Courant-Snyder formalism of longitudinal dynamics

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Originating from the stochastic nature of the photon emission process, the diffusion of the electron longitudinal coordinate exists even if the global phase slippage of a storage ring is zero, as we cannot zero all the local phase slippages simultaneously. This quantum diffusion is viewed as the most fundamental limit of the lowest bunch length realizable in an electron storage ring from the single-particle dynamics perspective. Here, we present an analysis of this effect using the Courant-Snyder parametrization in the longitudinal dimension. Analytical formulas for the longitudinal emittance, energy spread, and bunch length are derived. The same formalism is used to discuss the application of multiple radio frequency systems for longitudinal strong focusing. The presented work is expected to be useful in the development of novel light source mechanisms like steady-state microbunching, where a short bunch length or small longitudinal emittance is needed.

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I. INTRODUCTION

Short bunches and small longitudinal emittance are desired by synchrotron light source and lepton collider communities for coherent radiation generation and luminosity increase. One successful method of realizing short bunches in an electron storage ring is implementing a quasi-isochronous (low- α) lattice. The reason is the well-known $\sqrt{|\eta|}$ scaling law of the “zero-current” bunch length given by Sands [1], in which $\eta = \alpha - \frac{1}{\gamma^2}$ is the phase slippage factor and $\alpha = \frac{1}{C_0} \oint \frac{D_x(s)}{\rho(s)} ds$ is the momentum compaction factor of the ring, with γ the relativistic factor, C_0 the circumference of the ring, D_x the horizontal dispersion, and ρ the bending radius. However, from the single-particle dynamics perspective, there is a fundamental effect limiting the lowest bunch length realizable in an electron storage ring originating from the stochasticity of photon emission time, which was first originally analyzed by Shoji *et al.* [2,3]. This stochasticity results in a diffusion of the electron longitudinal coordinate z even if the global phase slippage of the ring is zero, as we cannot make all the local or partial phase slippages zero simultaneously. Therefore, there exists a bunch length limit, and the energy spread grows significantly when the bunch length is pushed

close to the limit. This quantum diffusion is of vital importance for new light source concepts like steady-state microbunching (SSMB) [4–13], which is now under active study, and other ideas invoking ultrashort electron bunches in storage rings.

While the treatment of this quantum excitation is intrinsically embedded in Chao’s solution by linear matrices (SLIM) formalism [14], which invokes 6×6 general transport matrices and applies to a 3D general coupled lattice without the assumption of longitudinal weak focusing, analytical formulas for this effect are still highly desired. The motivation is that these analytical results can help us to better understand its physics and the method to control this quantum diffusion to minimize the longitudinal emittance and bunch length, for example, in an SSMB storage ring. In this paper, we apply the longitudinal Courant-Snyder formalism in a planar uncoupled ring for the analysis of this effect, which can be viewed as an application of the SLIM formalism where a concise analytical result can be obtained. Formulas for the equilibrium longitudinal emittance, energy spread, and bunch length are derived, based on which some important points of this effect are analyzed. The application of multiple radio frequency (rf) cavities, or laser modulators in an SSMB storage ring, for longitudinal strong focusing are discussed using the same formalism with important observations made.

II. COURANT-SNYDER FORMALISM

The quantum diffusion in z in a storage ring emphasizes the fact that we should consider the quantum excitation on the longitudinal emittance rather than that on the particle

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energy alone. The key to understanding the effect is to change from the global viewpoint to a local one; i.e., the quantum excitation at different places around the ring contributes to the longitudinal emittance with different strengths, just like its transverse counterpart. For an accurate analysis of this effect, here we apply Chao's SLIM formalism [14]. 6D state vector $\mathbf{X} = (x, x', y, y', z, \delta)^T$ of the electron is used, with the superscript T representing the transpose. According to SLIM, the equilibrium eigenemittances and second moments of the beam in an electron storage ring are

$$\epsilon_k = \langle |A_k|^2 \rangle = \langle |A_{-k}|^2 \rangle = \frac{C_L \gamma^5}{c \alpha_k} \oint \frac{|\mathbf{E}_{k5}(s)|^2}{|\rho(s)|^3} ds \quad (1)$$

and

$$\Sigma_{ij}(s) = \langle \mathbf{X}_i \mathbf{X}_j \rangle(s) = 2 \sum_{k=I,II,III} \epsilon_k \text{Re}[\mathbf{E}_{ki}(s) \mathbf{E}_{kj}^*(s)], \quad (2)$$

respectively, where c is the speed of light in free space, α_k are the damping constants of the three eigenmodes, $C_L = 55 r_e \hbar / (48 \sqrt{3} m_e)$, with $\hbar$ the reduced Planck's constant, r_e the electron classical radius, and m_e the electron rest mass, $\mathbf{E}_k$ are the eigenvectors of the linear one-turn map $\mathbf{M}$, $*$ denotes complex conjugate, and $\text{Re}[\]$ means taking the real part of a complex number. The eigenemittances ϵ_k are defined as the positive eigenvalues of $i\Sigma\mathbf{S}$, where i is the imaginary unit, $\Sigma = \langle \mathbf{X}\mathbf{X}^T \rangle$ is the second moments matrix, and $\mathbf{S}$ is the symplectic form [15]. The subscripts I, II, III represent the three eigenmodes and in the planar uncoupled case correspond to the usual horizontal, vertical, and longitudinal dimensions x, y, z .

To simplify the discussion, here we consider only the horizontal and longitudinal dimensions and use the state vector $\mathbf{X} = (x, x', z, \delta)^T$. Under the assumptions that the ring is planar uncoupled and the rfs are placed at dispersion-free locations, which is the typical setup for present synchrotron sources, the betatron coordinate $\mathbf{X}_\beta = \mathbf{B}\mathbf{X}$ can be introduced to parametrize the transfer matrix in a diagonal form, with the dispersion matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & -D_x \\ 0 & 1 & 0 & -D'_x \\ D'_x & -D_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

The one-turn map $\mathbf{M}$ of $\mathbf{X}$ is related to the one-turn map $\mathbf{M}_\beta$ of $\mathbf{X}_\beta$ by

$$\mathbf{M} = \mathbf{B}^{-1} \mathbf{M}_\beta \mathbf{B}, \quad \mathbf{M}_\beta = \begin{pmatrix} \mathbf{M}_{x\beta} & 0 \\ 0 & \mathbf{M}_{z\beta} \end{pmatrix}, \quad (4)$$

with

$$\mathbf{M}_{x,z\beta} = \begin{pmatrix} \cos \Phi_{x,z} + \alpha_{x,z} \sin \Phi_{x,z} & \beta_{x,z} \sin \Phi_{x,z} \\ -\gamma_{x,z} \sin \Phi_{x,z} & \cos \Phi_{x,z} - \alpha_{x,z} \sin \Phi_{x,z} \end{pmatrix}, \quad (5)$$

in which $\Phi_x = 2\pi\nu_x$ and $\Phi_z = 2\pi\nu_z$ are the betatron and synchrotron phase advance per turn, respectively. The eigenvectors of $\mathbf{M}_\beta$ can be expressed using the Courant-Snyder functions [16] as

$$\mathbf{v}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\beta_x} \\ \frac{i-\alpha_x}{\sqrt{\beta_x}} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{\beta_z} \\ \frac{i-\alpha_z}{\sqrt{\beta_z}} \end{pmatrix}. \quad (6)$$

Therefore, the eigenvectors of $\mathbf{M}$ are

$$\mathbf{E}_x = \mathbf{B}^{-1} \mathbf{v}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\beta_x} \\ \frac{i-\alpha_x}{\sqrt{\beta_x}} \\ -\sqrt{\beta_x} D'_x + \frac{i-\alpha_x}{\sqrt{\beta_x}} D_x \\ 0 \end{pmatrix},$$

$$\mathbf{E}_z = \mathbf{B}^{-1} \mathbf{v}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i-\alpha_z}{\sqrt{\beta_z}} D_x \\ \frac{i-\alpha_z}{\sqrt{\beta_z}} D'_x \\ \sqrt{\beta_z} \\ \frac{i-\alpha_z}{\sqrt{\beta_z}} \end{pmatrix}. \quad (7)$$

According to SLIM, the equilibrium horizontal and longitudinal emittance are then

$$\epsilon_x = \frac{55}{96\sqrt{3}} \frac{\alpha_F \lambda_e^2 \gamma^5}{\alpha_H} \oint \frac{\mathcal{H}_x(s)}{|\rho(s)|^3} ds,$$

$$\epsilon_z = \frac{55}{96\sqrt{3}} \frac{\alpha_F \lambda_e^2 \gamma^5}{\alpha_L} \oint \frac{\beta_z(s)}{|\rho(s)|^3} ds, \quad (8)$$

respectively, in which α_H and α_L are the horizontal and longitudinal damping constants, respectively, $\alpha_F = \frac{1}{137}$ is the fine structure constant, $\lambda_e = \lambda_e / 2\pi = 386$ fm is the reduced Compton wavelength of the electron, and $\mathcal{H}_x = \gamma_x D_x^2 + 2\alpha_x D_x D'_x + \beta_x D'^2_x$ is the horizontal chromatic function.

After getting the eigenemittances, we can obtain the second moments of the beam according to Eqs. (2) and (7); more specifically,

$$\begin{aligned}\Sigma &= \mathbf{B}^{-1} \Sigma_\beta (\mathbf{B}^{-1})^T \\ &= \begin{pmatrix} \epsilon_x \beta_x + \epsilon_z \gamma_z D_x^2 & -\epsilon_x \alpha_x + \epsilon_z \gamma_z D_x D'_x & -(\epsilon_x \alpha_x + \epsilon_z \alpha_z) D_x - \epsilon_x \beta_x D'_x & \epsilon_z \gamma_z D_x \\ -\epsilon_x \alpha_x + \epsilon_z \gamma_z D_x D'_x & \epsilon_x \gamma_x + \epsilon_z \gamma_z D_x'^2 & \epsilon_x \gamma_x D_x + (\epsilon_x \alpha_x - \epsilon_z \alpha_z) D'_x & \epsilon_z \gamma_z D'_x \\ -(\epsilon_x \alpha_x + \epsilon_z \alpha_z) D_x - \epsilon_x \beta_x D'_x & \epsilon_x \gamma_x D_x + (\epsilon_x \alpha_x - \epsilon_z \alpha_z) D'_x & \epsilon_z \beta_z + \epsilon_x \mathcal{H}_x & -\epsilon_z \alpha_z \\ \epsilon_z \gamma_z D_x & \epsilon_z \gamma_z D'_x & -\epsilon_z \alpha_z & \epsilon_z \gamma_z \end{pmatrix}, \quad (9)\end{aligned}$$

where $\Sigma_\beta = \langle \mathbf{X}_\beta \mathbf{X}_\beta^T \rangle$. The distribution of a Gaussian beam is related to the second moments matrix according to

$$\begin{aligned}\psi(\mathbf{X}) &= \frac{1}{(2\pi)^2 \sqrt{\det \Sigma}} \exp \left(-\frac{1}{2} \mathbf{X}^T \Sigma^{-1} \mathbf{X} \right) \\ &= \frac{1}{(2\pi)^2 \epsilon_x \epsilon_z} \exp \left(-\frac{I_x}{\epsilon_x} - \frac{I_z}{\epsilon_z} \right), \quad (10)\end{aligned}$$

with $I_x = [\gamma_x(x - D_x \delta)^2 + 2\alpha_x(x - D_x \delta)(x' - D'_x \delta) + \beta_x(x' - D'_x \delta)^2]/2$ and $I_z = [\gamma_z(z + D'_x x - D_x x')^2 + 2\alpha_z(z + D'_x x - D_x x')\delta + \beta_z \delta^2]/2$.

III. SINGLE rf

A. Classical $\sqrt{|\eta|}$ scaling

Now we first reproduce the classical $\sigma_z \propto \sqrt{|\eta|}$ scaling using this longitudinal Courant-Snyder parameterization. To simplify the discussion further, in this section and the following, we focus on the longitudinal dimension only and the state vector $\mathbf{X} = (z, \delta)^T$ is used. We treat first the case where there is only one rf placed at a dispersion-free location. In this case, the linear longitudinal one-turn map in the middle of the rf cavity is

$$\begin{aligned}\mathbf{M} &= \begin{pmatrix} 1 & 0 \\ \frac{h}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\eta C_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{h}{2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{h}{2} \eta C_0 & -\eta C_0 \\ h - (\frac{h}{2})^2 \eta C_0 & 1 - \frac{h}{2} \eta C_0 \end{pmatrix} \quad (11)\end{aligned}$$

with $h = eV_{\text{rf}} k_{\text{rf}} \cos \phi_s / E_0$ quantifying the rf gradient, where e is the elementary charge, V_{rf} is the rf voltage, $k_{\text{rf}} = 2\pi/\lambda_{\text{rf}}$ is the rf wave number, ϕ_s is the synchronous phase, and $E_0 = \gamma m_e c^2$ is the electron energy. The $R_{56} = -\eta C_0$, a measure for the dependence of z on δ , of the ring and the rf kick h can be viewed as the longitudinal drift space and quadrupole, in correspondence to their transverse counterparts, respectively. Note, however, that the R_{56} can be either positive or negative, while the physical length of a drift space is always positive.

For rings working in the longitudinal weak focusing regime, $|\nu_s| \ll 1$, we then have

$$\begin{aligned}1 - \frac{h}{2} \eta C_0 &= \cos \Phi_z \approx 1 - \frac{\Phi_z^2}{2} \Rightarrow \Phi_z \\ &\approx \begin{cases} -\sqrt{h\eta C_0} & \text{if } \eta > 0, \\ \sqrt{h\eta C_0} & \text{if } \eta < 0. \end{cases} \quad (12)\end{aligned}$$

Therefore, β_z at the rf center is

$$\beta_{zS} = \frac{\mathbf{M}_{12}}{\sin \Phi_z} \approx \frac{-\eta C_0}{\Phi_z} \approx \sqrt{\eta C_0 / h}. \quad (13)$$

Here, we use the subscript S to denote that the result is the same with that obtained in Sands' analysis [1], although the method used here is different. As $|\nu_s| \ll 1$, therefore,

$$\beta_{zS} \gg |-\eta C_0|. \quad (14)$$

We will see later in Sec. IV that, in a longitudinal strong focusing ring, $|\nu_s|$ can be close to or even larger than 1, and β_z then can be the same level of or smaller than $|-\eta C_0|$.

Using this β_{zS} to represent β_z of the whole ring, we then get the longitudinal emittance obtained in Sands' analysis [1]:

$$\epsilon_{zS} = \frac{55}{96\sqrt{3}} \frac{\alpha_F \tilde{\lambda}_e^2 \gamma^5 \beta_{zS}}{\alpha_L} \oint \frac{1}{|\rho(s)|^3} ds. \quad (15)$$

For a ring consisting of isomagnets which bend always inward, ρ a positive constant and $\alpha_L = J_z U_0 / 2E_0 = J_z 2\pi \tilde{\lambda}_e \alpha_F \gamma^3 / 3\rho$, with U_0 the radiation loss per turn and J_z the longitudinal damping partition number [1], we have

$$\begin{aligned}\epsilon_{zS} &\approx \frac{55 \tilde{\lambda}_e}{J_z 32 \sqrt{3}} \frac{\gamma^2}{\rho} \sqrt{\frac{\eta C_0}{h}} \approx \sigma_{\delta S}^2 \beta_{zS}, \\ \sigma_{zS} &= \sqrt{\epsilon_{zS} \beta_{zS}} \approx \sqrt{\frac{55 \tilde{\lambda}_e}{J_z 32 \sqrt{3}} \frac{\gamma^2}{\rho}} \sqrt{\frac{\eta C_0}{h}} \approx \sigma_{\delta S} \beta_{zS}, \\ \sigma_{\delta S} &= \sqrt{\epsilon_{zS} \gamma_{zS}} \approx \sqrt{\epsilon_{zS} / \beta_{zS}} \approx \sqrt{\frac{55 \tilde{\lambda}_e}{J_z 32 \sqrt{3}} \frac{\gamma^2}{\rho}}. \quad (16)\end{aligned}$$

Therefore, to generate short bunches in an electron storage ring, we need to implement a quasi-isochronous lattice and high rf gradient.

B. Beyond the classical $\sqrt{|\eta|}$ scaling

Using a single β_{zs} to represent that of the whole ring is valid in usual rings where the relative variation of β_z is negligible and the electron distribution in the longitudinal phase space is always upright. But when η is small, the partial phase slippage can be significantly larger than the global one, and the variation of β_z and beam orientation in the longitudinal phase space around the ring can be significant; thus, the classical $\sqrt{|\eta|}$ scaling fails. Now we use the longitudinal Courant-Snyder formalism to conduct an analysis of this. At a position s_j , the ring can be divided into three parts, with their transfer matrices given by

$$\begin{aligned} \mathbf{T}(s_j, s_{\text{rf}-}) &= \begin{pmatrix} 1 & F(s_j, s_{\text{rf}-}) \\ 0 & 1 \end{pmatrix}, \\ \mathbf{T}(s_{\text{rf}-}, s_{\text{rf}+}) &= \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}, \\ \mathbf{T}(s_{\text{rf}+}, s_j) &= \begin{pmatrix} 1 & F(s_{\text{rf}+}, s_j) \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (17)$$

where

$$F(s_1, s_2) = - \int_{s_1}^{s_2} \left(\frac{D_x(s)}{\rho(s)} - \frac{1}{\gamma^2} \right) ds \quad (18)$$

is the partial R_{56} from s_1 to s_2 and

$$F(s_j, s_{\text{rf}-}) + F(s_{\text{rf}+}, s_j) = -\eta C_0, \quad (19)$$

with $+$ ($-$) denoting right after (before) the corresponding location. The one-turn map at s_j is

$$\mathbf{M}(s_j) = \mathbf{T}(s_{\text{rf}+}, s_j) \mathbf{T}(s_{\text{rf}-}, s_{\text{rf}+}) \mathbf{T}(s_j, s_{\text{rf}-}). \quad (20)$$

Therefore,

$$\begin{aligned} \beta_z(s_j) &= \frac{\mathbf{M}_{12}(s_j)}{\sin \Phi_z} \\ &= \frac{-\eta C_0 + F(s_{\text{rf}+}, s_j) F(s_j, s_{\text{rf}-}) h}{\sin \Phi_z}. \end{aligned} \quad (21)$$

As can be seen, the first term in the numerator is the usual global phase slippage (R_{56}) of the ring. The second term represents the impact of variation of the partial phase slippage on β_z . Sands' analysis means to drop the second term but keep only the first term. With both terms considered, the more accurate longitudinal emittance is then

$$\begin{aligned} \epsilon_z &= \epsilon_{zs} \left(1 + \frac{\langle F^2(s_{\text{rf}+}, s_j) \rangle + \eta C_0 \langle F(s_{\text{rf}+}, s_j) \rangle}{\eta C_0 / h} \right) \\ &\approx \epsilon_{zs} \left(1 + \frac{\langle F^2(s_{\text{rf}+}, s_j) \rangle + \eta C_0 \langle F(s_{\text{rf}+}, s_j) \rangle}{\beta_{zs}^2} \right), \end{aligned} \quad (22)$$

in which $\langle P \rangle$ means the radiation-weighted average of a parameter P around the ring, defined as

$$\langle P \rangle = \frac{\oint \frac{P}{|\rho(s)|^3} ds}{\oint \frac{1}{|\rho(s)|^3} ds}; \quad (23)$$

i.e., the average actually takes place only at the dipoles. After getting the longitudinal emittance and Courant-Snyder functions, the bunch length and energy spread at s_j can be obtained by

$$\sigma_z(s_j) = \sqrt{\epsilon_z \beta_z(s_j)}, \quad \sigma_\delta(s_j) = \sqrt{\epsilon_z \gamma_z(s_j)}. \quad (24)$$

For the bunch length, more accurately,

$$\sigma_z(s_j) = \sqrt{\epsilon_z \beta_z(s_j) + \epsilon_x \mathcal{H}_x(s_j)}, \quad (25)$$

considering the coupling from the transverse emittance [17] according to Eq. (9). In a planar uncouple ring which means there is only passive transverse-longitudinal coupling introduced by bending magnets, the transverse emittance always lengthens the bunch at places where $\mathcal{H}_x \neq 0$. But this conclusion is not generally true if we take advantage of transverse-longitudinal coupling actively for bunch compression when the transverse emittance is small [18–20]. The reason is that there is flexibility in projecting the three eigenemittances of a beam into different physical dimensions as shown in Eq. (2), although their values cannot be changed in a linear symplectic lattice [21].

As analyzed before, we know that in a longitudinal weak focusing ring $\epsilon_{zs} \propto \sqrt{\eta/h}$, but now, according to Eq. (22), there exists a specific $|\eta|$ where a minimum ϵ_z will be reached when adjusting $|\eta|$ by changing the dispersion function slightly if the overall pattern is given. Below this $|\eta|$, ϵ_z will grow with the decrease of $|\eta|$. The same argument applies also to the rf gradient h . Above a specific $|h|$, the emittance will grow, instead of decreasing, with the increase of $|h|$.

The above analysis has been confirmed by particle tracking simulation using a low-alpha lattice of the Metrology Light Source (MLS) storage ring [22,23] with the dispersion function pattern shown in Fig. 1. Also shown are $\beta_z(s)$ (for the case of $V_{\text{rf}} = 80$ MV) and $F(0, s)$ around the ring. The other related parameters for the simulation are given in Table I. The beam energy used is higher than the actual operation value of the MLS, and the rf voltages applied is also dramatically high. They are chosen here for

FIG. 1. The horizontal dispersion D_x , longitudinal beta function β_z , and $F(0, s)$ of the MLS lattice used for analysis and simulation. In this plot, the zero-length rf is placed at $s_{\text{rf}} = 0$ m and $V_{\text{rf}} = 80$ MV is applied. The dipoles are shown at the bottom as blue rectangles. Each dipole has a length of 1.2 m and bends the electron trajectory for an angle of $\pi/4$.

the numerical example to present the difference of this work and the classical analysis of Sands in a more notable way. The simulation results along with the theoretical predictions from Sands' analysis and this paper are shown in Fig. 2. Also shown is the theoretical calculation based on SLIM [14]. As can be seen from Fig. 2, our result and SLIM agree well with the simulation in the cases of ultrahigh rf voltages, while the predication of Sands deviates from the simulation.

C. Discussions

Now we make some more discussions. First, we observe that in Fig. 1 β_z changes notably only inside dipoles, which reflects the fact that $F(0, s)$, i.e., the effective longitudinal drift space, changes significantly only inside dipoles where $\rho \neq 0$. The minimum β_z around the ring is realized when $F(s_{\text{rf}+}, s_j) = F(s_j, s_{\text{rf}-}) = \frac{-\eta C_0}{2}$, and

$$\beta_{z,\text{min}} = \left(1 - \frac{h\eta C_0}{4}\right) \beta_{zS}. \quad (26)$$

Furthermore, in some cases, ϵ_z can actually be smaller than ϵ_{zS} . The physical meaning is then the average β_z at the dipoles is smaller than that at the rf. This, however, is usually not satisfied in a low-alpha lattice, where the change of $F(s_{\text{rf}+}, s_j)$ around the ring can be significantly larger than the global ring $R_{56} = -\eta C_0$. Therefore, β_z at other places can be significantly larger than that at the rf, just like the case shown in Fig. 1. For a novel light source mechanism like SSMB [4–13], an ultrasmall ϵ_z is desired. According to Eq. (8), a method to tailor the β_z around the ring can be envisioned for such a purpose [13], similar to the tailoring of $\mathcal{H}_x$ for the minimization of horizontal emittance. The theoretical minimum emittance (TME) can be analyzed, and the same scaling of $\epsilon_{z,\text{TME}} \propto \gamma^2 \theta^3$, where θ is the bending angle of each dipole, can be derived similar to the case of transverse dimension. Generally, a small

TABLE I. Parameters of the MLS lattice used in simulation.

Parameter	Value	Description
E_0	1.2 GeV	Beam energy
C_0	48 m	Ring circumference
η	2.9×10^{-4}	Phase slippage factor
f_{rf}	500 MHz	rf frequency
V_{rf}	2.5–160 MV	rf voltage
U_0	0.12 MeV	Radiation loss per turn
J_z	1.94	Longitudinal damping partition
τ_δ	1.6 ms	Radiation damping time
$\sigma_{\delta S}$	8.4×10^{-4}	Natural energy spread
$\langle F(s_{\text{rf}}, s_j) \rangle$	-0.19 m	...
$\sqrt{\langle F^2(s_{\text{rf}}, s_j) \rangle}$	0.26 m	...
$\sqrt{\langle F^2 \rangle - \langle F \rangle^2}$	0.18 m	...

FIG. 2. Bunch length saturation and energy spread and longitudinal emittance divergence when the bunch length is pushed to the limit given by quantum diffusion, in the simulation by increasing the rf voltage. The lattice optics and related parameters are shown in Fig. 1 and Table I. In each simulation, 1000 electrons are tracked for 50000 turns, which corresponds to about five longitudinal damping times, and the averaged result of the last 10000 turns are used to get the σ_z , σ_δ , and ϵ_z . The observation point is at the rf.

longitudinal emittance requires a small dipole bending angle and small dispersion at the dipoles. The application of transverse and longitudinal gradient bends, etc., can also be anticipated for reducing ϵ_z .

We point out that quantifying the impact of variation of β_z around the ring on ϵ_z using partial alpha or partial phase slippage variance $\langle F^2 \rangle - \langle F \rangle^2$, as that done in Ref. [2] and also our previous publication [11], is not generally accurate. Given the same dispersion function pattern, which means the same $\langle F^2 \rangle - \langle F \rangle^2$ as it is independent of the observation point, a different longitudinal beta function pattern can be generated if the rf is placed at a different location, therefore resulting in a different longitudinal emittance according to Eq. (8).

We can also view this from another way. Changing the rf location means shifting $F(s_{rf+}, s_j)$ up or down as a whole, which can also be done by artificially inserting a pair of $\pm r_{56}$ besides the rf. We assume that there is no synchrotron radiation within these artificially inserted $\pm r_{56}$, and, therefore, these manipulations do not change the R_{56} and also $\langle F^2 \rangle - \langle F \rangle^2$ of the ring, but they do affect β_z and, therefore, also ϵ_z . When $r_{56} = \langle F(s_{rf+}, s_j) \rangle + \frac{\eta C_0}{2}$, we arrive at the minimum longitudinal emittance

$$\epsilon_{z,\min} = \epsilon_{zS} \left(1 + \frac{\langle F^2 \rangle - \langle F \rangle^2 - (\frac{\eta C_0}{2})^2}{\eta C_0 / h} \right). \quad (27)$$

Equivalently, in the case of shifting rf location, $\epsilon_{z,\min}$ is reached when the rf is placed at a location such that $\langle F(s_{rf+}, s_j) \rangle = -\frac{\eta C_0}{2}$. The maximum longitudinal emittance is realized when the rf is placed at a place such that $|\langle F(s_{rf+}, s_j) \rangle + \frac{\eta C_0}{2}|$ reaches the maximum possible value.

To give the readers a more concrete feeling, here we present some calculations based on the lattice optics presented in Fig. 1 and other related parameters given in Table I. Figure 3 shows the emittance ratio ϵ_z/ϵ_{zS} as a

FIG. 3. The emittance ratio $\frac{\epsilon_z}{\epsilon_{zS}}$ as a function of the rf place s_{rf} (top) and the r_{56} artificially inserted in front of the rf whose location is fixed at $s_{rf} = 0$ m (bottom). The lattice optics and related parameters are shown in Fig. 1 and Table I. The rf voltage applied is $V_{rf} = 80$ MV.

function of the rf place s_{rf} and the $\pm r_{56}$ artificially inserted besides the rf whose location is fixed at $s_{\text{rf}} = 0$ m, respectively. An 80 MV rf voltage is applied in both calculations. As can be seen, indeed the place of the rf can have a crucial impact on the longitudinal emittance in this specific lattice optics and choice of rf voltage, and the impact of the artificially inserted $\pm r_{56}$ is also as expected.

When the minimum longitudinal emittance is reached, the bunch length at the rf is

$$\sigma_{z,\text{min}}(s_{\text{rf}}) = \sigma_{\delta\text{S}} \sqrt{\frac{\eta C_0}{h} + \langle F^2 \rangle - \langle F \rangle^2 - \left(\frac{\eta C_0}{2}\right)^2}. \quad (28)$$

In the case of ultrasmall η ,

$$\sigma_{z,\text{min}}(s_{\text{rf}}) \approx \sigma_{\delta\text{S}} \sqrt{\langle F^2 \rangle - \langle F \rangle^2}. \quad (29)$$

Therefore, the variance of F can be viewed as a parameter to quantify the lowest possible contribution of this effect to the equilibrium bunch length at the rf with a dispersion function pattern given [2], if we can choose the location of the rf. However, in a real machine, the rf location is fixed, and Eqs. (22), (24), and (25) should be used instead. The analysis of shifting rf location or adding artificial $\pm r_{56}$ helps us to better understand the physics, and we recognize the fact that the rf can actually not be placed inside dipoles and also that the Courant-Snyder functions lose their well-defined meaning if the rf is placed at a dispersive location.

D. Campbell's theorem

Now we explain in more detail why $\langle F^2 \rangle - \langle F \rangle^2$ is not a generally accurate criterion. The reason is that, while the photon emission process is stochastic, the evolution of F around the ring is deterministic. So the diffusion of z each turn due to quantum excitation is

$$\langle z^2 \rangle - \langle z \rangle^2 = \langle F^2 \rangle \langle \mathcal{N} \rangle \left\langle \frac{u^2}{E_0^2} \right\rangle \quad (30)$$

instead of

$$\langle z^2 \rangle - \langle z \rangle^2 = (\langle F^2 \rangle - \langle F \rangle^2) \langle \mathcal{N} \rangle \left\langle \frac{u^2}{E_0^2} \right\rangle, \quad (31)$$

where F is calculated with the final observation location as the ending point, $\langle \mathcal{N} \rangle$ is the expected number of emitted photons, u is the photon energy, and $\langle u^2 \rangle$ and later also $\langle u \rangle$ mean the average is taken with respect to the photon energy spectrum. This result can be understood with the help of Campbell's theorem [24].

From this theorem, some expectation result for the Poisson point process follows. For example, for the application in synchrotron radiation, we have $\delta = -\sum_i \frac{u_i}{E_0}$, where

i means the i th photon emission. Then, according to Campbell's theorem, we have

$$\begin{aligned} \langle \delta \rangle &= -\langle \mathcal{N} \rangle \left\langle \frac{u}{E_0} \right\rangle = -T_{\text{dipole}} \dot{\mathcal{N}} \left\langle \frac{u}{E_0} \right\rangle, \\ \langle \delta^2 \rangle - \langle \delta \rangle^2 &= \langle \mathcal{N} \rangle \left\langle \frac{u^2}{E_0^2} \right\rangle = T_{\text{dipole}} \dot{\mathcal{N}} \left\langle \frac{u^2}{E_0^2} \right\rangle, \end{aligned} \quad (32)$$

where $\dot{\mathcal{N}}$ is the number of photons emitted per unit time in the dipoles and T_{dipole} is the total time within dipoles.

Equation (32) is why $\dot{\mathcal{N}} \langle u^2 \rangle$ appears so often in the calculation of energy spread, emittance, etc., in electron storage ring physics. Note that the relation in Eq. (32) holds as long as the radiation is a Poisson point process. It is independent of whether $\langle u \rangle = 0$ or not, and is also independent of the detailed spectrum of the photon energy. In other words, the key of a Poisson point process is the randomness in whether there is a kick or not, i.e., the kick number, and not in the randomness of the size of the kicks. The importance of this theorem for electron dynamics was first pointed out by Sands [25]. A proof can be found in the article of Rice [26] and a less rigorous but simpler one in the lecture note of Jowett [27].

Now we can understand Eq. (30) like this. Suppose the rf is the observation point. We divide the ring into many sections, and in each section $F(s_j, s_{\text{rf}})$ does not change much. Then the change of electron longitudinal coordinate in one turn is $z = \sum_j z_j$, with $z_j = -\sum_i F(s_{ji}, s_{\text{rf}}) \frac{u_i}{E_0}$ the contribution due to photon emissions within the section j . So, according to Campbell's theorem, the variance of z_j is

$$\text{Var}(z_j) = F^2(s_j, s_{\text{rf}}) t_j \dot{\mathcal{N}} \left\langle \frac{u^2}{E_0^2} \right\rangle, \quad (33)$$

where t_j is the time within the dipoles in section j . As the photon emissions in different sections are uncorrelated, then the variance of z is the sum of variance of z_j :

$$\begin{aligned} \langle z^2 \rangle - \langle z \rangle^2 &= \frac{\sum_j [F^2(s_j, s_{\text{rf}}) t_j]}{T_{\text{total}}} T_{\text{total}} \dot{\mathcal{N}} \left\langle \frac{u^2}{E_0^2} \right\rangle \\ &= \langle F^2(s_j, s_{\text{rf}}) \rangle \langle \mathcal{N} \rangle \left\langle \frac{u^2}{E_0^2} \right\rangle, \end{aligned} \quad (34)$$

in which $T_{\text{total}} = \sum_j t_j$ is the total time within the dipoles. In fact, the same argument can be applied to explain why $|E_{k5}(s)|^2$ in Eq. (1), instead of their variance, are used in SLIM to calculate the increase of eigenemittances due to quantum excitation.

IV. MULTIPLE rfs

A. Analysis

The analysis in the above section considers the case with only a single rf. When there are multiple rfs, for the longitudinal dynamics, it is similar to implement multiple quadrupoles in the transverse dimension, and the beam dynamics can have more possibilities. A longitudinal strong focusing scheme, for example, can be invoked [6,28] not unlike its transverse counterpart which almost all the storage rings today implement. The linear beam dynamics with multiple rfs can be treated the same way as that with a single rf using SLIM. When all the rfs are placed at dispersion-free locations, the Courant-Snyder parametrization can be applied as analyzed in Sec. II. Here, we use a setup with two rfs as an example to show the scheme of manipulating β_z around the ring. The schematic layout of the ring is shown in Fig. 4. The treatment of cases with more rfs is similar.

We divide the ring into five sections, i.e., three longitudinal drifts (R_{56}) and two longitudinal quadrupole kicks (h), with the transfer matrices given by

$$\begin{aligned} \mathbf{T}_{D1} &= \begin{pmatrix} 1 & R_{56}^{(1)} \\ 0 & 1 \end{pmatrix}, & \mathbf{T}_{rf1} &= \begin{pmatrix} 1 & 0 \\ h_1 & 1 \end{pmatrix}, \\ \mathbf{T}_{D2} &= \begin{pmatrix} 1 & R_{56}^{(2)} \\ 0 & 1 \end{pmatrix}, & \mathbf{T}_{rf2} &= \begin{pmatrix} 1 & 0 \\ h_2 & 1 \end{pmatrix}, \\ \mathbf{T}_{D3} &= \begin{pmatrix} 1 & R_{56}^{(3)} \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (35)$$

Then the one-turn map at the radiator center is

$$\mathbf{M}_R = \mathbf{T}_{D3} \mathbf{T}_{rf2} \mathbf{T}_{D2} \mathbf{T}_{rf1} \mathbf{T}_{D1}. \quad (36)$$

Linear stability requires that the absolute value of $\text{Tr} \mathbf{M}_R < 2$, where Tr means the trace of. For generation

FIG. 4. Schematic layout of a storage ring using two rf systems for longitudinal strong focusing and an example beam distribution evolution in the longitudinal phase space. Note that the tilted angles of the beam distribution and bunch length ratios at different places do not strictly correspond to the parameters in Table II but only to present the qualitative characteristics.

of coherent radiation, we usually want the bunch length to reach its minimum at the radiator; then we need $\alpha_z = 0$ for $\mathbf{M}_R$.

With the primary purpose to present the principle, instead of a detailed design, here for simplicity we discuss only one special case: $R_{56}^{(1)} = R_{56}^{(3)}$, $h_1 = h_2 = h$. The treatment of more general cases with different signs and magnitudes of $R_{56}^{(1)}$ and $R_{56}^{(3)}$ and h_1 and h_2 is similar, but the same-signed $R_{56}^{(1)}$ and $R_{56}^{(3)}$ might be easier for a real lattice to fulfill. For example, if $R_{56}^{(1)}, R_{56}^{(3)} > 0$, a possible realization of them are chicanes.

For the special case of $R_{56}^{(1)} = R_{56}^{(3)}$, $h_1 = h_2 = h$ and denote $1 + R_{56}^{(1)}h = \xi_1$, $2 + R_{56}^{(2)}h = \xi_2$, we then have

$$\mathbf{M}_R = \begin{pmatrix} \xi_1 \xi_2 - 1 & \frac{\xi_1^2 \xi_2 - 2\xi_1}{h} \\ h \xi_2 & \xi_1 \xi_2 - 1 \end{pmatrix}. \quad (37)$$

The linear stability requires $|\xi_1 \xi_2 - 1| < 1$, and the synchrotron tune is

$$\nu_s = \begin{cases} \frac{1}{2\pi} \arccos[\xi_1 \xi_2 - 1] & \text{if } \frac{\xi_1^2 \xi_2 - 2\xi_1}{h} > 0, \\ 1 - \frac{1}{2\pi} \arccos[\xi_1 \xi_2 - 1] & \text{if } \frac{\xi_1^2 \xi_2 - 2\xi_1}{h} < 0. \end{cases} \quad (38)$$

Here, we give one example parameter set with a stable linear motion as shown in Table II. We recognize the fact that the large energy chirp strength h used is demanding, for SSMB only pulsed laser can now reach such a value, and here our primary goal is to present the principle based on which the interested readers can choose and optimize the parameters for their target applications. According to the longitudinal Courant-Snyder functions given in Table II (note the values of β_z and the signs of α_z), the evolution of

TABLE II. An example parameter set corresponding to the setup shown in Fig. 4.

Parameter	Value
$R_{56}^{(1)}$	$15 \mu\text{m}$
$R_{56}^{(2)}$	$-100 \mu\text{m}$
$R_{56}^{(3)}$	$15 \mu\text{m}$
h	$-5 \times 10^4 \text{ m}^{-1}$
ξ_1	0.25
ξ_2	7
ν_s	0.115
$\beta_z(s_{\text{rad}})$	$1.9 \mu\text{m}$
$\beta_z(s_{\text{rf}})$	$121 \mu\text{m}$
$\alpha_z(s_{\text{rad}})$	0
$\alpha_z(s_{\text{rf1-}})$	-7.9
$\alpha_z(s_{\text{rf1+}})$	-1.9
$\alpha_z(s_{\text{rf2-}})$	1.9
$\alpha_z(s_{\text{rf2+}})$	7.9

electron distribution in the longitudinal phase space (note the bunch lengths and orientations) around the ring is qualitatively shown in Fig. 4.

B. Discussions

Here, we make several observations from the above analysis and numerical example, which we believe are important. First, β_z in a longitudinal strong focusing ring is at the same level of or even smaller than the ring $|R_{56} = -\eta C_0|$, while in a longitudinal weak focusing ring $\beta_z \gg |-\eta C_0|$. Therefore, according to Eq. (8), the equilibrium longitudinal emittance in a longitudinal strong focusing ring can be much smaller than that in a longitudinal weak focusing ring [13]. Together with a smaller β_z , the bunch length can, thus, be much smaller than that in a longitudinal weak focusing ring. This is the reason behind the application of longitudinal strong focusing in SSMB to realize extreme short bunches [6,13].

Second, β_z changes significantly around the ring in the longitudinal strong focusing regime. Therefore, the bunch length and beam orientation in the longitudinal phase space vary greatly around the ring, as shown qualitatively in Fig. 4. This means the adiabatic approximation cannot be applied for the longitudinal dimension anymore [15]. Actually, the adiabatic approximation also breaks down in the case corresponding to Fig. 1, where the change of β_z around the ring is significant although the total synchrotron phase advance per turn is small. Therefore, the global tune is not a general criterion in the classification of a ring to be weak focusing or strong focusing. The evolution of β_z is more relevant.

The breakdown of adiabatic approximation can have crucial impacts on the study of both the single-particle and collective effects. For linear single-particle dynamics, the longitudinal and transverse dimensions should be treated the same way on equal footing, and SLIM formalism can be invoked. The treatment of nonlinear single-particle dynamics is more subtle, as the longitudinal dynamics now is strongly chaotic. For the collective effects, many classical treatments should be reevaluated, and some new formalism needs to be developed. For example, the Haissinski equation [29] for calculating the equilibrium beam distortion cannot be applied directly then. Also, to our knowledge, there is no discussion on coherent synchrotron radiation (CSR)-induced microwave instability in a longitudinal strong focusing ring. The scaling law obtained in the longitudinal weak focusing [30] cannot be applied directly. 3D CSR effects and also the impact of bunch lengthening from the transverse emittance on CSR need more in-depth study. This is especially true for an SSMB ring, considering the fact that the beam width there is much larger than the microbunch length, while the contrary is true in a conventional ring. The contribution from horizontal emittance can easily dominate the bunch length at many places in an SSMB ring. This, on the other hand, will be

helpful to suppress unwanted CSR and may also be helpful in mitigating intrabeam scattering (IBS), as extreme short bunches occur only at limited locations. The IBS in a longitudinal strong focusing ring and a general coupled lattice also deserves special attention. To our knowledge, the IBS formalism of Nash [31] and that of Kubo and Oide [32] can be applied for such purposes, as they are based on 6×6 general transport matrices, while the classical formalism of Piwinski [33] and Bjorken and Mtingwa [34] both adopt approximations which hold only in longitudinal weak focusing rings. An IBS formalism can be developed based on Chao's SLIM formalism [14], in which eigenanalysis has been invoked and applies to 3D general coupled lattice with longitudinal strong focusing.

V. SUMMARY

In this paper, we have presented the longitudinal Courant-Snyder formalism for electron storage ring linear dynamics analysis. We used the formalism first to reproduce the classical $\sigma_z \propto \sqrt{|\eta|}$ scaling. We then analyzed the case beyond this classical scaling, considering the fact that the longitudinal beta function can change significantly around the ring in a quasi-isochronous ring, due to the variation of partial phase slippage. Formulas for the equilibrium longitudinal emittance, energy spread, and position-dependent bunch length are derived, based on which several important points concerning the analysis are discussed. Finally, we used the same formalism to analyze the application of multiple rfs in which the scheme of longitudinal strong focusing is invoked and make important observations on this new working regime of a storage ring. We expect that the method and discussions presented in this paper will be useful for the storage ring physics, especially the longitudinal dynamics study, and the development of new light sources like steady-state microbunching.

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- [1] M. Sands, SLAC Technical Report No. SLAC-121, 1970.
 - [2] Y. Shoji, H. Tanaka, M. Takao, and K. Soutome, *Phys. Rev. E* **54**, R4556 (1996).
 - [3] K. Soutome, M. Takao, H. Tanaka, and Y. Shoji, in *Proceedings of the 6th European Particle Accelerator Conference (EPAC'98), Stockholm, 1998* (Institute of Physics, London, 1998), pp. 1008–1010.
 - [4] D. F. Ratner and A. W. Chao, *Phys. Rev. Lett.* **105**, 154801 (2010).

- [5] Y. Jiao, D. F. Ratner, and A. W. Chao, *Phys. Rev. Accel. Beams* **14**, 110702 (2011).
- [6] A. Chao, E. Granados, X. Huang, D. Ratner, and H.-W. Luo, in *Proceedings of the 7th International Particle Accelerator Conference (IPAC'16), Busan, Korea, 2016* (JACoW, Geneva, 2016), pp. 1048–1053.
- [7] C. Tang, X. Deng, A. Chao, W. Huang, T. Rui, J. Feikes, J. Li, M. Ries, A. Hoehl, D. Ratner *et al.*, in *Proceedings of the 60th ICFA Advanced Beam Dynamics Workshop on Future Light Sources (FLS'18), Shanghai, China, 2018* (JACoW, Geneva, 2018), pp. 166–170.
- [8] Z. Pan, T. Rui, W. Wan, A. Chao, X. Deng, Y. Zhang, W. Huang, and C. Tang, in *Proceedings of the 39th International Free Electron Laser Conference (FEL'19), Hamburg, Germany, 2019* (JACoW, Geneva, 2019), pp. 700–703.
- [9] C. Li, C. Feng, B. Jiang, and A. Chao, in *Proceedings of the 10th International Particle Accelerator Conference (IPAC'19), Melbourne, Australia, 2019* (JACoW, Geneva, 2019), pp. 1507–1509.
- [10] X. J. Deng, R. Klein, A. W. Chao, A. Hoehl, W. H. Huang, J. Li, J. Lubeck, Y. Petenev, M. Ries, I. Seiler, C. X. Tang, and J. Feikes, *Phys. Rev. Accel. Beams* **23**, 044001 (2020).
- [11] X. J. Deng, A. W. Chao, J. Feikes, W. H. Huang, M. Ries, and C. X. Tang, *Phys. Rev. Accel. Beams* **23**, 044002 (2020).
- [12] X. Deng, A. Chao, J. Feikes, A. Hoehl, W. Huang, R. Klein, A. Kruschinski, J. Li, A. Matveenko, Y. Petenev *et al.*, *Nature (London)* **590**, 576 (2021).
- [13] Y. Zhang, X. J. Deng, Z. L. Pan, Z. Z. Li, K. S. Zhou, W. H. Huang, R. K. Li, C. X. Tang, and A. W. Chao, *Phys. Rev. Accel. Beams* **24**, 090701 (2021).
- [14] A. W. Chao, *J. Appl. Phys.* **50**, 595 (1979).
- [15] A. W. Chao, *Lectures on Accelerator Physics* (World Scientific, Singapore, 2020).
- [16] E. D. Courant and H. S. Snyder, *Ann. Phys. (N.Y.)* **3**, 1 (1958).
- [17] Y. Shoji, *Phys. Rev. Accel. Beams* **7**, 090703 (2004).
- [18] H. Deng and C. Feng, *Phys. Rev. Lett.* **111**, 084801 (2013).
- [19] C. Feng and Z. Zhao, *Sci. Rep.* **7**, 4724 (2017).
- [20] X. J. Deng, W. H. Huang, Z. Z. Li, and C. X. Tang, *Nucl. Instrum. Methods Phys. Res., Sect. A* (to be published).
- [21] A. J. Dragt, F. Neri, and G. Rangarajan, *Phys. Rev. A* **45**, 2572 (1992).
- [22] J. Feikes, M. von Hartrott, M. Ries, P. Schmid, G. Wüstefeld, A. Hoehl, R. Klein, R. Müller, and G. Ulm, *Phys. Rev. Accel. Beams* **14**, 030705 (2011).
- [23] M. Ries, Ph.D. dissertation, Humboldt University of Berlin, Berlin, Germany, 2014.
- [24] N. Campbell, *Proc. Cambridge Philos. Soc.* **15**, 117 (1909).
- [25] M. Sands, *Phys. Rev.* **97**, 470 (1955).
- [26] S. O. Rice, *Bell Syst. Tech. J.* **23**, 282 (1944).
- [27] J. M. Jowett, *AIP Conf. Proc.* **153**, 864 (1987).
- [28] C. Biscari, *Phys. Rev. Accel. Beams* **8**, 091001 (2005).
- [29] J. Haissinski, *Nuovo Cimento Soc. Ital. Fis. B* **18**, 72 (1973).
- [30] K. L. F. Bane, Y. Cai, and G. Stupakov, *Phys. Rev. Accel. Beams* **13**, 104402 (2010).
- [31] B. Nash, Ph.D. thesis, Stanford University, Stanford, CA, 2006.
- [32] K. Kubo and K. Oide, *Phys. Rev. Accel. Beams* **4**, 124401 (2001).
- [33] A. Piwinski, in *Proceedings of the 9th International Conference on High Energy Accelerators, Stanford, CA (SLAC, Stanford, 1974)*, p. 405.
- [34] J. D. Bjorken and S. K. Mtingwa, *Part. Accel.* **13**, 115 (1983), <https://cds.cern.ch/record/140304/files/p115.pdf>.