Erratum: Higher-order moments, cumulants, and spectra of continuous quantum noise measurements [Phys. Rev. B **98, 205143 (2018)]**

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We wrongly claimed in our derivation of the four-time cumulant $C_4(z(t_1), z(t_2), z(t_3), z(t_4))$ that the terms α_{13} and α_{31} cancel in the original paper. A correct treatment starts with (we use the short notation of Appendix F)

$$\alpha_{13} + \alpha_{31} = T(AG'_{43}A\rho_0)T(AG'_{21}A\rho_0)\theta_{32} - T(AG'_{43}A\rho_0)T(AG'_{21}A\rho_0)/2. \quad (1)$$

The contribution of the sum $\text{Sum}(\alpha_{13} + \alpha_{31})$ to C_4 contains all 24 permutations of the indices 1 to 4. Assuming a preliminary time-order $t_4 > t_3 > t_2 > t_1$, one finds that most terms vanish [e.g., $-T(AG'_{23}A\rho_0)T(AG'_{41}A\rho_0)/2$ is zero since $G'_{23} = 0$ due to $t_3 > t_2$]. The surviving terms give rise to

$$\begin{aligned} \text{Sum}(\alpha_{13} + \alpha_{31}) &= -T(AG'_{42}A\rho_0)T(AG'_{31}A\rho_0) - T(AG'_{41}A\rho_0)T(AG'_{32}A\rho_0) \\ &= -T(AG'_{43}G'_{32}A\rho_0)T(AG'_{32}G'_{21}A\rho_0) - T(AG'_{43}G'_{32}G'_{21}A\rho_0)T(AG'_{32}A\rho_0). \end{aligned} \quad (2)$$

The notation in the last line guarantees that, for all time orders other than $t_4 > t_3 > t_2 > t_1$, the right-hand side is zero. We, therefore, can relax the restriction $t_4 > t_3 > t_2 > t_1$ and are left with

$$\text{Sum}(\alpha_{13} + \alpha_{31}) = -\text{Sum}[T(AG'_{43}G'_{32}A\rho_0)T(AG'_{32}G'_{21}A\rho_0)] - \text{Sum}[T(AG'_{43}G'_{32}G'_{21}A\rho_0)T(AG'_{32}A\rho_0)], \quad (3)$$

which is valid for all time orders. The correct expression for the four-time cumulant [Eq. (100) in our paper] follows as:

$$\begin{aligned} C_4(z(t_1), z(t_2), z(t_3), z(t_4)) &= \beta^8 \sum_{\text{prm. } t_j} \text{Tr}[\mathcal{A}'\mathcal{G}'(t_4 - t_3)\mathcal{A}'\mathcal{G}'(t_3 - t_2)\mathcal{A}'\mathcal{G}'(t_2 - t_1)\mathcal{A}'\rho_0] \\ &\quad - \beta^8 \sum_{\text{prm. } t_j} \text{Tr}[\mathcal{A}'\mathcal{G}'(t_4 - t_3)\mathcal{G}'(t_3 - t_2)\mathcal{A}'\rho_0]\text{Tr}[\mathcal{A}'\mathcal{G}'(t_3 - t_2)\mathcal{G}'(t_2 - t_1)\mathcal{A}'\rho_0] \\ &\quad - \beta^8 \sum_{\text{prm. } t_j} \text{Tr}[\mathcal{A}'\mathcal{G}'(t_4 - t_3)\mathcal{G}'(t_3 - t_2)\mathcal{G}'(t_2 - t_1)\mathcal{A}'\rho_0]\text{Tr}[\mathcal{A}'\mathcal{G}'(t_3 - t_2)\mathcal{A}'\rho_0]. \end{aligned} \quad (100)$$

with additional lines two and three originating from $\alpha_{13} + \alpha_{31} \neq 0$. The superoperator $\mathcal{A}$ in Eq. (3) was replaced in Eq. (4) by $\mathcal{A}' = \mathcal{A} - \text{Tr}(\mathcal{A}\rho_0)$ with the same arguments that were given in the original paper before Eq. (99). The corresponding fourth-order spectrum $S^{(4)}$ (trispectrum) [Eq. (109)] follows via Fourier transformation of the four-time C_4 (compare Sec. XIV) as

$$\begin{aligned} S^{(4)}(\omega_1, \omega_2, \omega_3, \omega_4 = -\omega_1 - \omega_2 - \omega_3) &= \beta^8 \sum_{\text{prm. } \omega_1, \omega_2, \omega_3, \omega_4} \text{Tr}[\mathcal{A}'\mathcal{G}'(\omega_4)\mathcal{A}'\mathcal{G}'(\omega_3 + \omega_4)\mathcal{A}'\mathcal{G}'(\omega_2 + \omega_3 + \omega_4)\mathcal{A}'\rho_0] \\ &\quad - \frac{\beta^8}{2\pi} \sum_{\text{prm. } \omega_1, \omega_2, \omega_3, \omega_4} \int \text{Tr}[\mathcal{A}'\mathcal{G}'(\omega_4)\mathcal{G}'(\omega_3 + \omega_4 - \omega)\mathcal{A}'\rho_0]\text{Tr}[\mathcal{A}'\mathcal{G}'(\omega)\mathcal{G}'(\omega_2 + \omega_3 + \omega_4)\mathcal{A}'\rho_0] d\omega \\ &\quad - \frac{\beta^8}{2\pi} \sum_{\text{prm. } \omega_1, \omega_2, \omega_3, \omega_4} \int \text{Tr}[\mathcal{A}'\mathcal{G}'(\omega_4)\mathcal{G}'(\omega_2 + \omega_3 + \omega_4)\mathcal{G}'(\omega_3 + \omega_4 - \omega)\mathcal{A}'\rho_0]\text{Tr}[\mathcal{A}'\mathcal{G}'(\omega)\mathcal{A}'\rho_0] d\omega, \end{aligned} \quad (109)$$

where the additional terms $S^{(4,2)}$ and $S^{(4,3)}$ (lines two and three, respectively) require the calculation of a convolution integral.

The *numerical* evaluation of, e.g., $S^{(4,3)}$ is, however, more conveniently performed in the following way: $S^{(4,3)}$ is obtained from the function g_3 (instead of g , compare Sec. XIV),

$$g_3(\tau_1, \tau_2, \tau_3) = -\text{Tr}[\mathcal{A}'\mathcal{G}'(\tau_3 + \tau_2 + \tau_1)\mathcal{A}'\rho_0]\text{Tr}[\mathcal{A}'\mathcal{G}'(\tau_2)\mathcal{A}'\rho_0], \quad (4)$$

FIG. 1. (a) Replacement of Fig. 4(a): Fourth-order correlation spectrum $S^{(4)}(\omega_1, -\omega_1, \omega_2, -\omega_2)$ of the spin-spin system in ZnO:In at $B_x = 0$ mT. (b) Contribution of $S^{(4,2)}$ and $S^{(4,3)}$.

via its Fourier transformation $\tilde{g}_3(\nu_1, \nu_2, \nu_3)$. For a numerical evaluation of $\tilde{g}_3$, we rewrite g_3 (for a definition of Λ , see our discussion in Sec. XV),

$$g_3(\tau_1, \tau_2, \tau_3) = -\beta^8 \text{Tr}[\mathcal{A}' \Lambda \mathcal{D}(\tau_3 + \tau_2 + \tau_1) \Lambda^{-1} \mathcal{A}' \rho_0] \text{Tr}[\mathcal{A}' \Lambda \mathcal{D}(\tau_2) \Lambda^{-1} \mathcal{A}' \rho_0], \quad (5)$$

with diagonal matrix $[\mathcal{D}(\tau)]_{k,k} = e^{\lambda_k \tau}$ and $[\mathcal{D}(\tau)]_{m,m} = 0$ if $\lambda_m = 0$. We introduce the matrix $(\mathcal{S}_k)_{i,j} = \delta_{i,k} \delta_{j,k} (1 - \delta_{m,k})$ with a single nonzero entry at (k, k) for $k \neq m$. After defining

$$s_k = \text{Tr}[\mathcal{A}' \Lambda \mathcal{S}_k \Lambda^{-1} \mathcal{A}' \rho_0], \quad (6)$$

we can write

$$g_3(\tau_1, \tau_2, \tau_3) = -\beta^8 \sum_{k \neq m, l \neq m} s_k s_l e^{\lambda_k(\tau_3 + \tau_2 + \tau_1)} e^{\lambda_l \tau_2}. \quad (7)$$

The Fourier transformation then obtains the form

$$\tilde{g}_3(\nu_1, \nu_2, \nu_3) = -\beta^8 \sum_{k \neq m, l \neq m} s_k s_l \frac{-1}{(\lambda_k + i\nu_1)(\lambda_k + i\nu_3)(\lambda_k + \lambda_l + i\nu_2)}. \quad (8)$$

Consequently, the contribution $S^{(4,3)}$ to $S^{(4)}$ is (compare Sec. XIV),

$$S^{(4,3)}(\omega_1, \omega_2, \omega_3, \omega_4) = \beta^8 \sum_{k \neq m} \sum_{l \neq m \text{ perm. } \omega_j} s_k s_l \frac{1}{(\lambda_k + i\nu_1)(\lambda_k + i\nu_3)(\lambda_k + \lambda_l + i\nu_2)} \Big|_{\nu_1=\omega_2+\omega_3+\omega_4, \nu_2=\omega_3+\omega_4, \nu_3=\omega_4}. \quad (9)$$

FIG. 2. (a) Replacement of Fig. 6 (0°): Fourth-order correlation spectrum $S^{(4)}(\omega_1, -\omega_1, \omega_2, -\omega_2)$ of the spin-spin system in ZnO:In at $B = 10$ mT. (b) Contribution of $S^{(4,2)} + S^{(4,3)}$.

FIG. 3. (a) Replacement of Fig. 6 (30°): Fourth-order correlation spectrum $S^{(4)}(\omega_1, -\omega_1, \omega_2, -\omega_2)$ of the spin-spin system in ZnO:In at $B = 10$ mT. (b) Contribution of $S^{(4,2)} + S^{(4,3)}$.

A very similar calculation for $S^{(4,2)}$ yields

$$g_2(\tau_1, \tau_2, \tau_3) = -\beta^8 \sum_{k \neq m, l \neq l} s_k s_l e^{\lambda_k(\tau_3 + \tau_2)} e^{\lambda_l(\tau_2 + \tau_1)}, \quad (10)$$

and, eventually,

$$S^{(4,2)}(\omega_1, \omega_2, \omega_3, \omega_4) = \beta^8 \sum_{k \neq m, l \neq m \text{ perm. } \omega_j} s_k s_l \frac{1}{(\lambda_l + i\nu_1)(\lambda_k + i\nu_3)(\lambda_k + \lambda_l + i\nu_2)} \Big|_{\nu_1=\omega_2+\omega_3+\omega_4, \nu_2=\omega_3+\omega_4, \nu_3=\omega_4}. \quad (11)$$

The same expression for s_k [Eq. (6)] is used for both, $S^{(4,2)}$ and $S^{(4,3)}$ and needs to be evaluated only once for each k .

Application of the correct expression Eq. (109) for $S^{(4)}$ to the spin-spin system in ZnO:In leads to only small changes as compared to the incomplete previous expression. The largest change is found for $B = 0$ mT where a correction at $(\omega_1, \omega_2) = (0.5, 0.5)$ GHz occurs [Fig. 1(b)]. Very small changes occur for $B = 10$ mT at out-of-plane angles of 0° and 30° , see Figs. 2 and 3. The correction is located in the spectrum around (0.025, 0.025) GHz and assumes both positive and negative values. The conclusions of our paper are not affected as the corrections are always very small for finite magnetic fields.

C_4 and $S^{(4)}$ are within the very general stochastic master equation approach the correct expressions for a continuous quantum measurement of the operator $\mathcal{A}$ on a system ρ_0 (equilibrium with environment and measurement apparatus) for arbitrary measurement strengths β . The corresponding expressions for C_2 , $S^{(2)}$, C_3 , and $S^{(3)}$ can be found in the original paper.