Composite order in $SU(N)$ theories coupled to an Abelian gauge field

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We consider $SU(N)$ -symmetric Ginzburg-Landau models coupled to a noncompact Abelian gauge field focusing on the case $N > 2$ at finite temperature. We show that, at least for sufficiently large gauge-field coupling constants, these models have two phase transitions. The intermediate phase between the symmetric and low-temperature phases is a state with composite neutral order and no Meissner effect. In this neutral phase the system spontaneously breaks only the symmetry associated with phase differences and density differences between components. For $N > 2$, in contrast to the $SU(2)$ case, the neutral state cannot be mapped onto an $O(M)$ model. We term this state the $SU(N)_n$ phase.

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I. INTRODUCTION

Phase transitions in $U(1)$ Abelian gauge theories, such as superconductors, is an old problem which is fairly well understood in the single-component case [1,2]. The phase diagrams of multicomponent superconductors beyond mean-field approximation are more complex and less studied. Multicomponent gauge theories with Abelian gauge fields appear in the context of multicomponent electronic and nuclear superconductivity (see, e.g., Refs. [3–6]), and as effective field theories for other quantum systems [7–11]. The principal difference is that multicomponent models feature additional phases in which there is order only in various products of the complex fields, while individual phases are disordered.

The most investigated cases are two- and three-dimensional $U(1) \times U(1)$ superconductors described by two complex fields $\psi_{1,2}$. They have, under certain conditions, phase transitions to “paired” states where order remains only in products of the fields such as $\langle \psi_1 \psi_2 \rangle \neq 0$ or $\langle \psi_1 \psi_2^* \rangle \neq 0$ while individual phases are disordered: $\langle \psi_i \rangle = 0$ [3,8,12–15]. These new states represent (i) neutral superfluids, where only counterflow of charged components is allowed (termed metallic superfluid) and (ii) charge-4e superconductors. Analogous states of composite order occur for different microscopic reasons in strongly interacting superfluid mixtures on a lattice [16–18] and in pair-density-wave superconductors [19–21].

The origin of these transitions and additional phases is the fact that in these systems the composite topological defects, i.e., bound states of vortices in different components, have lower energy than the simplest vortices in the individual fields $\psi_{1,2}$. The entropically driven proliferation of composite vortices disorders the phases of the individual fields but does not restore the symmetry completely: the system retains order in

products of the fields up to a higher temperature where the elementary topological defects proliferate (for a discussion of the principle see Ref. [22]).

Besides representing new types of states, paired phases have attracted interest in models connected with the discussion of the orders of the phase transitions in the so-called deconfined quantum criticality problem [8–11,23]. Phases with composite order have also been found in $SU(2)$ models [9,10,24,25]. A theory for an $SU(2)$ double of complex fields coupled to an Abelian gauge field can be thought of as a model with $U(1)_{\text{local}} \times O(3)_{\text{global}}$ symmetry.

The situation is more subtle in the $SU(2)$ case than in the $U(1) \times U(1)$ case since it is believed that there are no energetically stable composite vortexlike topological defects in type-2 $SU(2)$ gauge theories [26]. Nonetheless, for a sufficiently strong coupling constant it has been shown that such $SU(2)$ models have two transitions: at lower temperature the Meissner effect disappears but the system retains broken global $O(3)$ symmetry [9,10,24,25] which is restored only at a higher temperature. Currently there is no known duality argument relating phase transitions to statistical mechanics of topological defects in $SU(N)$ systems of the kind that exists for $U(1)$ systems [1,2]. However, it should be noted that even the lack of energetic stability of composite vortices in the ground state does not necessarily exclude the possibility of entropically excited such vortices driving phase transitions. In that respect an instructive example is the magnetic response of $SU(2)$ gauge theory [27]: Strictly speaking, $SU(2)$ gauge theory is not a superconductor in the same sense as a $U(1)$ superconductor, as one cannot use topological arguments based on $U(1)$ invariants [22] to prove nonrelaxation of superconducting current, and individual vortices carrying one flux quantum are not stable in the type 2 regime [26]. Nonetheless, in an external field the system forms a vortex lattice [27] stabilized by intervortex forces. Likewise vortices can exist under certain conditions in neutral $SU(2)$ systems [28].

The above raises the question of what the phase diagrams are in $SU(N)$ gauge theories with $N > 2$. Such higher- N theories have attracted recent interest in the context of effective field theory for the quantum J-Q Heisenberg model,

[29–31] which catalyzed renewed interest in $SU(N)$ gauge theories [32–36]. Another motivation to study these states is the prospect of the creation of artificial gauge fields in ultracold atoms [37,38], which can open up new possibilities of realizing new $SU(N)$ generalizations of superconducting states.

II. MODELS WITH FIXED TOTAL DENSITY

We consider $SU(N)$ -symmetric Ginzburg-Landau models in three spatial dimensions, given by the Hamiltonian density

$$h = \frac{1}{2} \sum_i |(\nabla + iq\mathbf{A})\psi_i|^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2$$

with the constraint $\sum_i |\psi_i|^2 = 1$. (1)

Different complex fields $\psi_i = |\psi_i|e^{i\phi_i}$ are here coupled by the vector potential $\mathbf{A}$. The amplitudes are subjected to the constraint that the total superconducting density $\sum_i |\psi_i|^2 = 1$, but are otherwise allowed to fluctuate.

In order to elucidate the properties of the model (1) we note that it can be rewritten as [39]

$$h = \frac{1}{2} \mathbf{j}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} \sum_{i < j} |\psi_i|^2 |\psi_j|^2 (\nabla \phi_{ij})^2 + \frac{1}{2} \sum_i (\nabla |\psi_i|)^2, \quad (2)$$

where $\phi_{ij} = \phi_j - \phi_i$ and $\mathbf{j}$ is the density of charged supercurrent:

$$\mathbf{j} = \sum_i |\psi_i|^2 (\nabla \phi_i + q\mathbf{A}). \quad (3)$$

In Eq. (2), the terms on the first line are, respectively, associated with electrically charged currents and with magnetic fields. The terms on the second line are, respectively, associated with electrically neutral currents consisting of counterflows of charged components, and with gradients of relative density.

Note that, quite generically, when all components have nonzero density, only the vortices that have winding in the phase of each component (composite vortices) will have finite energy per unit length [40], as do vortices in ordinary single-component superconductors. These are therefore the energetically cheapest vortices that can be thermally excited. However, proliferation of such objects cannot restore symmetries associated with phase and density differences between components. Below we numerically investigate the possibility of this scenario.

III. MONTE CARLO SIMULATION METHODS

We discretize the model (1) on a three-dimensional simple cubic lattice with L^3 sites and lattice constant $a = 1$. The discretized model is given by the Hamiltonian density

$$h = \frac{1}{2} \sum_{k < l} F_{kl}^2 - \sum_{i,k} |\psi_i(\mathbf{r})| |\psi_i(\mathbf{r} + \mathbf{k})| \cos \chi_{i,k}(\mathbf{r}), \quad (4)$$

where

$$F_{kl} = A_k(\mathbf{r}) + A_l(\mathbf{r} + \mathbf{k}) - A_k(\mathbf{r} + \mathbf{l}) - A_l(\mathbf{r}) \quad (5)$$

is a lattice curl,

$$\chi_{i,k}(\mathbf{r}) = \phi_i(\mathbf{r} + \mathbf{k}) - \phi_i(\mathbf{r}) + qA_k(\mathbf{r}) \quad (6)$$

is a gauge-invariant phase difference, k and l signify coordinate directions, and $\mathbf{k}$ is a vector pointing from a lattice site to the next site in the k direction. Periodic boundary conditions are applied in all three spatial directions. The thermal probability distribution for states of the system at inverse temperature β is given by the Boltzmann weight

$$e^{-\beta H}, \quad H = \sum_{\mathbf{r}} h(\mathbf{r}). \quad (7)$$

We generate representative samples from these thermal distributions using Monte Carlo simulation.

The simulations are performed using the Metropolis-Hastings algorithm with local updates of each of the degrees of freedom. The sizes of the local updates of the vector potential are adjusted during the equilibration in order to make the acceptance probability 0.5. Each phase is updated separately by simply selecting a new value for the phase with uniform distribution on the range from $-\pi$ to π . The amplitude degrees of freedom on a given site are updated together by randomly choosing a new density distribution. Care must be taken to use the correct measure when updating the amplitudes subject to the constraint of fixed total density. This can be done as follows: The probability distribution $\rho(s)$ for the value of the first density s chosen is

$$\rho(s) \propto (1 - s)^{N-2}, \quad (8)$$

where N is the number of components (whence $N - 2$ is the number of remaining degrees of freedom in the densities when the first density has been picked). Note that once one has selected the first density $|\psi_1|^2$, the problem of selecting the second density is simply the problem of selecting the first density in an $N - 1$ component system with total density fixed to be $1 - |\psi_1|^2$, and so on. Given that one can draw a random number x with uniform distribution on the range $[0, 1]$, how does one obtain a random number s drawn from the distribution (8)? The answer is to let

$$s = 1 - x^{1/(N-1)}. \quad (9)$$

To see this, note that

$$dx \propto (1 - s)^{N-2} ds, \quad (10)$$

whence a uniform distribution for x corresponds to the desired distribution for s .

We also use parallel-tempering swaps between systems with neighboring temperatures; typically one set of swaps is proposed every 24 sweeps. The simulated temperatures are adjusted within a fixed interval in order to make the acceptance ratios for parallel-tempering swaps equal for all pairs of neighboring temperatures. We use reweighting between simulated temperatures in order to improve our data (more on this below).

Equilibration is checked by comparing results obtained using the first and second halves of the data gathered after

equilibration, by comparing inverse-temperature derivatives obtained using finite differences and statistical estimators, and by inspecting obtained time series of Monte Carlo data. Errors are estimated by bootstrapping, and stated errors correspond to one standard error.

IV. PHASE DIAGRAMS

We consider the phase diagrams, in terms of electric charge q and inverse temperature β , for the cases of two, three, and four components. These phase diagrams involve two types of order: superconducting order and a neutral order that we term $SU(N)_n$ order (where the subscript n indicates the word neutral). We first describe the methods we use to locate phase transitions of each type, and then present and discuss our results.

A. Locating superconducting transitions

While systems that do not have local U(1) symmetry are not superconductors in the usual sense [i.e., do not have a conserved U(1) topological invariant [22]], they can exhibit magnetic field screening (the Meissner effect). Correspondingly, we define a superconducting phase transition as a transition where the Meissner effect disappears. In order to locate superconducting transitions, we use the dual stiffness [9,25,41]

$$\rho^\mu(\mathbf{q}) = \left\langle \frac{|\sum_{\mathbf{r}, \nu, \lambda} \epsilon_{\mu\nu\lambda} [\Delta_\nu A_\lambda(\mathbf{r})] e^{i\mathbf{q}\cdot\mathbf{r}}|^2}{(2\pi)^2 L^3} \right\rangle, \quad (11)$$

where $\epsilon_{\mu\nu\lambda}$ is the Levi-Civita symbol, Δ_ν is a difference operator (lattice gradient), and $\langle \cdot \rangle$ is a thermal expectation value. Specifically, we measure the dual stiffness in the z direction evaluated at the smallest relevant wave vector in the x direction $\mathbf{q}_{\min}^x = (2\pi/L, 0, 0)$, i.e., $\rho^z(\mathbf{q}_{\min}^x)$; we denote this quantity simply as ρ . The quantity ρ will (in the thermodynamic limit) be zero in the Meissner state in which fluctuations of the magnetic field are suppressed, and nonzero in the normal phase. In the sense that it is zero in the low-temperature phase and nonzero in the high-temperature phase, it is thus a dual order parameter.

Generally, our estimates of superconducting critical temperatures are given by finite-size crossings of the quantity $L\rho$ (dual stiffness scaled by system size), extrapolated to the thermodynamic limit [9,25]. (Although the transitions we study are not necessarily continuous, we expect that such crossings will converge to the critical temperature.) We estimate the locations of the crossings by simple linear interpolation between simulated inverse temperatures. For some of the transitions we study, the crossings do not appear to be sufficiently well behaved for extrapolation to the thermodynamic limit to be possible. Therefore, we also consider the inflection points of $\rho(\beta)$, again extrapolated to the thermodynamic limit. We estimate the location of such an inflection point by reweighting from the simulated inverse temperature with the largest absolute value of the inverse-temperature derivative. In some cases, the quality of the data is not sufficient for this second method to work. In those cases in which both methods appear to work, the results are consistent.

B. Locating neutral transitions

The model has a neutral sector that we call the $SU(N)_n$ sector. Again, because there is no conserved U(1) topological invariant the order in that sector is not strictly speaking superfluid, in contrast to the composite order in $U(1)^N$ models [12,22]. In order to locate neutral transitions we use the helicity modulus, which measures the free-energy cost of an infinitesimal phase twist.

Consider imposing a twist in a certain linear combination $\sum_i a_i \phi_i$ of the phases, i.e., replacing the phase ϕ_i by

$$\phi'_i(\mathbf{r}) = \phi_i(\mathbf{r}) - a_i \delta \cdot \mathbf{r}. \quad (12)$$

The helicity modulus for this phase combination is then

$$\Upsilon_{\mu, \{a_i\}} = \frac{1}{L^3} \frac{\partial^2 F[\phi'_i]}{\partial \delta_\mu^2}, \quad (13)$$

where F is the free energy. Using the fundamental relations

$$F = -T \ln Z, \quad Z = \text{Tr } e^{-\beta H}, \quad (14)$$

one can derive an expression for the second derivative of the free energy F in terms of the first and second derivatives of the Hamiltonian H with respect to the phase-twist parameter δ :

$$\frac{\partial^2 F}{\partial \delta^2} = \left\langle \frac{\partial^2 H}{\partial \delta^2} \right\rangle - \beta \left\langle \left(\frac{\partial H}{\partial \delta} \right)^2 \right\rangle. \quad (15)$$

In deriving the above, we have used that $\langle \partial H / \partial \delta \rangle = 0$.

Looking at the Hamiltonian (4) it is clear that any first derivative $\partial H / \partial \delta$ will be a linear combination of sine functions, and any second derivative $\partial^2 H / \partial^2 \delta$ will be a linear combination of cosine functions. In both cases, the coefficients will depend on the a_i . Because of the square in the second term in Eq. (15), the helicity modulus for a given linear combination of phases will depend not only on the corresponding single-phase helicity moduli, but also on certain cross terms that involve pairs of components [15,42]. In our case the components are all equivalent, and consequently each of the single-phase helicity moduli are equal, as are each of the cross-term helicity moduli. Furthermore, because the phase sum couples to the vector potential, its helicity modulus is necessarily zero. From this it follows that the ratio between the value of a single-phase helicity modulus and the value of a cross-term helicity modulus is given by simple combinatorics, and that all helicity moduli are proportional to each other (the phase-sum helicity modulus being the special case where the constant of proportionality is zero).

Because of the aforementioned proportionality of all helicity moduli, we could, in principle, equivalently use any helicity modulus (except that for the phase sum) to locate neutral transitions. We choose to use the average of all single-phase moduli and the negatives of all cross-term moduli, as this would appear to minimize statistical error. We denote this quantity, when rescaled to equal a single-phase modulus, simple as Υ . In order to improve statistical precision, we determine a helicity modulus for a given temperature as a weighted average of the modulus determined from the data for that temperature and moduli obtained by reweighting from neighboring temperatures. We do this as follows: When reweighting from the nearest n temperatures on either

FIG. 1. Phase diagrams for $N = 2$ (blue, lower diagram) $N = 3$ (red, middle diagram) and $N = 4$ (green, upper diagram). For large values of β , i.e., low temperatures, the system is in a fully ordered state in which all (global and local) symmetries are broken. Decreasing β at a fixed low value of the gauge coupling q , the system undergoes a single (as far as numerical accuracy permits to judge) phase transition to a fully symmetric state. For high enough electric charge q there are new $SU(N)_n$ phases. The physical meaning of these phases is that order is retained in the phase-difference and relative-density degrees of freedom whilst superconducting order is absent. Errors are estimated to be smaller than symbol sizes, and lines are a guide to the eye.

side, the unweighted modulus has relative weight $n + 1$ and the relative weights of the other moduli decrease by 1 for each step away from the temperature to which reweighting is performed. Since the true value of the phase-sum helicity modulus is zero, the average over all temperatures of the squared phase-sum helicity modulus is a measure of the statistical error in the helicity moduli. The number n is chosen to be the first natural number for which $n + 1$ gives a larger error. However, we limit n to a maximal value of $n = 8$ in order to limit the computational power required. Also, in some cases we measure the helicity moduli in each coordinate direction and average them in order to further improve statistical precision. All this is motivated by the fact that for the systems studied here helicity moduli tend to have large statistical errors compared to other quantities that we study.

Our estimates of neutral critical temperatures are given by finite-size crossings of the quantity $L\Upsilon$ (helicity modulus scaled by system size), extrapolated to the thermodynamic limit. This is similar to how we use finite-size crossings of $L\rho$ in the superconducting case. Again, we use linear interpolation between simulated inverse temperatures to estimate the locations of the crossings.

C. Results

Phase diagrams for the two-, three- and four-component cases are shown in Fig. 1. Our main conclusion is that the systems have split transitions and phases with composite order for $N > 2$. This occurs for high values of the coupling constant; if the temperature is increased the system first undergoes a phase transition where the Meissner effect disappears. This phase transition is driven by excitations that disorder only the sum of the phases, leaving order in the phase differences

FIG. 2. Finite-size crossings of $L\Upsilon$ (helicity modulus scaled by system size) versus inverse temperature β (top) and of $L\rho$ (dual stiffness scaled by system size) versus inverse temperature (middle) as well as heat capacity $c = L^{-3} dE/dT$ versus inverse temperature (bottom) for $N = 3$, $q = 3$ and $L = 12, 16, 20, 24, 32, 40, 48$. Note that the scale for the inverse-temperature axis is different for the heat-capacity plot than for the other two plots; the crossings show much smaller finite-size corrections than the heat-capacity peaks. Error estimates are indicated by (narrow) shaded error regions.

and relative densities. Such excitations are vortices for which all phases wind by the same multiple of 2π . In the $SU(N)$ case, this type of vortex is not intrinsically stable in a non-fluctuating type-II system [26]; because of this aspect we call it nontopological, although it does have certain topological characteristics.

The remaining broken symmetry in the system is described by the following neutral terms in the original model:

$$h_{\text{neutral}} = \frac{1}{2} \sum_{i < j} |\psi_i|^2 |\psi_j|^2 (\nabla \phi_{ij})^2 + \frac{1}{2} \sum_i (\nabla |\psi_i|)^2. \quad (16)$$

These phases, in contrast to the $U(1)^N$ and $SU(2)$ cases, cannot be mapped onto $O(N)$ models. Instead these are $SU(N)_n$ phases where there is order associated with the spontaneous

FIG. 3. Finite-size crossings of $L\Upsilon$ (helicity modulus scaled by system size) versus inverse temperature β (top) and heat capacity versus inverse temperature (bottom) for $N = 3$, $q = 6$ and $L = 12, 16, 20, 24, 32, 40$. Error estimates are indicated by (narrow) shaded error regions.

FIG. 4. Finite-size crossings of $L\rho$ (dual stiffness scaled by system size) versus inverse temperature (top) and heat capacity versus inverse temperature (bottom) for $N = 3$, $q = 6$ and $L = 12, 16, 20, 24, 32, 40$. Error estimates are indicated by (narrow) shaded error regions.

FIG. 5. Histograms of action density $\beta H/L^3$ for temperatures in the vicinity of the neutral transition for $N = 4$ and $q = 7$; the temperatures chosen are those for which the bimodality is most pronounced. The bimodality becomes more pronounced with increasing system size, which shows that the transition is discontinuous.

breaking of symmetry only in the phase differences and relative densities.

For the superconducting phase transitions at the highest charge simulated ($q = 7$), the shifts in transition temperature with increasing component number are small, as compared to the corresponding shifts for the direct transitions at lower charges. The reason for this is that the superconducting transitions are driven by composite integer-flux excitations: when charge is increased, these objects become more tightly bound composites of fractional flux excitations [cf. the $U(1)^N$ case [13,40,43]].

We now present more detailed results for some representative examples of the phase transitions in the aforementioned phase diagrams. We choose to consider the case $N = 3$, and the charges $q = 3$ for which there is a single transition and $q = 6$ for which there are two separate transitions. In Fig. 2 we show the helicity moduli, dual stiffnesses, and heat capacities for $q = 3$ and system sizes in the range $L = 12-48$. Within our numerical uncertainty, the two transitions occur at the same temperature. Also, there are clear peaks in heat capacity that increase with system size. In Fig. 3 we show the helicity moduli and heat capacities for the neutral transition for $q = 6$ with system sizes in the range $L = 12-40$, and in Fig. 4 we show the heat capacities and dual stiffnesses for the charged transition in systems with the same charge and sizes. The two transitions are clearly separated in temperature. Again, the transitions are associated with clear peaks in heat capacity that become more pronounced with increasing system size.

An interesting question is that of the order of the neutral phase transition in the $SU(N)$ case [i.e., the transition from the symmetric phase to the $SU(N)_n$ phase]. A mean-field argument used for the $U(1) \times U(1)$ and $SU(2)$ cases indicates a second-order transition [8,24]. An argument for the $U(1) \times U(1)$ case [43] suggests that it may be first order, but no discontinuities have been resolved for systems with that symmetry. We find that in the $SU(4)$ case at $q = 7$ the neutral transition is first order. The evidence for this is bimodality in energy histograms, with the bimodality becoming more pronounced with increasing system size (Fig. 5).

V. CONCLUSION

$SU(N)$ theories with noncompact Abelian gauge fields are of interest as effective theories of various condensed-matter systems. The most investigated case is that of $SU(2)$. We have considered $SU(N)$ -symmetric Ginzburg-Landau models at finite temperature with a focus on the case $N > 2$.

We demonstrated that these systems have states with composite order that appear at sufficiently large coupling constants. The principle difference, as compared to the composite order in the previously discussed paired phases in $U(1) \times U(1)$ and $SU(2)$ theories, is that the order considered here cannot be mapped onto S^1 or S^2 neutral systems and is not described by a simple order parameter. Also, in the present case it is nontopological vortices that drive the transitions into these states, in contrast to the case of $U(1) \times U(1)$ systems. The new phases display no Meissner effect, but have a spontaneously broken symmetry associated with the relative phases φ_{ij} and relative densities $|\psi_i|/|\psi_j|$ of the components.

We term these phases $SU(N)_n$ states. These states result from proliferation of vortices that carry phase winding in all components. Such excitations disorder the phase sum and eliminate the Meissner effect, yet cannot disorder the $SU(N)_n$ sector of the model. Our results raise questions about other properties of this type of ordering, which may be realized with further progress in the creation of artificial gauge fields in $SU(N)$ ultracold atomic systems.

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