Untying links through anti-parity-time-symmetric coupling

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We reveal how the vector field links are untied under the influence of anti-parity-time-symmetric couplings in a dissipative sublattice-symmetric topological photonic crystal lattice. The topology of the quasi-one-dimensional two-band system is encoded in the geometric topology of the vector fields associated with the Bloch Hamiltonian. The linked vector fields reflect the topology of the nontrivial phase. The topological phase transition occurs concomitantly with the untying of the vector field link at the exceptional points. Counterintuitively, more dissipation constructively creates a nontrivial topology. The linking number predicts the number of topological photonic zero modes.

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I. INTRODUCTION

The robust topological structures of monopole, skyrmion, and vortex have attracted much research interest in optics, quantum physics, and condensed matter physics. The zero-intensity lines of light field in the form of isolated knotted nodal lines are observed as optical vortices in experiments [1]. The creation, observation, and investigation on the static and dynamical properties of quantum knots in Bose-Einstein condensates are reported [2]. Besides, topological phases are extremely stable [3–11]. In topological systems, zero-energy Fermi surfaces can form knotted or linked nodal lines [12–18]; alternatively, the fictitious magnetic field of a topological system with topological defects, associated with vortex or antivortex textures, reflects nontrivial topology [19,20]. In this Rapid Communication, a nontrivial topology is extracted from the spin polarization, which is associated with a two-band topological system and defines a vector field. The vector field exhibits a link or knot topology in the topologically nontrivial phase. We visualize the vortex links, present alterations of the vector field topology under the influence of anti-parity-time-symmetric (anti- $\mathcal{PT}$ -symmetric) couplings, and reveal the untying of vector field links associated with a topological phase transition at the exceptional point (EP) of the non-Hermitian topological system. Counterintuitively, we demonstrate that more dissipation constructively creates nontrivial topology.

II. DISSIPATIVE PHOTONIC CRYSTAL LATTICE

Photonic crystal is an excellent platform for the study of topological physics [6]. The non-Hermitian quasicrystal

[21,22], high-order topological phase [23–27], robust edge state [28–36], and topological lasing [37–43] in photonic crystals were discovered. Here, we consider a photonic crystal lattice as schematically illustrated in Fig. 1(a), which is implementable in many experimental platforms including the hybrid silicon microcavities [36–38], the optical waveguides [32,44,45], and the dielectric microwave resonators [29,39]. The auxiliary resonators in white and gray are evanescently coupled to the primary resonators in red and blue. The primary resonators have an identical resonant frequency ω_c . The auxiliary resonators induce the effective couplings between primary resonators. The blue, red, and white resonators are passive; the gray resonators have dissipation. The dissipation rate is much larger than the coupling strength between the gray auxiliary resonators and the colored primary resonators. The large dissipation enables the adiabatical elimination of the auxiliary resonator light field and results in an anti- $\mathcal{PT}$ -symmetric reciprocal coupling $-i\gamma$ between two adjacent primary resonators [46]; under the on-resonance condition, the coupling strength γ equals to the product of two evanescent couplings between the gray auxiliary resonator and its two adjacent primary resonators divided by the dissipation rate of the gray auxiliary resonator. The other two effective coupling strengths induced by the white auxiliary resonators are denoted as J and κ .

In coupled mode theory [47], the equations of motion for the coupled resonator array are

$$i\frac{d}{dt}\psi_{A,j} = \omega_c\psi_{A,j} - i\gamma\psi_{B,j} + J\psi_{B,j-1} + \kappa\psi_{B,j-2}, \quad (1)$$

$$i\frac{d}{dt}\psi_{B,j} = \omega_c\psi_{B,j} - i\gamma\psi_{A,j} + J\psi_{A,j+1} + \kappa\psi_{A,j+2}, \quad (2)$$

where $\psi_{A,j}$ and $\psi_{B,j}$ are the wave-function amplitudes for the corresponding sites j of the sublattices A and B . The schematic of the quasi-one-dimensional (quasi-1D) bipartite lattice is shown in Fig. 1(b).

Applying the Fourier transformation, the equations of motion for the eigenmode with momentum k

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FIG. 1. (a) Quasi-1D coupled resonator array. The primary resonators (in blue and red) are effectively coupled by the auxiliary resonators (in gray and white). (b) Schematic of (a). The open boundary of (b) corresponds to the sidelining cut of (a), indicated by the solid red lines.

reduce to

$$i \frac{d}{dt} \psi_{A,k} = \omega_c \psi_{A,k} + (-i\gamma + J e^{-ik} + \kappa e^{-2ik}) \psi_{B,k}, \quad (3)$$

$$i \frac{d}{dt} \psi_{B,k} = \omega_c \psi_{B,k} + (-i\gamma + J e^{ik} + \kappa e^{2ik}) \psi_{A,k}. \quad (4)$$

After the common resonant frequency ω_c is removed for both sublattices, the two-band system is described by the Bloch Hamiltonian $h_k = (\vec{d} - i\vec{\Gamma}) \cdot \vec{\sigma}$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrix for spin-1/2 and $\vec{d} - i\vec{\Gamma}$ is the fictitious magnetic field including the real part $\vec{d} = (J \cos k + \kappa \cos(2k), J \sin k + \kappa \sin(2k), 0)$ and the imaginary part $\vec{\Gamma} = (\gamma, 0, 0)$ [48,49]. Notably, the quasi-1D lattice and its Bloch Hamiltonian h_k are not anti- $\mathcal{PT}$ -symmetric. The two energy bands are

$$\varepsilon_k^\pm = \pm [(d_x - i\gamma)^2 + d_y^2]^{1/2}. \quad (5)$$

The corresponding eigenmodes are

$$|\psi_k^\pm\rangle = [1, (d_x - i\gamma + id_y)/\varepsilon_k^\pm]^T / \mathcal{N}, \quad (6)$$

which are normalized to unity $\langle \psi_k^\pm | \psi_k^\pm \rangle = 1$. The average values of Pauli matrices under the eigenmode yield a physical observable of spin polarization [19,20] and define a momentum dependent *unity* 3D vector field $\mathbf{F}_\pm(k) = (F_{\pm,x}(k), F_{\pm,y}(k), F_{\pm,z}(k))$. All of the three components $F_{\pm,x,y,z}(k) = \langle \psi_k^\pm | \sigma_{x,y,z} | \psi_k^\pm \rangle$ are gauge independent periodic functions of the momentum k and can be expressed as

$$\mathbf{F}_\pm(k) = (\sin \theta_\pm \cos \phi_\pm, \sin \theta_\pm \sin \phi_\pm, \cos \theta_\pm). \quad (7)$$

The information of the vector field is encoded in $(\theta_\pm, \phi_\pm)$. In knot theory, 1D topology describes 1D closed curves in 3D space [50].

III. LINKING TOPOLOGY

Figure 2 shows the vector fields $\mathbf{F}_+(k)$ and $\mathbf{F}_-(k)$ in different topological phases; they are mapped onto a torus as two closed curves, and their interrelation provides a visualization of the band topology of h_k : the linked vector field curves represent the nontrivial topology. In Fig. 2, the torus has a major radius of R_0 , which is the distance from the center of the tube to the center of the torus. r_0 is the minor radius of the tube. k is taken as the toroidal direction, and $\mathbf{R}_\pm(k) = (R_0 + r_\pm \cos \phi_\pm) \vec{e}_r + r_\pm \sin \phi_\pm \vec{e}_z$; $r_\pm = r_0 \sin \theta_\pm$ is selected to ensure that $\mathbf{R}_\pm(k)$ forms a closed curve. The curve detaches from the surface of the torus when $|r_\pm| \neq r_0$. The braiding of the two vector field curves is characterized by the linking number

$$L = \frac{1}{4\pi} \oint_k \oint_{k'} \frac{\mathbf{R}_+(k) - \mathbf{R}_-(k')}{|\mathbf{R}_+(k) - \mathbf{R}_-(k')|^3} \cdot [d\mathbf{R}_+(k) \times d\mathbf{R}_-(k')]. \quad (8)$$

Different links are topologically inequivalent and cannot be continuously deformed into one another without a topological phase transition.

FIG. 2. Vector field curves $\mathbf{F}_\pm(k)$ and energy bands for distinct topological phases. (a) $\gamma = 1$, (b) $\gamma = 1.274$, (c) $\gamma = 2$, (d) $\gamma = 2.715$, and (e) $\gamma = 3$. The linking numbers are $L = -2, -1$, and 0 for (a), (c), and (e); (b) and (d) are the gapless phases. Other parameters are $J = 1$ and $\kappa = 2$. The energy bands are fully complex except at the crossing points of their imaginary parts.

In the Hermitian case ($\gamma = 0$), the band energy is real and $F_{\pm,z}(k) = 0$; subsequently, $\theta_{\pm} = \pm\pi/2$ and $|r_{\pm}(k)| = r_0$. Both vector field curves are located on the surface of the torus. The orthogonality of eigenmodes in the Hermitian system yields $r_{\pm}(k) = \pm r_0$. In the presence of non-Hermiticity ($\gamma \neq 0$), the energy band is complex, and $\theta_{\pm} \neq \pm\pi/2$. The minor radius for the vector field curve is $|r_{\pm}(k)| \neq r_0$, because of the nonorthogonality of eigenmodes in the non-Hermitian system, and the two vector field curves detach from the torus surface.

In Fig. 2(a), the two vector field curves form a Solomon's knot with the linking number $L = -2$. As non-Hermiticity increases, two independent vector field curves meet at the EPs [51–53] and form a network [Fig. 2(b)]. The EPs are the nodes of the network with $\mathbf{F}_+(k) = \mathbf{F}_-(k)$ as a consequence of eigenmode coalescence. As non-Hermiticity increases, the curves reform as two independent closed loops; the vector field link changes through the reconnection associated with the EPs, and the linking number of the vector field curves is reduced by one, forming a Hopf link with linking number $L = -1$ [Fig. 2(c)]. This exhibits the untying of the link topology through the anti- $\mathcal{PT}$ -symmetric coupling. After another EP occurs at even greater anti- $\mathcal{PT}$ -symmetric coupling strength [Fig. 2(d)], the two vector field curves are unlinked [Fig. 2(e)]. The linking number associated with this phase is $L = 0$ and the band is topologically trivial. Notably, the anti- $\mathcal{PT}$ -symmetric coupling induces the topological phase transition at the EP, which dramatically differs from the topological phase transitions in other non-Hermitian systems with $\mathcal{PT}$ -symmetric gain and loss [28–34], as well as in non-Hermitian systems with asymmetric coupling strengths [54–59], where topological phase transition is irrelevant to the EP. Counterintuitively, increasing the dissipation can create nontrivial topology instead of behaving destructively: more dissipation reduces the strength of the anti- $\mathcal{PT}$ -symmetric coupling, tying vector field link and increasing the linking number.

The linking topology is closely related to the vortex associated with the topological defects in the vector field. For the photonic crystal lattice under consideration, the real part of the fictitious magnetic field satisfies $d_x(k) = d_x(-k)$, $d_y(k) = -d_y(-k)$, and $d_z(k) = 0$. The types of non-Hermiticities and their manners of appearance are important for the non-Hermitian topological systems [34,54]. If the non-Hermiticity is $\vec{\Gamma} = (0, 0, \gamma)$, the topological phase transition and the existence of the edge state are unaltered because of the pseudo-anti-Hermiticity protection [34]; the topological properties of the non-Hermitian system are inherited by the EPs (exceptional rings or exceptional surfaces in 2D or 3D) [60–67]. If the non-Hermiticity is $\vec{\Gamma} = (0, \gamma, 0)$, the non-Hermitian skin effect occurs under open boundary condition [54–59,68–78], the non-Hermitian Aharonov-Bohm effect under periodical boundary condition invalidates the conventional bulk-boundary correspondence [54], and the non-Bloch band theory is developed for topological characterization [77–82]. Here, the dissipation induced anti- $\mathcal{PT}$ -symmetric coupling corresponds to the imaginary part $\vec{\Gamma} = (\gamma, 0, 0)$. h_k lacks pseudo-anti-Hermiticity protection. The inversion symmetry of h_k ensures the absence of a non-Hermitian Aharonov-Bohm effect and the validity of conventional bulk-boundary corre-

FIG. 3. (a) Complex spectrum of h_k for $\vec{d} - i\vec{\Gamma} = (d_x - i\gamma, d_y, 0)$ at $\gamma = 2$. The color indicates the imaginary part. The solid cyan line is the bulk Fermi arc that connects two EPs (crosses). The black curve is the effective magnetic field for the photonic crystal lattice of Fig. 1 at $J = 1, \kappa = 2$; its projection is shown at the top of (b) $(\mathbf{F}_{+,x}, \mathbf{F}_{+,y})$. (c) Phase diagram. (d) Energy spectrum under open boundary.

spondence. The anti- $\mathcal{PT}$ -symmetric coupling γ in h_k splits the diabolic point ($d_x = d_y = 0$) in the Hermitian case ($\gamma = 0$) into a pair of EPs [cyan crosses in Fig. 3(a)]. As γ changes, the system switches among topologically trivial phases and different topologically nontrivial phases; therefore, the anti- $\mathcal{PT}$ -symmetric coupling induces topological phase transition and unties the vortex link.

Figure 3(a) depicts the complex energy bands for $h_k = [d_x(k) - i\gamma]\sigma_x + d_y(k)\sigma_y$ as functions of d_x and d_y when $\gamma = 2$ is fixed. The Riemann surface spectrum has two EPs connected by the bulk Fermi arc, where the real part of the band gap closes. Two EPs located at $(0, \pm\gamma)$ in the d_x - d_y plane are split from the diabolic point [at the origin $(0, 0)$] for $\gamma = 0$. The integer topological charge associated with the diabolic point splits into two equal half-integer topological charges [51] in the presence of anti- $\mathcal{PT}$ -symmetric coupling. The two-component planar vector field $(F_{+,x}(k), F_{+,y}(k))$ is depicted in Fig. 3(b) [83]. The linking number of the vector field curves $\mathbf{F}_{\pm}(k)$ equals the winding number of the planar vector field, $w_{\pm} = (2\pi)^{-1} \int_0^{2\pi} \nabla_k \phi_{\pm}(k) dk = -L$, where $\phi_{\pm}(k) = \arctan[F_{\pm,y}(k)/F_{\pm,x}(k)]$. The half-integer topological charge associated with the EP is revealed from the argument of the planar vector field $\mathbf{F}_{\pm}(k)$. The topological charge associated with the diabolic point or EP is equal to the global Zak phase [33], which is the summation of the winding numbers of the real fictitious magnetic field loop encircling each individual EP. Notably, the global Zak phase can always be zero and invalid for topological characterization at an inappropriate gauge of the eigenmodes [84]. Thus, topological

characterization employing the vector field also benefits from its gauge independent feature.

The topological phase transition occurs at the EPs, where the band gap closes at exact zero energy. The topological phase transition points are

$$\gamma^2 = \kappa^2 + J^2/2 \pm (J/2)\sqrt{J^2 + 8\kappa^2}. \quad (9)$$

The phase diagram in the κ - γ plane is shown in Fig. 3(c). The black curve in Fig. 3(a) indicates one energy branch of the photonic crystal lattice. The projection of the black curve in the d_x - d_y plane is the real fictitious magnetic field loop $\vec{d}$ [Fig. 3(b)], which exhibits time-reversal symmetry and mirror symmetry with respect to the horizontal axis, $d_x(k) = d_x(-k)$ and $d_y(k) = -d_y(-k)$. The real fictitious magnetic field loop $\vec{d}$ passes the EPs $(0, \pm\gamma)$ at the gapless phase. In Fig. 3(b), the colored crosses indicate the EPs, and the colored lines indicate the bulk Fermi arcs at different γ . Each time the loop $\vec{d}$ comes across the bulk Fermi arc, the real part of the energy gap closes and two eigenmodes switch [85,86]. For weak non-Hermiticity, the real part of the energy bands is gapped; the fictitious magnetic field loop passes $d_x(k) = 0$ four times as a result of the winding number $w_{\pm} = 2$. Consequently, the magnetic field loop encircles the two EPs (green crosses) twice, $F_{\pm}(k)$ form a Solomon's knot, and the imaginary part of the energy band intersects four times [Fig. 2(a)] at fixed momenta $d_x(k) = 0$ independent of non-Hermiticity. As non-Hermiticity increases, the band gap diminishes; the vector field loop passing $(0, \pm d_y(k_0))$ may cross the bulk Fermi arc when $|\gamma| > |d_y(k_0)|$. The crossing indicates that the band gap of the real energy closes, and the real part instead of the imaginary part of the energy band crosses at $k = \pm k_0$; this occurs for large non-Hermiticity. We notice two and four crossings in the real part of the energy bands in Figs. 2(c) at $\gamma = 2$ and 2(e) at $\gamma = 3$, respectively. In Fig. 2(c), the magnetic field loop encircles the two EPs (cyan crosses) once; in Fig. 2(e), the magnetic field loop does not encircle either of the two EPs (blue crosses).

IV. TOPOLOGICAL PHOTONIC ZERO MODES

Topological phase transition is inextricably related to the change of topological photonic edge modes. Untying the link by one reduces one pair of edge modes. Figure 1 is a quasi-1D coupled Su-Schrieffer-Heeger chain; the termination under the open boundary condition is indicated by the red lines, the open boundary spectrum as a function of the anti- $\mathcal{PT}$ -symmetric coupling strength is depicted in Fig. 3(d). The edge modes are zero modes, which are analytically solvable at the limitation of infinite size, and satisfy the steady-state equations of motion

$$0 = -i\gamma\psi_{A,j} + J\psi_{A,j+1} + \kappa\psi_{A,j+2}, \quad (10)$$

$$0 = -i\gamma\psi_{B,j} + J\psi_{B,j-1} + \kappa\psi_{B,j-2}. \quad (11)$$

The edge modes decay to zero from one boundary to the other. The left (right) edge modes localize within the sublattice A (B) in blue (red). The steady-state equations of motion are recurrence formulas. The localized edge modes can be regarded as

FIG. 4. Edge states in the topologically nontrivial phases. The parameters are (a) $\gamma = 1$ with $|\rho_+| < 1$, (b) $\gamma = 1$ with $|\rho_-| < 1$, (c) $\gamma = 1.274$, and (d) $\gamma = 2$. Other parameters are $J = 1$ and $\kappa = 2$. $P(j) = \langle \psi_j | \psi_j \rangle$. The blue (red) bar indicates the probability of the left (right) photonic edge modes on the sublattice A (B).

plane waves $e^{\pm ikj}$ with complex momenta k , which are zeros of the complex function

$$f(\rho) \equiv -i\gamma + J\rho + \kappa\rho^2 = 0, \quad (12)$$

where $\rho = e^{\pm ik_0} = -(J \pm \sqrt{4i\kappa\gamma + J^2})/(2\kappa)$ yields the localization length of the edge mode $\xi = |\text{Im}(k_0)|^{-1}$ [87]. The number of edge modes at either boundary is equal to the number of zeros $f(\rho) = 0$ within the region $|\rho| < 1$, being identical to the linking number of the two vector field links. The gapless phase is the boundary of two gapped phases, where topological phase transition occurs. The number of topologically protected edge modes at the gapless phase is equal to that at the gapped phase with a small winding number; two (zero) topological edge modes present in the gapless phase between topological phases $L = -2$ ($L = 0$) and $L = -1$. The left edge mode amplitude is $\psi_{A,j} = \rho^j$ and the right edge mode amplitude is $\psi_{B,N+1-j} = \rho^j$. The two pairs of edge modes at $\gamma = 1$ are depicted in Figs. 4(a) and 4(b), respectively. The pair of edge modes at $\gamma = 1.274$ and $\gamma = 2$ are depicted in Figs. 4(c) and 4(d).

V. CONCLUSIONS

We investigate a dissipative quasi-1D topological photonic crystal lattice of coupled resonators. The coupled resonator array exhibits sublattice symmetry and the dissipation induced anti- $\mathcal{PT}$ -symmetric couplings alternatively presented. Counterintuitively, more dissipation constructively generate nontrivial topology from trivial topology. The nontrivial topology is established from the linking geometry of the vector fields, which are the average values of the Pauli matrices under the Bloch Hamiltonian. The anti- $\mathcal{PT}$ -symmetric coupling unties the vector field links, alters the band topology, and induces topological phase transition at the EPs. The linking number determines the number of topologically protected

photonic zero modes. The dissipation may not affect the dissipation rate of the photonic zero modes although it may alter the band topology. Our findings deepen the understanding of the influence of dissipation induced anti- $\mathcal{PT}$ -symmetric coupling and the non-Hermitian 1D topology.

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