Paraxial optics of charged particles in solenoids

Timofey Gorlov^{*}

Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831, USA

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Paraxial optics of charged particles in solenoid magnets with axisymmetric field is considered. Analytic and symplectic transverse maps for charged particle motion through solenoids with 3D magnetic fields have been derived. The formula for transverse maps contains both linear and nonlinear effects caused by entrance and exit fringe fields of the solenoid. The explicit transfer map can be used for applications of accelerator physics that use solenoid magnets.

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I. INTRODUCTION

We derived an analytic nonlinear-linear transverse map for single charged particle dynamics through solenoid magnets in a paraxial approximation. Solenoid magnets have been used in electron optics with relatively small energies of electrons [1]. Also, solenoids have wide application in high energy ion accelerators and colliders [2–5]. Recently, there is particular interest in using the solenoids for integrable optics [6,7] and for self-consistent beams [8].

In accelerator physics, for calculation of single particle dynamics in solenoids, the most commonly used solution in the form of a transfer map can be found, for example in [1]. This transfer map is a good linear approximation of beam dynamics but it considers only the linear effects and does not take into account the detailed distribution of 3D magnetic field in the solenoid. In modern physics, precise linear and nonlinear beam dynamics calculations are performed by using numerical simulations of particles in given 3D fields [9,10]. An analytic approach for nonlinear beam dynamics in solenoids has been performed in [11]. Simple and effective nonlinear kick estimate can be performed by using analytic integration of motion equations like in [12,13].

In this paper we derive an analytic nonlinear transfer map for beam dynamics in solenoids directly from Newton's force equation for charged particle motion in a 3D magnetic field. For solving the equation, we developed and applied a method of iterations that was initially used in [14] for solving a quadrupole magnet. The method was originally applied to find the second order solution in magnetic field.

Here we developed this method to find the N th order solution that depends on parameter N in explicit analytic form. Then by calculating $N \rightarrow \infty$ we found precise analytic solution (with some limitations) for particle dynamics. The solution has been represented in the form of transfer maps widely used in accelerator physics.

II. PARAXIAL MOTION IN AXIAL-SYMMETRIC MAGNETIC FIELDS

We use Newton's relativistic force equation for motion of a charged particle in a static magnetic field:

$$\ddot{\mathbf{r}} = \frac{q}{m\gamma} \dot{\mathbf{r}} \times \mathbf{B}. \quad (1)$$

The axially symmetric static magnetic field in Cartesian coordinates has the following form:

$$\mathbf{B} = \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x(x^2+y^2)^i}{2^{2i+1} i!(i+1)!} \frac{\partial^{2i+1} B(z)}{\partial z^{2i+1}} \\ \sum_{i=0}^{\infty} \frac{(-1)^{i+1} y(x^2+y^2)^i}{2^{2i+1} i!(i+1)!} \frac{\partial^{2i+1} B(z)}{\partial z^{2i+1}} \\ \sum_{i=0}^{\infty} \frac{(-1)^i (x^2+y^2)^i}{4^i (i!)^2} \frac{\partial^{2i} B(z)}{\partial z^{2i}} \end{pmatrix} \quad (2)$$

where $B(z)$ is the distribution of the z -component of the $\mathbf{B}$ -field along the z axis. We now truncate magnetic field (2) for calculation of particle dynamics near z -axis:

$$\mathbf{B} = \begin{pmatrix} -\frac{1}{2} \frac{\partial B}{\partial z} x + \frac{1}{16} x(x^2+y^2) \frac{\partial^3 B}{\partial z^3} \\ -\frac{1}{2} \frac{\partial B}{\partial z} y + \frac{1}{16} y(x^2+y^2) \frac{\partial^3 B}{\partial z^3} \\ B - \frac{1}{4} (x^2+y^2) \frac{\partial^2 B}{\partial z^2} \end{pmatrix}. \quad (3)$$

Substituting (3) into (1) we obtain a 3D time-dependent system of equations for particle motion in a 3D magnetic field:

^{*}tg4@ornl.gov

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$$\begin{aligned}
\ddot{x} &= +\frac{q}{m\gamma} \left[B\dot{y} + \frac{y}{2} \frac{\partial B}{\partial z} \dot{z} - \frac{x^2 + y^2}{16} \left(y\dot{z} \frac{\partial^3 B}{\partial z^3} + 4\dot{y} \frac{\partial^2 B}{\partial z^2} \right) \right] \\
\ddot{y} &= -\frac{q}{m\gamma} \left[B\dot{x} + \frac{x}{2} \frac{\partial B}{\partial z} \dot{z} - \frac{x^2 + y^2}{16} \left(x\dot{z} \frac{\partial^3 B}{\partial z^3} + 4\dot{x} \frac{\partial^2 B}{\partial z^2} \right) \right] \\
\ddot{z} &= \frac{q}{m\gamma} (y\dot{x} - x\dot{y}) \left(\frac{x^2 + y^2}{16} \frac{\partial^3 B}{\partial z^3} - \frac{1}{2} \frac{\partial B}{\partial z} \right) \quad (4)
\end{aligned}$$

where the particle vector $\mathbf{r} = \{x, y, z\} \equiv \{x(t), y(t), z(t)\}$ depends on time.

Equations (4) can be rewritten using a paraxial approximation that is frequently used in accelerator physics in the case of high longitudinal particle momentum. In this approximation $\dot{y}, \dot{x} \ll \dot{z} = v = \beta c = \text{const}$. In the general case, transformation of $\ddot{x}$ and $\ddot{y}$ in (4): $\ddot{\mathbf{r}} \rightarrow \mathbf{r}''(z)\dot{z}^2 + \mathbf{r}'(z)\dot{z}$ produces term $\mathbf{r}'(z)\dot{z}$ with the second order angular aberrations $x'^2, y'^2, x'y'$. For typical accelerator values $r' < 50$ mrad, the relative angular nonlinear aberration is much smaller than spatial one $r'/(r/R) \ll 1$ where R is an aperture of solenoid. In this paper we omit this term for simplicity of final solution. In the end we will perform benchmark that compares approximation from (4) to (5). Substituting $\ddot{\mathbf{r}} \rightarrow \mathbf{r}''(z)\beta^2 c^2$ and a new variable $z = \beta c t$ into (4) we obtain 2D equations for transverse motion of a particle in a 3D magnetic field:

$$\begin{aligned}
\frac{d^2 x}{dz^2} &= +k \left[b \frac{dy}{dz} + \frac{y}{2} \frac{db}{dz} - \frac{x^2 + y^2}{16} \left(y \frac{d^3 b}{dz^3} + 4 \frac{dy}{dz} \frac{d^2 b}{dz^2} \right) \right] \\
\frac{d^2 y}{dz^2} &= -k \left[b \frac{dx}{dz} + \frac{x}{2} \frac{db}{dz} - \frac{x^2 + y^2}{16} \left(x \frac{d^3 b}{dz^3} + 4 \frac{dx}{dz} \frac{d^2 b}{dz^2} \right) \right] \\
k &= \frac{qB_0}{mc\beta\gamma}. \quad (5)
\end{aligned}$$

Here, we introduced a parameter k and the “normalized” longitudinal function $b \equiv b(z) = B(z)/B_0$ where $B_0 = B(z_{\max})$ is the amplitude of the magnetic field in solenoid (see Fig. 1).

System (5) requires initial conditions for orbit calculation. In accelerator physics a particle is considered to come from $z = -\infty$ and it is convenient to determine initial conditions such that the particle orbit would match a drift with given parameters $x_0, x_{p0} \equiv dx/dz, y_0, y_{p0} \equiv dy/dz$ at some abstract entrance plane of the solenoid located at $z = 0$:

$$\mathbf{r}(z)_{B_0=0} = \mathbf{r}_0 + \mathbf{r}_{p0} z = \begin{pmatrix} x(z) \\ y(z) \end{pmatrix}_{B_0=0} = \begin{pmatrix} x_0 + x_{p0} z \\ y_0 + y_{p0} z \end{pmatrix} \quad (6)$$

FIG. 1. Schematic of particle dynamics in magnetic field of solenoid. The blue dashed line represents an abstract drift of particle coming from $z = -\infty$ in the case of zero magnetic field [see Eq. (6)]. The red dashed curve represents particle orbit $\mathbf{r}(z)$ in solenoidal magnetic field. The gray arrow lines represent the magnetic field distribution in the solenoid. The vertical axis represents relative scale for all of schematics listed here.

After that we obtain an integrodifferential equations from (5) that contain the initial conditions (6):

$$\begin{aligned}
x(z) &= x_0 + x_{p0} z + k \int_{-\infty}^z \int_{-\infty}^z \left[b \frac{dy}{dz} + \frac{y}{2} \frac{db}{dz} - \frac{x^2 + y^2}{16} \left(y \frac{d^3 b}{dz^3} + 4 \frac{dy}{dz} \frac{d^2 b}{dz^2} \right) \right] dz dz \\
y(z) &= y_0 + y_{p0} z - k \int_{-\infty}^z \int_{-\infty}^z \left[b \frac{dx}{dz} + \frac{x}{2} \frac{db}{dz} - \frac{x^2 + y^2}{16} \left(x \frac{d^3 b}{dz^3} + 4 \frac{dx}{dz} \frac{d^2 b}{dz^2} \right) \right] dz dz. \quad (7)
\end{aligned}$$

The system (7) can be solved in form of a series expansion of $x(z)$ and $y(z)$ in powers of the k parameter:

$$x(z) = \sum_{n=0}^{\infty} x_n(z) k^n \quad y(z) = \sum_{n=0}^{\infty} y_n(z) k^n. \quad (8)$$

After substituting (8) into (7) and equating terms with equal powers of k we obtain explicit recurrence formulas for n th series terms $x_n(z)$ and $y_n(z)$ on the left side that depend on lower order terms $i = 0 \dots n-1$ on the right side of the expression:

$$\begin{aligned}
x_n(z) &= + \int_{-\infty}^z \int_{-\infty}^z \left[b \frac{dy_{n-1}}{dz} + \frac{y_{n-1}}{2} \frac{db}{dz} - \sum_{i=0}^{n-1} \sum_{j=0}^i \frac{x_j x_{i-j} + y_j y_{i-j}}{16} \left(x_{n-i-1} \frac{d^3 b}{dz^3} + 4 \frac{dx_{n-i-1}}{dz} \frac{d^2 b}{dz^2} \right) \right] dz dz \\
y_n(z) &= - \int_{-\infty}^z \int_{-\infty}^z \left[b \frac{dx_{n-1}}{dz} + \frac{x_{n-1}}{2} \frac{db}{dz} - \sum_{i=0}^{n-1} \sum_{j=0}^i \frac{x_j x_{i-j} + y_j y_{i-j}}{16} \left(y_{n-i-1} \frac{d^3 b}{dz^3} + 4 \frac{dy_{n-i-1}}{dz} \frac{d^2 b}{dz^2} \right) \right] dz dz.
\end{aligned} \quad (9)$$

The formulas (9) have an explicit form that suggests an iterative algorithm that we chosen in this paper. We use Cartesian coordinates for the sake of simplicity. For example, a cylindrical system of coordinates yields a similar recurrence equation with $r_n(z)$ in both the left and the right side, but more complicated calculations are required to extract it from there. Terms $x_0(z)$, $y_0(z)$ are trivial drift solution (6) and $x_1(z)$, $y_1(z)$ are calculated by substituting $x_0(z)$, $y_0(z)$ into (7) and integrating twice. The first order solution in k for the particle orbit and its slope is:

$$\begin{aligned}
x(z) &= x + x_p z + \frac{k}{16} \left[4x_p(x_p y - y_p x) \int_{-\infty}^z z b' dz - b'(y + y_p z)(x + x_p z)^2 - b'(y + y_p z)^3 + 2b y_p (y^2 - x^2 + 4z^2) \right. \\
&\quad \left. + 2b y z (4 - x_p^2 + y_p^2) + 2(y + y_p z)(2b x x_p - (4 - x_p^2 - y_p^2) \int_{-\infty}^z z b' dz) \right] \\
x_p(z) &= \frac{dx(z)}{dz}
\end{aligned} \quad (10)$$

And there are similar expressions for $y(z)$, $y_p(z)$. Here we write $\{x, x_p, y, y_p\} \equiv \{x_0, x_{p0}, y_0, y_{p0}\}$ for simplicity. For higher orders the expansion solution (10) becomes cumbersome and practically unusable so it needs to be transformed into the form of a *transfer map*.

III. TRANSFER MAPS

Accelerator physics calculates single-particle beam dynamics mostly by using transfer map $\mathbf{M}$ that sends initial phase space coordinates $\vec{z}_i$ into final coordinates $\vec{z}_f = \mathbf{M}(\vec{z}_i)$ without using z or time-dependent orbit formalism. The linear transfer map for a solenoid written in matrix form can be found for example in [1,15]:

$$\mathbf{M}_{\text{solenoid}} = \mathbf{M}_{\text{exit}} \mathbf{M}_{\text{body}} \mathbf{M}_{\text{entrance}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{k}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{k}{2} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\sin kL}{k} & 0 & \frac{1-\cos kL}{k} \\ 0 & \cos kL & 0 & \sin kL \\ 0 & -\frac{1-\cos kL}{k} & 1 & \frac{\sin kL}{k} \\ 0 & -\sin kL & 0 & \cos kL \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{k}{2} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{k}{2} & 0 & 0 & 1 \end{pmatrix} \quad (11)$$

This map formally consists of three parts (see Fig. 1): an entrance fringe field $\mathbf{M}_{\text{entrance}}$ located at $z = 0$, $\mathbf{M}_{\text{body}}$ that represents the exact analytic solution of particle motion in the longitudinal magnetic field B_0 between $0 < z < L$ and an exit fringe field $\mathbf{M}_{\text{exit}}$ located at $z = L$. Particles drift in free space through drift transfer maps before and after the map.

A. Entrance transfer map

In this section we will find transfer map for “half-solenoid” from $z = -\infty$ to $z = z_{\text{max}}$ in the following form of sequential maps:

$$\mathbf{S}_{\text{entrance}} = \mathbf{K}_{\text{kick}} \mathbf{M}_{\text{body}} \mathbf{T}_{\text{entrance}} \mathbf{D}_{\text{drift}} \quad (12)$$

After that, we can easily write the map for the other half of solenoid from $z = z_{\text{max}}$ to $z = \infty$ using symmetry properties. In Eq. (12) we basically need to find a so called “fringe field” map $\mathbf{T}_{\text{entrance}}$ because all other maps are considered to be known: $\mathbf{D}_{\text{drift}}$ is the linear

drift map [1,15] that transfer particle with initial conditions (6) from $z = -\infty$ to $z = 0$ and finishes with coordinates $\{x_0, x_{p0}, y_0, y_{p0}\}$ at the entrance of the thin element $\mathbf{T}_{\text{entrance}}$. $\mathbf{M}_{\text{body}}$ is the linear transfer matrix (11) but with parameter of length L replaced by z_{max} has not been defined yet. $\mathbf{K}_{\text{kick}}$ is a thin nonlinear transfer map at the center of the solenoid z_{max} that we define as:

$$\mathbf{K}_{\text{kick}} \triangleq \begin{pmatrix} x^f \\ x_p^f \\ y^f \\ y_p^f \end{pmatrix} = \begin{pmatrix} x \\ x_p - \frac{k}{16} y (x^2 + y^2) b''(z_{\text{max}}) \\ y \\ y_p + \frac{k}{16} x (x^2 + y^2) b''(z_{\text{max}}) \end{pmatrix} \quad (13)$$

Formally, it is possible to define any kind of transfer map construction (12) for example without $\mathbf{K}_{\text{kick}}$ but the ultimate goal is to replace the orbit formalism (10) in such

a way that the map sequence (12) and especially $\mathbf{T}_{\text{entrance}}$ is as simple as possible. Now, to find the transfer map, we equate the final coordinates of both formalisms (12), (10) at the center of solenoid:

$$\begin{pmatrix} x(z_{\text{max}}) \\ x_p(z_{\text{max}}) \\ y(z_{\text{max}}) \\ y_p(z_{\text{max}}) \end{pmatrix} = \mathbf{K}_{\text{kick}} \mathbf{M}_{\text{body}} \mathbf{T}_{\text{entrance}}(\vec{z}) \quad (14)$$

This gives us an equation for the $\mathbf{T}_{\text{entrance}}$ map. Here, $\vec{z}$ denotes input coordinates $\{x, x_p, y, y_p\}$ that are equal to $\{x_0, x_{p0}, y_0, y_{p0}\}$ from initial condition (6). The solution for $\mathbf{T}_{\text{entrance}}$ can be written out in a general form:

$$\mathbf{T}_{\text{entrance}}(\vec{z}) = (\mathbf{K}_{\text{kick}} \mathbf{M}_{\text{body}})^{-1} \begin{pmatrix} x(z_{\text{max}}) \\ x_p(z_{\text{max}}) \\ y(z_{\text{max}}) \\ y_p(z_{\text{max}}) \end{pmatrix} \quad (15)$$

By substituting expressions (13), (11) and (10) into (15) and taking into account that $b(z_{\text{max}}) = 1$ and $b'(z_{\text{max}}) = 0$ we obtain an explicit form of the transfer map (15). Also we can define the integrals in (10) to be zero for simplicity:

$$\int_{-\infty}^{z_{\text{max}}} z b'(z) dz = 0 \quad (16)$$

This will automatically define position of the center of solenoid z_{max} relative to the entrance fringe field that we placed in $z = 0$ by the definition:

$$z_{\text{max}} = \int_{-\infty}^{z_{\text{max}}} b(z) dz \quad (17)$$

For calculation of (15), because we used only the first order map (10) in k we also need to truncate resulting expression (15) to the first order in k and the resulting map is

$$\begin{pmatrix} x^f \\ x_p^f \\ y^f \\ y_p^f \end{pmatrix} = \begin{pmatrix} x + \frac{k}{8}(2xyx_p + y_p(y^2 - x^2)) \\ x_p + \frac{k}{8}(4y + 2x_p y_p x + y(y_p^2 - x_p^2)) \\ y - \frac{k}{8}(2xyy_p + x_p(x^2 - y^2)) \\ y_p - \frac{k}{8}(4x + 2x_p y_p y + x(x_p^2 - y_p^2)) \end{pmatrix}. \quad (18)$$

The transfer map for higher orders in k can be calculated by calculating the corresponding higher order orbit (10). The first order map (18) agrees with the first order nonlinear map calculated by the Lie methods through integration of Hamiltonian [16]. Detailed consideration of the transformation (15) shows that (18) can be obtained directly from (10) by using the rules $b = 1, b' = 0, b'' = 0$ and the integrals (16) must be equal to zero. Also all z -polynomial terms of the form $F(z)z^i$ or cz^i must be equal to zero.

B. Exit transfer map

After tracking through the entrance half of solenoid by (12), a particle moves through the exit half of solenoid from $z = z_{\text{max}}$ to $z = +\infty$. If the magnetic field profile $b(z)$ is symmetric, i.e., $b(z_{\text{max}} + \Delta z) = b(z_{\text{max}} - \Delta z)$ for any Δz then we can immediately write out the transfer map for the exit half using the property of T-symmetry in classical mechanics:

$$\mathbf{S}_{\text{exit}} = \mathbf{I} \mathbf{S}_{\text{entrance}}^{-1} \mathbf{I} \quad (19)$$

$$\mathbf{I} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

that basically amounts to finding the inverse map of $\mathbf{S}_{\text{entrance}}$. Substituting (12) into (19) and after performing operator transformations we get the exit transfer map from $z = z_{\text{max}}$ to $z = +\infty$:

$$\mathbf{S}_{\text{exit}} = \mathbf{D}_{\text{drift}} \mathbf{T}_{\text{exit}} \mathbf{M}_{\text{body}} \mathbf{K}_{\text{kick}}^{-1} \quad (20)$$

$\mathbf{T}_{\text{exit}}$ is found by using the transformation (19):

$$\mathbf{T}_{\text{exit}} = \mathbf{I} \mathbf{T}_{\text{entrance}}^{-1} \mathbf{I} \quad (21)$$

because all other transfer maps of (20) are known.

C. Full solenoid transfer map

The maps (12) and (20) can be applied sequentially if its needed to get particle phase space parameters at the center of the solenoid z_{max} . However, it is usually necessary to track particle through the whole solenoid. In this case the total transfer map of solenoid will be

$$\mathbf{S}_{\text{total}} = \mathbf{S}_{\text{exit}} \mathbf{S}_{\text{entrance}} = \mathbf{T}_{\text{exit}} \mathbf{M}_{\text{body}} \mathbf{T}_{\text{entrance}} \quad (22)$$

and is presented here without drifts. $\mathbf{M}_{\text{body}}$ is the matrix (11) with parameter $L = 2z_{\text{max}}$ or

$$L = L_{\text{eff}} = \int_{-\infty}^{+\infty} b(z) dz = \frac{1}{B_0} \int_{-\infty}^{+\infty} B(z) dz \quad (23)$$

which is the effective length of the solenoid (see Fig. 1). Here we recall again that k is defined by (5) through the amplitude of magnetic field $B_0 = B(z_{\text{max}})$.

IV. LEADING ORDER TRANSFER MAP

The transfer map (18) for higher order expansions in the k -parameter is much simpler than (10) but still very cumbersome and not symplectic. By using the recurrence formula (9) we can analyze higher orders terms deriving and keeping only so-called *leading order* $b(z)^n$ terms in (10) that gives unit $b(z)^n \rightarrow 1$ in the resulting transfer map. After transformations (9) we will get a recurrence formula for the n th order expansion of the transfer map that will contain a pure dependence on input parameters $\{x, x_p, y, y_p\}$ because $b(z) = 1$:

$$\begin{aligned}
x_n &= \frac{1}{16n} \sum_{i=0}^{n-2} \sum_{j=0}^i x_{n-i-2} (x_j x_{i-j} + y_j y_{i-j}) + \frac{1}{8n} \sum_{i=0}^{n-1} \sum_{j=0}^i (y'_{n-i-1} (y_j y_{i-j} - x_j x_{i-j}) + 2x'_{n-i-1} x_j y_{i-j}) \\
x'_n &= + \left(1 - \frac{1}{2n}\right) y_{n-1} - \frac{1}{16n} \sum_{i=0}^{n-2} \sum_{j=0}^i (x'_{n-i-2} (3x_j x_{i-j} + y_j y_{i-j}) + 2y'_{n-i-2} x_j y_{i-j}) \\
&\quad + \frac{1}{8n} \sum_{i=0}^{n-1} \sum_{j=0}^i (y_{n-i-1} (y'_j y'_{i-j} - x'_j x'_{i-j}) + 2x_{n-i-1} x'_j y'_{i-j}) \\
y_n &= \frac{1}{16n} \sum_{i=0}^{n-2} \sum_{j=0}^i y_{n-i-2} (x_j x_{i-j} + y_j y_{i-j}) + \frac{1}{8n} \sum_{i=0}^{n-1} \sum_{j=0}^i (x'_{n-i-1} (y_j y_{i-j} - x_j x_{i-j}) - 2y'_{n-i-1} x_j y_{i-j}) \\
y'_n &= - \left(1 - \frac{1}{2n}\right) x_{n-1} - \frac{1}{16n} \sum_{i=0}^{n-2} \sum_{j=0}^i (y'_{n-i-2} (3y_j y_{i-j} + x_j x_{i-j}) + 2x'_{n-i-2} x_j y_{i-j}) \\
&\quad + \frac{1}{8n} \sum_{i=0}^{n-1} \sum_{j=0}^i (x_{n-i-1} (y'_j y'_{i-j} - x'_j x'_{i-j}) - 2y_{n-i-1} x'_j y'_{i-j})
\end{aligned} \tag{24}$$

Finally, to calculate the leading order transfer map for $\mathbf{T}_{\text{entrance}}$ we need to calculate infinite series:

$$\mathbf{T}_{\text{entrance}} \triangleq \begin{pmatrix} x^f \\ x_p^f \\ y^f \\ y_p^f \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} x_n k^n \\ \sum_{n=0}^{\infty} x'_n k^n \\ \sum_{n=0}^{\infty} y_n k^n \\ \sum_{n=0}^{\infty} y'_n k^n \end{pmatrix} \tag{25}$$

We skip all intermediate calculations in this paper and present solution that can be verified by (24) after the Taylor series expansion in k :

$$\begin{aligned}
\mathbf{T}_{\text{entrance}} \triangleq \begin{pmatrix} x^f \\ x_p^f \\ y^f \\ y_p^f \end{pmatrix} &= \begin{pmatrix} R \cos \Phi \\ \frac{Q}{R} \cos \Phi - \left(\frac{J}{R} - \frac{kR}{2}\right) \sin \Phi \\ R \sin \Phi \\ \frac{Q}{R} \sin \Phi + \left(\frac{J}{R} - \frac{kR}{2}\right) \cos \Phi \end{pmatrix} \\
\Phi &= \phi - Q \left[\frac{k}{8} - \frac{R^2}{J^3} (4(1 - e^{\frac{kJ}{4}}) + kJ \left(1 + \frac{kJ}{8}\right)) \right] \\
R &= \sqrt{\frac{J^2}{4 + kJ + e^{\frac{kJ}{4}} (J^2 - 4)}} \\
Q &= q \left(1 - (4 - e^{-\frac{kJ}{4}} (4 + kJ)) \frac{r^2}{J^2} \right) \\
\phi &= \arctan(x, y) \\
r &= \sqrt{x^2 + y^2} \\
q &= xx_p + yy_p \\
J &= xy_p - yx_p.
\end{aligned} \tag{26}$$

The fringe field map (26) is sometimes called the hard-edge model because it depends only on the amplitude of magnetic field $B_0 = \int B'(z) dz$ in form of k parameter without considering effect of the more detailed magnetic field shape $B(z)$. The exit fringe field can be calculated in analytic form by using (21) but we will do it in the next section for a more advanced transfer map.

V. SOFT-EDGE TRANSFER MAP

Apparently, the leading order terms $b(z)$ impact only the nonlinear effect because the transfer map (26) turns into (11): $T_{\text{entrance}} \rightarrow M_{\text{entrance}}$ when taking the linear part of its input parameters $\{x, x_p, y, y_p\}$. Calculating the second and higher order transfer map (18) reveals many different coefficients of the following type:

$$\begin{aligned}
m_i &= \int_{-\infty}^{z_{\max}} z^i b(z) b'(z) dz \\
p_i &= \int_{-\infty}^{z_{\max}} z^i b'(z)^2 dz \\
s_i &= \int_{-\infty}^{z_{\max}} z^i b''(z)^2 dz \\
b(z) &= \frac{B(z)}{B_{\max}}
\end{aligned} \tag{27}$$

Recall again that the frame of integration (27) satisfies conditions (16), (17). The parameter $m_0 = b(z_{\max})^2/2 = 1/2$ is one of the hard-edge terms that has been taken into account in the previous section. In this paper we will take into account only four soft-edge coefficients: m_1, m_2, p_0, s_0 that is sufficient to derive an accurate solution of the original equation (5) that we are solving. Generally, the

error due to magnetic field truncation (3), (5) is much larger than the error due to the limited system of the soft-edge parameters.

The soft-edge transfer map can be calculated exactly like the leading order map in the previous section. In this section we skip all intermediate transformations and calculations and present final solution (25) for the entrance transfer map depending on m_1, m_2, p_0, s_0 :

$$\begin{aligned} \mathbf{T}_{\text{entrance}} &\triangleq \begin{pmatrix} x^f \\ x_p^f \\ y^f \\ y_p^f \end{pmatrix} = \begin{pmatrix} R \cos \Phi \\ \frac{Q}{R} \cos \Phi - \left(\frac{J}{R} - \frac{kR}{2}\right) \sin \Phi \\ R \sin \Phi \\ \frac{Q}{R} \sin \Phi + \left(\frac{J}{R} - \frac{kR}{2}\right) \cos \Phi \end{pmatrix} \\ \Phi &= \phi - Q \left[\frac{k}{8} - \frac{R^2}{J^3} \left(4(1 - e^{\frac{kJ}{4}}) + kJ \left(1 + \frac{kJ}{8} \right) \right) \right] \\ R &= \sqrt{\frac{J^2}{4 + kJ + e^{\frac{kJ}{4}} \left(\frac{J^2}{r^2} e^{\frac{k^2 m_2}{2}} - 4 \right)}} \\ Q &= \left(1 - (4 - e^{-\frac{kJ}{4}} (4 + kJ)) \frac{r^2}{J^2} e^{-\frac{k^2 m_2}{2}} \right) \\ &\quad \times \left(r e^{-\frac{k^2 m_2}{4}} \sqrt{\frac{2}{3}} \tan \left[\sqrt{\frac{3}{8}} k^2 m_1 r e^{-\frac{k^2 m_2}{4}} \right] \right. \\ &\quad \left. - \frac{k^2}{8} r^4 p_0 - \frac{3k^2}{256} r^6 s_0 + q \right) \\ \phi &= \arctan(x, y) \\ r &= \sqrt{x^2 + y^2} \\ q &= xx_p + yy_p \\ J &= xy_p - yx_p \end{aligned} \quad (28)$$

Exit map for symmetric solenoids $b(z_{\text{max}} + \Delta z) = b(z_{\text{max}} - \Delta z)$ can be derived directly from (28) by (21):

$$\begin{aligned} \mathbf{T}_{\text{exit}} &\triangleq \begin{pmatrix} x^f \\ x_p^f \\ y^f \\ y_p^f \end{pmatrix} = \begin{pmatrix} R \cos \Phi \\ \frac{Q}{R} \cos \Phi - \frac{J}{R} \sin \Phi \\ R \sin \Phi \\ \frac{Q}{R} \sin \Phi + \frac{J}{R} \cos \Phi \end{pmatrix} \\ \Phi &= \phi + q \left[\frac{k}{8} - \frac{r^2}{J^3} \left(4(1 - e^{\frac{kJ}{4}}) + kJ \left(1 + \frac{kJ}{8} \right) \right) \right] \\ R &= e^{\frac{k^2 m_2}{4}} \sqrt{\frac{J^2}{4 + e^{-\frac{kJ}{4}} \left(\frac{J^2}{r^2} - 4 - kJ \right)}} \\ Q &= q \left(1 - (4 - e^{-\frac{kJ}{4}} (4 + kJ)) \frac{R^2}{J^2} e^{-\frac{k^2 m_2}{2}} \right)^{-1} \\ &\quad + R e^{-\frac{k^2 m_2}{4}} \sqrt{\frac{2}{3}} \tan \left[\sqrt{\frac{3}{8}} k^2 m_1 R e^{-\frac{k^2 m_2}{4}} \right] \\ &\quad - \frac{k^2}{8} R^4 p_0 - \frac{3k^2}{256} R^6 s_0 \\ \phi &= \arctan(x, y) \\ r &= \sqrt{x^2 + y^2} \\ q &= xx_p + yy_p \\ J &= xy_p - yx_p + \frac{k}{2} (x^2 + y^2). \end{aligned} \quad (29)$$

For symmetric $B(z)$ fields all coefficients m_1, m_2, p_0, s_0 in (28), (29) are the same with the same sign. If the field $B_{ns}(z)$ is not symmetric then coefficients for the entrance fringe field (28) must be calculated as in (27) by using $B_{ns}(z)$, and for the exiting field by using its mirror transformation relative to the z_{max} point and integrating (27) as usual from $-\infty$ to z_{max} where $z_{\text{max}} = B_0^{-1} \int_{z_{\text{max}}}^{+\infty} B_{ns}(z) dz$ will have slightly different value than (17).

A. Linear transfer map for solenoid

We will write out soft-edge linear transfer map for solenoid by calculating linear coefficients (28) and (29) for $\{x, x_p, y, y_p\}$ phase space coordinates:

$$\begin{aligned} \mathbf{M}_{\text{solenoid}} &= \mathbf{M}_{\text{exit}} \mathbf{M}_{\text{body}} \mathbf{M}_{\text{entrance}} \\ &= \begin{pmatrix} e^{\frac{k^2 m_2}{4}} & 0 & 0 & 0 \\ \frac{k^2 m_1}{2} e^{-\frac{k^2 m_2}{4}} & e^{-\frac{k^2 m_2}{4}} & -\frac{k}{2} e^{-\frac{k^2 m_2}{4}} & 0 \\ 0 & 0 & e^{\frac{k^2 m_2}{4}} & 0 \\ \frac{k}{2} e^{-\frac{k^2 m_2}{4}} & 0 & \frac{k^2 m_1}{2} e^{-\frac{k^2 m_2}{4}} & e^{-\frac{k^2 m_2}{4}} \end{pmatrix} \begin{pmatrix} 1 & \frac{\sin kL}{k} & 0 & \frac{1 - \cos kL}{k} \\ 0 & \cos kL & 0 & \sin kL \\ 0 & -\frac{1 - \cos kL}{k} & 1 & \frac{\sin kL}{k} \\ 0 & -\sin kL & 0 & \cos kL \end{pmatrix} \begin{pmatrix} e^{-\frac{k^2 m_2}{4}} & 0 & 0 & 0 \\ \frac{k^2 m_1}{2} e^{-\frac{k^2 m_2}{4}} & e^{\frac{k^2 m_2}{4}} & \frac{k}{2} e^{-\frac{k^2 m_2}{4}} & 0 \\ 0 & 0 & e^{-\frac{k^2 m_2}{4}} & 0 \\ -\frac{k}{2} e^{-\frac{k^2 m_2}{4}} & 0 & \frac{k^2 m_1}{2} e^{-\frac{k^2 m_2}{4}} & e^{\frac{k^2 m_2}{4}} \end{pmatrix} \end{aligned} \quad (30)$$

All zero elements in the fringe field matrices (30) really depend on the m_3 and m_4 coefficients that we did not take into account because of its small linear effect.

VI. PROPERTIES OF TRANSFER MAPS

a. Symplecticity. The nonlinear and linear maps of the solenoid are symplectic [1] analytically in canonical coordinates. Transverse canonical coordinates depend on transverse vector potential A_x and A_y are equal to zero outside of the solenoid. In this case input and output canonical coordinates of the full solenoid (22) are equal to normal ones $\{x, x_p, y, y_p\}, \{x^f, x_p^f, y^f, y_p^f\}$ and the nonlinear (22), (28), (29) or linear (30) maps of the full solenoid are symplectic:

$$\mathbf{M}^\top \mathbf{J} \mathbf{M} = \mathbf{J} \quad (31)$$

where $\mathbf{M}$ is Jacobian of the map and $\mathbf{J}$ is the unit symplectic matrix [1]:

$$\mathbf{M} = \frac{\partial(x^f, x_p^f, y^f, y_p^f)}{\partial(x, x_p, y, y_p)} \quad (32)$$

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Individual maps $\mathbf{T}_{\text{entrance}}$, or $\mathbf{M}_{\text{entrance}}$ are not symplectic in normal coordinates but symplectic in canonical coordinates because of nonzero transverse vector potential inside of solenoid. When the final coordinates in the Jacobian matrix $\mathbf{M}$ are replaced by canonical ones:

$$\mathbf{M} = \frac{\partial(x^f, x_p^f - k \frac{y^f}{2}, y^f, y_p^f + k \frac{x^f}{2})}{\partial(x, x_p, y, y_p)} \quad (33)$$

the individual maps $\mathbf{T}_{\text{entrance}}$ and $\mathbf{M}_{\text{entrance}}$ become symplectic (31).

b. Canonical angular momentum is a constant of motion that is equal to:

$$\text{const} = x(z)y'(z) - y(z)x'(z) + \frac{k}{2}(x(z)^2 + y(z)^2)b(z) - \frac{k}{16}(x(z)^2 + y(z)^2)^2 b''(z) \quad (34)$$

in terms of the equation of motion (5) with given truncated magnetic field. Substituting final/output coordinates of (12) into (34) with $z = z_{\text{max}}$ confirms conservation of canonical angular momentum:

$$x^f y_p^f - y^f x_p^f + \frac{k}{2}(x^{f2} + y^{f2}) - \frac{k}{16}(x^{f2} + y^{f2})^2 b''(z_{\text{max}}) = x y_p - y x_p \quad (35)$$

where the resulting expression $x y_p - y x_p$ is the canonical angular momentum for $k = 0$ before the particle enters the solenoid.

c. Hard-edge fringe field limit defined as the unit step function for input and output fringe field: $b'(z) \rightarrow \delta(z)$ produces divergent integrals $p_0, s_0 \rightarrow \infty$ (27) and hence the transfer maps (28), (29). The same kind of divergence has been found by integration of Hamiltonian in Lie algebra method [11]. Therefore the nonlinear beam dynamics of solenoid depends on fringe field shape that cannot be simplified by the hard-edge fringe field model.

d. Paraxial limitation. The expression under the square root in (28) must be much greater than zero for the paraxial motion approximation to be valid. If it is less than zero then particle will be reflected back by the entrance fringe field of solenoid. This can be demonstrated by solving the full 3D equation of motion (1). This can happen for example at small particle energy or in very strong magnetic fields $k^2(x^2 + y^2) > 8$ even if $x_p = y_p = 0$. This effect is called a magnetic mirror and must be estimated and taken into account before applying the paraxial model. The expression under the square root for the exit fringe field (29) is always positive. This means that particle will always exit solenoid with any initial conditions if it starts in its center.

VII. BENCHMARK OF PARTICLE OPTICS

We now perform a benchmark of our transfer maps for proton motion in the magnetic field of a single current loop with $R = 10$ cm radius. The current loop is located at the center $z = 0$ with maximum magnetic field $B_0 = 1$ T at the center of coordinates. 3D magnetic field of the current loop has a precise analytic solution [17] and shown in Fig. 2. The $B(0, 0, z)$ -field profile of the current loop has the following form:

$$B(0, 0, z) = B_0 b(z) = B_0 \frac{R^3}{(R^2 + z^2)^{\frac{3}{2}}} \quad (36)$$

The transfer map parameters for the magnetic field (36) are calculated to be the following:

FIG. 2. Simulation of proton dynamics in 3D field of the current loop with radius $R = 0.1$ m. The gray arrows show precise magnetic field distribution of the current loop. The proton starts with $\{x, 0, y, 0\}$ parameters where $r_0 = \sqrt{x^2 + y^2}$.

$$\begin{aligned}
 z_{\text{entrance}} &= -R \\
 z_{\text{exit}} &= +R \\
 L_{\text{eff}} &= L = 2R \\
 m_1 &= R \left(\frac{1}{2} - \frac{3\pi}{32} \right) \\
 m_2 &= R^2 \left(\frac{3}{4} - \frac{3\pi}{16} \right) \\
 p_0 &= \frac{1}{R} \frac{45\pi}{256} \\
 s_0 &= \frac{1}{R^3} \frac{1575\pi}{2048} \\
 b''(z_{\text{max}}) &= -\frac{3}{R^2}.
 \end{aligned} \tag{37}$$

A proton with input parameters $\{x, 0, y, 0\}$ will be focused by the magnetic field and will cross the z -axis at some point $z = z_f$ called the focal length. By applying the linear model (30) we can calculate the focal length as:

$$\begin{aligned}
 z_f &= z_{\text{exit}} - \frac{x^f}{x_p^f} = z_{\text{exit}} - \frac{y^f}{y_p^f} \\
 &= z_{\text{exit}} - \frac{2e^{\frac{k^2 m_2}{2}} (\cot[\frac{kL}{2}] + km_1)}{k(2km_1 \cot[\frac{kL}{2}] - 1 + k^2 m_1^2)}.
 \end{aligned} \tag{38}$$

Note that it does not depend on the input parameter $r_0 = \sqrt{x^2 + y^2}$ in linear approximation. Table I shows

TABLE I. Benchmark of the focal length for different proton energies T_p and different models and simulations.

T_p (MeV)	z_f^a (m)	z_f^b (m)	z_f^c (m)	z_f^d (m)	z_f^e (m)
1	0.73873	0.73873	0.44891	0.76209	0.73670
10	7.15776	7.15776	4.23124	7.18318	7.15606
100	74.7005	74.7005	44.0173	74.7261	74.6988
1000	1086.73	1086.73	640.150	1086.75	1086.72

^a2D z-dependent tracking (5) with truncated field.

^b3D t-dependent tracking (1) in 3D field of current loop [17].

^cModel (11) with $L = \int b(z) dz$ and $B_0 = B_{\text{max}} = 1T$.

^dModel (11) with $L = \frac{(\int b(z) dz)^2}{\int b(z)^2 dz}$ and $B_0 = B_{\text{max}} \frac{\int b(z)^2 dz}{\int b(z) dz}$ [18].

^eFocal length (38) obtained by the soft-edge linear model (30).

the focal length obtained by different models and simulations. The initial transverse coordinate of the proton $r_0 = 0.1$ mm was chosen to be small for numerical simulations *a* and *b* in order to avoid non-linear aberrations. The linear model (30) agrees best when compared with numerical simulations *a* and *b*. Larmor rotation angle of solenoid that equals to $\delta\phi = -kL/2$ is the same for all linear models (a–d) of Table I. Nonlinear aberrations start at larger r_0 , as is shown in Fig. 3 for the case of protons with 10 MeV energy. The nonlinear map (22), (28), (29) has a perfect agreement with the 2D simulation (5) and a good agreement with the 3D simulation (1) for $r_0 < R/2$. Figure 3 also shows that there is no difference between 2D and 3D case if magnetic field is precisely calculated.

FIG. 3. Focal length as a function of input coordinate $r_0 = \sqrt{x^2 + y^2}$ for protons with 10 MeV energy. Two lower matching curves show 2D (5) and 3D (1) simulations in case of precisely defined magnetic field of the current loop [17]. Two upper matching curves are for 2D simulations (5) with truncated magnetic field and for the nonlinear soft-edge transfer maps (22), (28), (29) derived from this 2D equation. The single horizontal blue dashed line represents the hard-edge model (26) that is equivalent to the soft-edge model with $m_1 = m_2 = p_0 = s_0 = 0$.

It follows that for more accuracy of the transfer map we need to take more or infinite series expansion (2) for solving 2D equation (5). The hard edge model represents only 1% change for the focal length within the solenoid aperture so its not applicable for any aberration estimate.

VIII. SUMMARY

We derived an analytic solution for paraxial charged particle motion in solenoidal magnetic fields that works accurately within the 2D paraxial approximation and for beam sizes up to half the aperture of the solenoid, $r_{\text{beam}} < 0.5R_{\text{solenoid}}$. The accuracy is limited by the precision of the magnetic field given in the form of a truncated series expansion near z -axis. For greater accuracy, it is necessary to take the full infinite series expansion of the magnetic field which will be a subject of future works. The method developed in this paper can be extended for other magnets widely used in accelerator physics such as dipoles, quadrupoles, sextupoles etc. Also, it is possible to introduce space charge into the equation of motion and obtain solutions for intense and nonlinear charged-particle beams dynamics in magnetic fields.

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