-space wave functions are normally expressed in terms of the light-front momentum coordinates $\frac{k_i^+}{P^+}$ and $\mathbf{k}_i$ of the i th constituent. These coordinates are internal momentum coordinates, independent of the hadron's momentum P^μ and the choice of the observer's Lorentz frame. As a result, their Fourier transforms, the canonically conjugate internal spatial coordinates $\mathbf{b}_i$, and $\tilde{z}_i$, given by the variable,

$$\tilde{z}_i = P^+ x_i^- \quad (4)$$

are also independent of the frame of the hadron. Our $\tilde{z}_i$ seems similar to the variable z of Ref. [26], but its origin and meaning are different. The canonical spatial coordinate occurs for *each* of the constituents of a Fock-space component of a hadronic wave function. In the following, we use $\tilde{z}$ instead of $\tilde{z}_i$ to simplify the notation. See also Refs. [30,31].

Making a standard Fourier transform yields the coordinate space wave function given by

$$\psi_n(\tilde{z}, \mathbf{b}) = \frac{1}{\sqrt{2\pi}} \int_0^1 dx \psi_n(x, \mathbf{b}) e^{i\tilde{z}x}, \quad (5)$$

or the mixed form

$$\psi_n(\tilde{z}, \mathbf{k}) = \frac{1}{\sqrt{2\pi}} \int_0^1 dx \psi_n(x, \mathbf{k}) e^{i\tilde{z}x}. \quad (6)$$

These LF wave functions are matrix elements of field operators between Fock-space components [12] using standard light-front coordinates.

It is worthwhile to compare the present approach with the concept that the longitudinal direction is Lorentz contracted to zero in the infinite momentum frame. Contraction occurs if one identifies the longitudinal coordinate as x^- , the coordinate canonically conjugate to the momentum variable k^+ . This leads to a frame-dependent coordinate-space wave function from the relation:

$$\chi_P^+(x^-, b) = \sqrt{\frac{P^+}{2\pi}} \int_0^1 dx \psi_n(x, \mathbf{b}) e^{ix^-P^+x}. \quad (7)$$

The resulting density $\rho_{P^+}(x^-, b) = |\chi_{P^+}(x^-, b)|^2$ in the infinite momentum frame is obtained by taking $P^+ \rightarrow \infty$. Taking the limit carefully [32] yields $\rho(x^-, b) =$

$\int dx |\psi_n(x, b)|^2 \delta(x^-)$, a result that corresponds to a picture in which generalized parton distribution functions (GPDs) are represented as disks [33]. See also Ref. [34]. There is one caveat: The contraction occurs *only* for matrix elements of the independent quark-field operators involving the so-called “good” operator γ^+ . The coordinate x^- is frame dependent. Instead, a boost-invariant distribution is obtained using the longitudinal coordinate $\tilde{z} = P^+ x^-$.

Quark distribution functions and $\tilde{z}$. The frame-independent quark distribution function $q_n(x)$ of Eq. (3) can be expressed in terms of the longitudinal coordinate $\tilde{z}$ using the inverse Fourier transform of Eq. (6) so that

$$q_n(x) = \int \frac{d\tilde{z} d\tilde{z}'}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \psi_n^*(\tilde{z}', \mathbf{k}) \psi_n(\tilde{z}, \mathbf{k}) e^{i(\tilde{z}-\tilde{z}')x}. \quad (8)$$

Let $\tilde{Z} \equiv (\tilde{z} + \tilde{z}')/2$, $\Delta\tilde{z} \equiv \tilde{z} - \tilde{z}'$ and use Eq. (6) to express the wave functions in terms of integrals of standard momentum fractions (y, y') and integrate over $\tilde{Z}$. The result sets $y = y'$ so that the quark distribution $q_n(y)$ appears. Then, use the real-valued nature of $q_n(x)$ of Eq. (8) to obtain

$$q_n(x) = \int_{-\infty}^{\infty} d(\Delta\tilde{z}) g_n(\Delta\tilde{z}, x), \quad (9)$$

with

$$g_n(\Delta\tilde{z}, x) = \frac{1}{2\pi} \int_0^1 dy q_n(y) \cos \Delta\tilde{z}(y - x). \quad (10)$$

The function $g_n(\Delta\tilde{z}, x)$ is a measure of the contribution to quark (antiquark) distribution functions that occur at a particular value of $\Delta\tilde{z}$. In contrast with the distributions of Ref. [26], this quantity is real valued. The variables $\Delta\tilde{z}, x$ are canonically related. Therefore, observe that $g_n(\Delta\tilde{z}, x)$ is a relativistic version of the distribution that Wigner [35] introduced a long time ago.

Models to further our understanding of $g(\Delta\tilde{z}, x)$. The first model considered is that of a pseudoscalar meson with massless quarks and one valence $q\bar{q}$ Fock-space component; the LF holographic model for the massless pion in the chiral $m_q = 0$ limit. The eigenfunction of the holographic light-front Hamiltonian [36] is given by

$$\psi_M(x, \mathbf{b}) = \frac{\kappa}{\sqrt{\pi}} \sqrt{x(1-x)} e^{-[\mathbf{b}^2 \kappa^2 x(1-x)]/2}. \quad (11)$$

The transverse variable [36] $\zeta^2 = b^2 x(1-x)$ is canonically conjugate to $\frac{k^2}{x(1-x)}$, and the wave functions are simplified if this variable is used. Here, we take another path by exhibiting the separate dependence of $\tilde{z}$ and transverse coordinates.

The momentum-space version of Eq. (11) is obtained from the Fourier transform to the canonically conjugate $\mathbf{k}$ so that

$$\psi_M(x, \mathbf{k}) = \frac{2\sqrt{\pi}}{\kappa\sqrt{x(1-x)}} e^{-\{\mathbf{k}^2/[2\kappa^2 x(1-x)]\}}. \quad (12)$$

Using Eq. (3), one finds that the parton distribution is constant for this model $q^M(x) = 1$. For $m_q \neq 0$, one [37] models the mass dependence so that $q^M(x) = \exp(-\frac{m_q^2}{\kappa^2 x(1-x)})$.

The coordinate-space wave function is obtained by using Eq. (11) in Eq. (5). It is useful to define a light-front

FIG. 2. The density $\rho_M^S(\tilde{z}, b)$ values of $\kappa b = 0, 1, 3$. Values of $\rho_M^S(\tilde{z}, b)$ fall as κb increases.

coordinate-space density,

$$\rho_M(\tilde{z}, b) \equiv |\psi_M^S(\tilde{z}, b)|^2. \quad (13)$$

The variable b is the transverse distance between the struck quark and the transverse center of momentum [5]. The coordinate $\tilde{z}$ is the dimensionless longitudinal spatial variable canonically conjugate to the momentum variable $k^+/P^+ = x$. To explain, consider a quark (1)-spectator S system. The total (+) momentum is $P^+ = p_1^+ + p_S^+$. The value of the canonical spatial variable $X^- = (x_1^- + x_S^-)/2$ is undetermined because the states we consider have a definite value of P^+ . However, the difference $x^- \equiv (x_1^- - x_S^-)$ is canonically conjugate to $(p_1^+ - p_S^+)/2$, and $x_{1,2}^- = X^- \pm x^-/2$. Thus, x^- is the difference between the longitudinal positions of the quark and the spectator, and $\tilde{z} = x^- P^+$ is the dimensionless frame-independent version of x^- .

Obtaining $\rho_M(\tilde{z}, b)$ provides a new way of examining hadronic wave functions. This is shown for $m_q = 0$ in Fig. 2.

FIG. 3. $g_\tau(\Delta\tilde{z}, 0)$. The numbers refer to the value of τ , the number of constituents in the Fock state. (a) $0 \leq \Delta\tilde{z} \leq 10$, and (b) $10 \leq \Delta\tilde{z} \leq 200$.

FIG. 4. $\rho(\tilde{z}, b)$. The numbers refer to the value of b in units of GeV^{-1} .

An interesting feature is that, for each value of κb , the density vanishes at values of $\tilde{z} \approx 7.6$.

We may gain understanding of this vanishing by obtaining an approximate closed form expression for $\psi_M^S(\tilde{z}, b)$. Use Eq. (11) in Eq. (5). Then, change variables to $u = x - 1/2$. Then,

$$\psi_M^S(\tilde{z}, b) = \frac{2\kappa}{\sqrt{2\pi}} e^{-b^2\kappa^2/8} e^{i\tilde{z}/2} \times \int_0^{1/2} du \sqrt{1/4 - u^2} e^{b^2\kappa^2 u^2/2} \cos \tilde{z}u, \quad (14)$$

so that a term $e^{i\tilde{z}/2} e^{-b^2\kappa^2/8}$ factors out of the integral. One may expand the $e^{b^2\kappa^2 u^2/2}$ in powers of $b^2 u^2$ to get a sum of closed-form expressions. The most important term is the one obtained by replacing the exponential by unity, so that

$$\frac{\pi}{\sqrt{2\kappa}} \psi_M^S(\tilde{z}, b) \approx \frac{\pi}{4} e^{i\tilde{z}/2} e^{-b^2\kappa^2/8} \frac{J_1(\tilde{z}/2)}{\tilde{z}}, \quad (15)$$

which is reasonably accurate for $\tilde{z} > b^2\kappa^2$. In that case, the oscillations cause cancellations of higher-order terms in $b^2 u^2$. The first zero of $J_1(x)$ occurs at $x = 3.8171$ so that one obtains a qualitative understanding of the zero crossing at $\tilde{z} \approx 7.6$ shown in Fig. 2. The result Eq. (15) means that, for $\tilde{z} > b^2\kappa^2$, the density falls only as $1/\tilde{z}^3$, approximately modulated by $\cos^2(\tilde{z} + \pi/4)$. The existence of such a long-distance tail indicates that using the $\tilde{z}$ variable has the potential to reveal interesting aspects of hadronic physics.

Next, we determine the function $g_n(\Delta\tilde{z}, x)$ for the model of Eq. (11). Use Eq. (10) with $q_n(y) = 1$ as given above to find

$$g_M(\Delta\tilde{z}, x) = \frac{1}{2\pi} \frac{\sin(\Delta\tilde{z}x) + \sin[\Delta\tilde{z}(1-x)]}{\Delta\tilde{z}}. \quad (16)$$

Observe the slow $1/\Delta\tilde{z}$ falloff for all values of x .

Similarly, a slow falloff is also obtained for models with massive quarks. In the limit of large quark masses, defined by $\gamma \equiv m_q^2/\kappa^2 > 1$, we find that

$$g_M(\Delta\tilde{z}, x) \approx \frac{e^{-4\gamma} e^{-(\Delta\tilde{z}^2/64\gamma)} \cos[\Delta\tilde{z}(x - \frac{1}{2})]}{8\sqrt{\pi}\sqrt{\gamma}}. \quad (17)$$

Universal light-front wave functions. The model given in Eq. (11) is very simple with $q_n(x) = 1$ for $m_q = 0$. A recent paper [38] presents a universal description of generalized parton distributions obtained from light-front holographic QCD, and we will use its models. These are presented as functions of the number τ of constituents of a Fock-space component. Regge behavior at small x and inclusive counting rules as $x \rightarrow 1$ are incorporated. Nucleon and pion valence quark distribution functions are obtained in precise agreement with global fits. The model is defined by the quark distribution $q_\tau(x)$ and the profile function $f(x)$ with

$$q_\tau(x) = \frac{1}{N_\tau} [1 - w(x)]^{\tau-2} w(x)^{-(1/2)} w'(x), \quad (18)$$

$$f(x) = \frac{1}{4\lambda} \left[(1-x) \ln\left(\frac{1}{x}\right) + a(1-x)^2 \right], \quad (19)$$

and $w(x) = x^{1-x} e^{-a(1-x)^2}$.

The value of the universal scale λ is fixed from the ρ mass: $\sqrt{\lambda} = \kappa = m_\rho/\sqrt{2} = 0.548 \text{ GeV}$ [37,39]. The flavor-independent parameter $a = 0.531 \pm 0.037$. The u and d quark distributions of the proton are given by a linear superposition of q_3 and q_4 whereas those of the pion are obtained from q_2 and q_4 .

Given these distributions, we may study the function $g(\Delta\tilde{z}, x)$ of as a function of τ using Eq. (10) with q_τ replacing q_n .

Figure 3 shows $g(\Delta\tilde{z}, 0)$ as a function of $\Delta\tilde{z}$, the dimensionless separation between the emission and the absorption of the photon r of Fig. 1. In the laboratory frame, the proton radius corresponds to about $\Delta\tilde{z} = 3$. One observes a slow falloff with increasing $\Delta\tilde{z}$: $g_\tau(\Delta\tilde{z}) \approx \frac{1}{\sqrt{\Delta\tilde{z}}}$ for all values of τ . This qualitative behavior can be understood analytically. The function $q_\tau(x) \approx 1/\sqrt{x}$ for small values of x and $(1-x)^{2\tau-3}$. A useful approximation for $q_\tau(x)$ is given by the product of the two forms. In that case, one may consider, e.g., an important contribution to $g(r, x)$, $\mathcal{G}(\Delta\tilde{z}, x)$ with

$$\mathcal{G}(\Delta\tilde{z}, x) \equiv \int_0^1 \frac{dy}{\sqrt{y}} (1-y)^3 \cos \Delta\tilde{z}(y-x) \sim \frac{\sqrt{2\pi}(\Delta\tilde{z})^{3/2} \sin(x) + \sqrt{2\pi}(\Delta\tilde{z})^{3/2} \cos(x) + 3 \cos(x - \Delta\tilde{z})}{2(\Delta\tilde{z})^2}, \quad (20)$$

which, for all values of x , demonstrates $\Delta\tilde{z}^{-1/2}$ falloff, modulated by oscillatory behavior. An essential feature is that there is a significant probability that the deep inelastic-scattering

process occurs at large separations between the absorption and the emission of the virtual photon. This means that the traditional idea that long longitudinal distances (the Ioffe

FIG. 5. $\tilde{z}^{1.6}\rho(\tilde{z}, 0)$. The numbers refer to the value of τ , the number of constituents in the Fock state.

time) [24,26] underlies deep inelastic scattering at small x_{bj} is related to specific features of hadronic light-front wave functions.

Reference [38] also presents the universal LF wave function,

$$\psi_{\text{eff}}^{\tau}(x, \mathbf{b}) = \frac{1}{2\sqrt{\pi}} \sqrt{\frac{q_{\tau}(x)}{f(x)}} (1-x) \exp \left[-\frac{(1-x)^2}{8f(x)} \mathbf{b}^2 \right], \quad (21)$$

in the transverse impact space representation. The dependence on $\tilde{z}$ is contained in the wave-function $\psi_{\tau}(\tilde{z}, b)$, computed according to Eq. (5). The density $\rho(\tilde{z}, b) = |\psi_{\text{eff}}^{\tau}(\tilde{z}, \mathbf{b})|^2$ is shown in Fig. 4.

The main interest, here, is to study the dependence on $\tilde{z}$, which is displayed in Fig. 5 for $b = 0$. The same general behavior is seen for other values of b . The density falls roughly as $1/\tilde{z}^{1.6}$. This very slow falloff again indicates the large spatial extent of hadronic wave functions.

We examine how the transverse extent depends on $\tilde{z}$ by defining an expectation value $b_{\tau}^2(\tilde{z}) \equiv \frac{\int d^2b |\psi_{\tau}(\tilde{z}, b)|^2 b^2}{\int d^2b |\psi_{\tau}(\tilde{z}, b)|^2}$. The values shown in Fig. 6 generally increase with increasing $\tilde{z}$, in contrast with intuition based on rotational invariance.

The reader will note that the results for the case of $\tau = 2$ shown in Figs. 3, 5, and 6 exhibit significant oscillations at large values of $\tilde{z}$. The oscillations in $\tilde{z}$ for large values of $\tilde{z}$ are due to contributions of values of x near unity in the integrand of Eq. (5). The function $\sqrt{q_2(x)/f(x)}(1-x) \approx \sqrt{1-x}$ for values of x near 1 and its Fourier transform $\approx i/\tilde{z} + e^{i(\tilde{z}+\pi/4)}/\tilde{z}^{3/2} \sqrt{\pi}/2$ which oscillates for large values of $\tilde{z}$. However, the function $1-w(x)$, [recall Eq. (19)] which appears only for $\tau = 3, 4$, approaches $1.531(1-x)^2$ for values of x near unity. This suppresses the integrand of the Fourier transform in the critical region. The oscillating term $e^{i\tilde{z}}$ has a coefficient of $1/\tilde{z}^{5/2}$ for $\tau = 3$ and $1/\tilde{z}^{7/2}$ for $\tau = 4$. The suppression is largest for $\tau = 4$, and some weaker oscillations persist for $\tau = 3$.

The unexpected rising $b_{\tau}^2(\tilde{z})$ behavior shown in Fig. 6 is a feature of its definition, given above, in terms of a ratio. The numerator rises with increasing values of $\tilde{z}$

FIG. 6. $b_{\tau}^2(\tilde{z})$ in units of GeV^{-2} . The numbers refer to the value of τ and the number of constituents in the Fock state. The dashed curve is for $\tau = 3$. The smoothest curve is for $\tau = 4$.

faster than the denominator. Another interesting quantity is the average value of $\mathbf{b}^2$ as a function of x : $b^2(x) \equiv \int d^2b b^2 |\psi_{\text{eff}}(x, \mathbf{b})|^2 / \int d^2b |\psi_{\text{eff}}(x, \mathbf{b})|^2 = \frac{4f}{(1-x)^2}$. This quantity is independent of the value of τ and ranges from about 1.1 fm^2 at $x = 0$ to 0.23 fm^2 at $x = 1$. This decreasing behavior arises from the vanishing of $f(x)$ as x approaches 1. Indeed, $\lim_{x \rightarrow 1} f(x) \text{ GeV}^2 = 1.27454(1-x)^2 + 0.416245(1-x)^3 + \dots$. The mean-square transverse size decreases with increasing x . Similarly, the mean-square transverse momentum,

$$\begin{aligned} \mathbf{k}^2(x) &\equiv \int d^2k k^2 |\psi_{\text{eff}}(x, \mathbf{k})|^2 / \int d^2k |\psi_{\text{eff}}(x, \mathbf{k})|^2 \\ &= (1-x)^2/(4f) \\ &= 1/b^2(x) \end{aligned} \quad (22)$$

increases with increasing x . This behavior, a specific feature of the model of Ref. [38], is completely opposite from that obtained from the simpler form of Eq. (11) as well as that of many models of GPDs.

Summary and outlook. A longitudinal spatial variable $\tilde{z}$ has been introduced in Eq. (4), thus, allowing a representation of light-front wave functions in terms of all three frame-independent spatial coordinate variables. Both the valence model of Eq. (11) and the universal light-front model which incorporates Regge behavior Eq. (21) provide a light-front coordinate-space density Eq. (13), that has a long tail in the longitudinal separation between the struck constituent and the spectators. This allows the absorption-emission separation distance x^- occurring in deep inelastic scattering to be very long. The result Eq. (10) shows how given regions of x^- contribute to the quark distribution at each value of Bjorken x .

Acknowledgments. This work has been partially supported by U.S. D.O.E. Grant No. DE-FG02-97ER-41014 and by U.S. D.O.E. Contract No. DE-AC02-76SF00515. We thank T. Liu, R. Suffian, A. Radyushkin, D. Richards, and J. Qiu for useful discussions.

- [2] X. Ji, Gauge-Invariant Decomposition of Nucleon Spin, *Phys. Rev. Lett.* **78**, 610 (1997).
- [3] A. V. Radyushkin, Scaling limit of deeply virtual Compton scattering, *Phys. Lett. B* **380**, 417 (1996).
- [4] D. E. Soper, The parton model and the bethe-salpeter wave function, *Phys. Rev. D* **15**, 1141 (1977).
- [5] M. Burkardt, Impact parameter space interpretation for generalized parton distributions, *Int. J. Mod. Phys. A* **18**, 173 (2003).
- [6] G. A. Miller, Charge Density of the Neutron, *Phys. Rev. Lett.* **99**, 112001 (2007).
- [7] G. A. Miller, Transverse charge densities, *Annu. Rev. Nucl. Part. Sci.* **60**, 1 (2010).
- [8] P. A. M. Dirac, Forms of relativistic dynamics, *Rev. Mod. Phys.* **21**, 392 (1949).
- [9] G. P. Lepage and S. J. Brodsky, Exclusive processes in perturbative quantum chromodynamics, *Phys. Rev. D* **22**, 2157 (1980).
- [10] S. J. Brodsky, H. C. Pauli, and S. S. Pinsky, Quantum chromodynamics and other field theories on the light cone, *Phys. Rep.* **301**, 299 (1998).
- [11] S. J. Brodsky and G. P. Lepage, Exclusive processes in quantum chromodynamics, in *Perturbative QCD*, Advanced Series on Directions in High Energy Physics (World Scientific, Singapore, 1989), pp. 93–240.
- [12] S. J. Brodsky, D. S. Hwang, B. Q. Ma, and I. Schmidt, Light-cone representation of the spin and orbital angular momentum of relativistic composite systems, *Nucl. Phys. B* **593**, 311 (2001).
- [13] J. C. Collins and D. E. Soper, Parton distribution and decay functions, *Nucl. Phys. B* **194**, 445 (1982).
- [14] X. D. Ji, Off forward parton distributions, *J. Phys. G: Nucl. Part. Phys.* **24**, 1181 (1998).
- [15] J. C. Collins, What exactly is a parton density?, *Acta Phys., Pol. B* **34**, 3103 (2003).
- [16] M. Tanabashi *et al.*, (Particle Data Group), Review of particle physics, *Phys. Rev. D* **98**, 030001 (2018).
- [17] X. Ji, Parton Physics on a Euclidean Lattice, *Phys. Rev. Lett.* **110**, 262002 (2013).
- [18] C. Alexandrou, K. Cichy, M. Constantinou, K. Jansen, A. Scapellato, and F. Steffens, Light-Cone Parton Distribution Functions from Lattice QCD, *Phys. Rev. Lett.* **121**, 112001 (2018).
- [19] H.-W. Lin, J.-W. Chen, X. Ji, L. Jin, R. Li, Y.-S. Liu, Y.-B. Yang, J.-H. Zhang, and Y. Zhao, Proton Isovector Helicity Distribution on the Lattice at Physical Pion Mass, *Phys. Rev. Lett.* **121**, 242003 (2018).
- [20] R. S. Sufian, J. Karpie, C. Egerer, K. Orginos, J.-W. Qiu, and D. G. Richards, Pion valence quark distribution from matrix element calculated in lattice QCD, *Phys. Rev. D* **99**, 074507 (2019).
- [21] K. Cichy and M. Constantinou, A guide to light-cone PDFs from Lattice QCD: an overview of approaches, techniques and results, *Adv. High Energy Phys.* **2019**, 3036904 (2019).
- [22] C. Monahan, Recent developments in x -dependent structure calculations, PoS (**LATTICE2018**), 018 (2018).
- [23] V. N. Gribov, B. L. Ioffe, and I. Y. Pomeranchuk, What is the range of interactions at high-energies, *Yad. Fiz.* **2**, 768 (1965) [*Sov. J. Nucl. Phys.* **2**, 549 (1966)].
- [24] B. L. Ioffe, Space-time picture of photon and neutrino scattering and electroproduction cross-section asymptotics, *Phys. Lett. B* **30**, 123 (1969).
- [25] Y. V. Kovchegov and E. Levin, Quantum chromodynamics at high energy, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **33**, 1 (2012).
- [26] V. Braun, P. Gornicki, and L. Mankiewicz, Ioffe - time distributions instead of parton momentum distributions in description of deep inelastic scattering, *Phys. Rev. D* **51**, 6036 (1995).
- [27] V. Braun and D. Muller, Exclusive processes in position space and the pion distribution amplitude, *Eur. Phys. J. C* **55**, 349 (2008).
- [28] R. L. Jaffe, Parton distribution functions for twist four, *Nucl. Phys.* **B229**, 205 (1983).
- [29] M. Diehl, Generalized parton distributions, *Phys. Rep.* **388**, 41 (2003).
- [30] A. V. Radyushkin, Quasi-parton distribution functions, momentum distributions, and pseudo-parton distribution functions, *Phys. Rev. D* **96**, 034025 (2017).
- [31] K. Orginos, A. Radyushkin, J. Karpie, and S. Zafeiropoulos, Lattice QCD exploration of parton pseudo-distribution functions, *Phys. Rev. D* **96**, 094503 (2017).
- [32] A. Dumitru, G. A. Miller, and R. Venugopalan, Extracting many-body color charge correlators in the proton from exclusive DIS at large Bjorken x , *Phys. Rev. D* **98**, 094004 (2018).
- [33] A. Accardi *et al.*, Electron Ion Collider: The Next QCD Frontier: Understanding the glue that binds us all, *Eur. Phys. J. A* **52**, 268 (2016).
- [34] P. Hoyer, Comments on the relativity of shape, in *Shape of Hadrons*, edited by C. N. Papanicolas and A. M. Bernstein, AIP Conf. Proc. No. 904 (AIP, New York, 2007), p. 65.
- [35] E. P. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* **40**, 749 (1932).
- [36] S. J. Brodsky and G. F. de Téramond, Light-front dynamics and AdS/QCD correspondence: The pion form factor in the space- and time-like regions, *Phys. Rev. D* **77**, 056007 (2008).
- [37] S. J. Brodsky, G. F. de Téramond, H. G. Dosch, and J. Erlich, Light-front holographic QCD and emerging confinement, *Phys. Rep.* **584**, 1 (2015).
- [38] G. F. de Téramond *et al.* (HLFHS Collaboration), Universality of Generalized Parton Distributions in Light-Front Holographic QCD, *Phys. Rev. Lett.* **120**, 182001 (2018).
- [39] S. J. Brodsky, G. F. de Téramond, H. G. Dosch, and C. Lorce, Universal effective hadron dynamics from superconformal algebra, *Phys. Lett. B* **759**, 171 (2016).