Systematical study on superconducting radio frequency elliptic cavity shapes applicable to future high energy accelerators and energy recovery linacs

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(Received 25 May 2016; published 25 October 2016)

Elliptic cavities at medium- and high- β range are receiving broader use in the particle accelerator applications. Optimizing the shape of these cavities is a complex and demanding process. In this paper we propose an optimization approach to minimize the ratio of peak magnetic field to the acceleration field H_{pk}/E_{acc} while keeping the ratio of peak surface electric field to the accelerating field E_{pk}/E_{acc} , aperture radius and wall slope angle α at some permitted values. We show that it is possible to substantially vary the cavity geometry without violating the constraints or deteriorating the objective of the optimization. This gives us freedom in designing the geometry to overcome problems such as multipactor while maintaining the minimal H_{pk}/E_{acc} . The optimization is then performed to find a set of optimized geometries with minimum H_{pk}/E_{acc} for different β 's ranging from 0.4 to 1, different peak surface electric fields, wall slope angles and aperture radii. These data could be generally used as a suitable starting point in designing elliptic cavities.

DOI: [10.1103/PhysRevAccelBeams.19.102002](https://doi.org/10.1103/PhysRevAccelBeams.19.102002)

I. INTRODUCTION

As it was stated in [1] the interest in proton accelerators has necessitated the development of structures that bridge the gap between $\beta = 1$ cavities for electrons and low- β resonators for heavy ions. All the superconducting radio-frequency (SRF) cavities in operation today for the speed-of-light particles are of the same design: they are so-called elliptic cavities. Their geometry can be extended to lower β by reducing the length of the cells.

The attractive feature of the elliptic cavities is their simple geometry and therefore relative cheapness compared to other types of SRF cavities used for lower β [2], like different $\lambda/4$ or $\lambda/2$ transmission line cavities or multielement spoke cavities. So, the attempt to expand the area of elliptic cavities to lower β 's is natural, and it is realized in many laboratories.

Any superconducting cavity is an expensive structure and it should be properly optimized to obtain maximal acceleration rate E_{acc} . A hard limiting barrier in the SRF cavities is the maximal (or peak) surface magnetic field H_{pk} . So, minimization of the value of H_{pk}/E_{acc} is the first task for the cavity optimization. Minimization of this value is closely related to the decrease of rf losses in the cavity which defines the cost of operation. However, some

circumstances make necessary deviations from the optimal shape: higher order modes, multipactor, Lorentz force detuning and others.

In this paper, minimization of the H_{pk}/E_{acc} is done for given values of E_{pk}/E_{acc} , where E_{pk} is the peak surface electric field. The proposed algorithm is interchangeable, and, conversely, minimization of the E_{pk}/E_{acc} can be done for given values of H_{pk}/E_{acc} if the field emission is the dominated factor for the operation field gradient. However, we see a little reason to do this work because the peak electric field, or, more precisely, E_{pk}/E_{acc} , to prevent field emission, is already limited from above as a primary parameter in the present paper.

In this work we try to systematically compare the properties of elliptic cavities at different β , to understand limitations from the lower values of β and to suggest some solutions of the multipactor problem.

We found that some substantial deviations of the shape can be done without deterioration of the value of H_{pk}/E_{acc} . This property opens the way to overcome above-mentioned problems with multipactor, higher order modes, and Lorentz forces keeping minimal peak magnetic field and cryogenic losses.

II. ELLIPTIC GEOMETRY AND SURFACE FIELDS

Optimization of an elliptic cavity consists of the proper choice of geometrical parameters for a contour described by two elliptic arcs and a straight segment tangent to both of them, Fig. 1.

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Three of these parameters, length of the half-cell L , the aperture radius R_a , and the equatorial radius R_{eq} are not a subject of optimization. They are defined by other physical requirements: L should be equal to the quarter of the wave length, so that the particle could be effectively accelerated in each cell of a multicell cavity, the aperture is defined by requirements to coupling between cells and by the level of wake fields that could be allowed for a given project, the equatorial radius R_{eq} is used for tuning the cavity for a given frequency.

From the remaining six parameters shown in Fig. 1 (A , B , a , b , d , γ), four only are needed to fully describe the geometry. Here A and B , also as a and b are the half-axes of the equatorial and iris constitutive ellipses, respectively.

Some authors follow [3] and use combinations of these parameters: $R = B/A$, the equator ellipse aspect ratio and $r = b/a$, the iris ellipse aspect ratio, also as the wall distance from the iris plane d and the wall angle inclination γ . So, we have four dissimilar primary parameters: R , r , d , and γ .

The reason of this choice is explained [3] as follows: R defines a local minimum of the peak surface magnetic field, r defines a local minimum of the peak surface electric field, d allows to reduce the capacitive volume in favor of the magnetic volume and vice versa, in order to balance the peak surface magnetic and electric fields on the cavity walls, γ influences the mechanical behavior of the cavity and controls its inductive volume.

This explanation looks convincing and physically feasible. However, change of the angle γ or the permitted value of E_{pk}/E_{acc} leads to the necessity to optimize again the values of R and r . Besides, there is a dependence of the local minimum of the magnetic field on the iris aspect ratio though not so strong as on the equator aspect ratio, and conversely, the electric peak field minimum depends on the

FIG. 1. Cell shape parametrization.

FIG. 2. Geometry of the elliptic half-cell: non-reentrant—(a), and reentrant—(b) and (c). Angle $\alpha = 90^\circ + \gamma$, compare to Fig. 1.

equator ellipse aspect ratio. On the other hand, the choice of four half-axes lengths (A , B , a , and b) as the primary parameters for optimization simplifies the description and does not increase the work for optimization.

Recalculation from A/B , a/b , d and γ to A , B , a and b can be easily done solving a system of algebraic equations.

Equations defining half-axes of the smaller ellipse, a and b , and also coordinates of the point of conjugation (x_2 , y_2) (point x_2 in Fig. 2), taking as given the values of γ , L , R_a , d and r , are the following:

$$\begin{aligned} \tan \gamma &= \frac{L - d - x_2}{y_2 - R_a}, \\ \frac{(L - x_2)^2}{a^2} + \frac{(y_2 - R_a - b)^2}{b^2} &= 1, \\ b(L - x_2) \tan \gamma &= a \sqrt{a^2 - (L - x_2)^2}, \\ b/a &= r. \end{aligned} \quad (1)$$

Half-axes of the bigger ellipse, A and B , and coordinates of the point of conjugation (x_1 , y_1) (x_1 in Fig. 2), taking additionally as given the values of R_{eq} and R , are the following:

$$\begin{aligned} \tan \gamma &= \frac{x_2 - x_1}{y_1 - y_2}, \\ \frac{x_1^2}{A^2} + \frac{(y_1 - R_{eq} + B)^2}{B^2} &= 1, \\ Bx_1 \tan \gamma &= A \sqrt{A^2 - x_1^2}, \\ B/A &= R. \end{aligned} \quad (2)$$

We will use this set of primary geometric parameters for optimization (A , B , a , and b), as well as it is used in many other projects [4–6]. Some details of the geometry used in the future discussion are shown in Fig. 2. We will measure the angle of the wall inclination as the angle α between the

FIG. 3. Comparison of three known geometries of the elliptic half-cells.

axis of rotation and the straight segment of the wall, $l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, like it was done in our earlier publications. The cavity with $\alpha < 90^\circ$ is known as the reentrant cavity.

As an example, three widely known geometries of the inner half-cells are shown in Fig. 3: TESLA [4] from DESY, the low-loss cavity [5] from JLab, and the reentrant cavity by Cornell [6].

The surface electric and magnetic fields in these cavities are shown in Figs. 4 and 5. One can see that the electric

FIG. 4. Electric field on the surface of the cavities from Fig. 3.

FIG. 5. Magnetic field on the surface of the cavities from Fig. 3.

field has irregularities at the points P and Q on the surface of non-reentrant cavities—where the arc conjugates with the segment of the straight line. We believe that the best shape should have more regular dependencies for these curves, but for this purpose we have, possibly, to abandon the paradigm of an elliptic shape and to pass to a smoother contour, e.g., using some kind of splines [7].

In the case of the reentrant cavity, the maximum is not flat as it is on non-reentrant geometry. One can suppose that flattening of this maximum will decrease the peak electric field but the E_{acc} should not change significantly.

Comparing the magnetic fields of the TESLA cell and the reentrant cell having the same aperture we can see that decrease of the maximal field is obtained through the lengthening of the maximal field region. Again, better flattening of the maximal values of this field could decrease the peak field though not much. The area under these curves is approximately the same in both cases. In the case of the low loss cavity the decrease of the field is obtained due to lesser aperture. As it will be shown below, H_{pk}/E_{acc} is a monotone function of the aperture radius and it is smaller here due to a smaller R_a .

Because of high cost of superconducting cavities even a several percent further decrease of H_{pk}/E_{acc} is worth the investigations, and we see that this decrease is possible.

III. SOME DEFINITIONS

The accelerating field E_{acc} is defined as the energy increment ΔU , in volts, after the passage of a particle through the inner cell of the cavity to the length of this cell. For $\beta = 1$ this length ($2L$, Figs. 2 and 3) is equal to $\lambda/2$, $L = \lambda/4$. In the case of $\beta < 1$ we have to define the accelerating field as $E_{acc} = \Delta U/(\beta\lambda/2)$ since the length of the half-cell is $L = \beta\lambda/4$. In the case when the particle velocity does not correspond to the length of the cell, the value of ΔU will differ from the value calculated for the geometrical β_{geo} . For fear of causing confusion, we will use E_{acc} corresponding to the geometrical beta. Usage of the particle velocity β_{eff} higher than β_{geo} is justified by the fact

that the transit time factor becomes higher than for $\beta_{\text{eff}} = \beta_{\text{geo}}$ if this exceedance is about 2%–5%.

The end cells do not have a physically defined exact length because the field propagates in the beam pipe exponentially decreasing. It appears logical for calculation of the accelerating field in the end cells to use the same definition: $E_{\text{acc}} = \Delta U / (\beta\lambda/2)$, so that the length of this cell is not required.

It is convenient to compare different cavities with a well known one. So, for the definition of the overvoltage $E_{\text{pk}}/E_{\text{acc}}$ where E_{pk} is maximal electric field on the cavity surface, we will use the value of $e = E_{\text{pk}}/E_{\text{acc}}/2$ so that for the well-known TESLA cavity $e = 1$.

Analogously, the values of $H_{\text{pk}}/E_{\text{acc}}$ are normalized to corresponding values of TESLA [42 Oe/(MV/m)] so that $h = H_{\text{pk}}/E_{\text{acc}}/42$ is equal to 1 for TESLA cells. {According to our calculations, the normalized magnetic field appears about 1.5% less than this value [41.4 Oe/(MV/m)]. In publications [8] of 1992 (Haebel) $H_{\text{pk}}/E_{\text{acc}} = 41.7$, later [9] (Edwards, 1995) this value is shown as 42, the last publication [4] (Aune, 2000) gives 42.6 Oe/(MV/m). Normalization for 42 is chosen because: (1) it is convenient to use round numbers, (2) some deviations in different references are about this value.}

The dimensions of cavities (always in millimeters)—unless specially noted otherwise—are given for the frequency of 1300 MHz, this is a “most popular” frequency of superconducting cavities. However, all the optimizations will be valid for any frequency because the discussed figures of merit, $E_{\text{pk}}/E_{\text{acc}}$, $H_{\text{pk}}/E_{\text{acc}}$, as well as R_{sh}/Q —the effective cavity impedance, and GR_{sh}/Q —the geometrical factor of losses, do not depend on frequency; dimensions should be scaled only, if a different frequency is used. Here R_{sh} is the shunt impedance of a cavity, Q is the quality factor, and G is the geometrical impedance.

IV. METHOD OF OPTIMIZATION

First of all it should be noted that for calculation of the secondary values for any geometrical parameters we used the SUPERLANS code by D. Myakishev and V. Yakovlev [10] and the TUNEDCELL envelope code written by Dmitri Myakishev [11]. These codes have the necessary accuracy needed for our optimization [12].

From the previous optimization [13] for the minimum h we know that this minimum is a monotone function of the overvoltage $E_{\text{pk}}/E_{\text{acc}}$, it is decreasing when this value increases, and a monotone function of the wall slope angle α , it is decreasing when the angle becomes smaller. So, the minimal value of h will be at the given permitted values of $E_{\text{pk}}/E_{\text{acc}}$ and α .

The goal of the cavity optimization is to find this minimum of h under given constraints: the value of e is less than or equal to a given one, and the value of α is bigger

FIG. 6. Brute force optimization does not lead to the minimum.

than or equal to a given one. We are going to minimize the function $h(A, B, a, b)$ under these conditions. The “brute force” or the “grid search” approach is to calculate h on a 4-dimensional grid around some point, to find the minimum of h between these points, to make this point of minimum the central point and repeat the procedure. When the central point is the best point for a given steps in all 4 directions, the value of this step can be decreased, so the minimum can be found with a needed accuracy. However, the example presented in Fig. 6 shows that this method is inefficient when the optimized function belongs to a class of so called ravine functions.

In order to prove this, we tried to solve the optimization problem with the Levenberg-Marquardt algorithm [14] which is gradient based. And in fact, as predicted by literature, due to the ravine shape of the problem, it did not show good convergence compared to a Monte-Carlo based algorithm. Eventually we used a Monte-Carlo scheme that uses information from precomputed values, which is often referred to as Markov-Chain Monte-Carlo in the literature [15]. Therefore we pick a starting point from which, by Monte-Carlo, a better point is searched.

At the points marked with crosses around the point A (Fig. 6), the level of the surface is higher than at the point A. This level is decreasing to points B, C, and D. In this example steps along Y should be two times bigger than in the direction X to find the correct minimum.

As it will be shown below (Figs. 8–11), we have exactly the case of a ravine function: a rapid descent to the boundary and gently inclined way along the boundary.

The Monte Carlo method to find the minimum was faster than the grid method but it also “stuck” very often not reaching the final minimal point. To check the correctness of the found minima $h(A, B, a, b)$ for each value of the slope angle α , as presented in Figs. 17 and 19 to 24, the

FIG. 7. Dependencies of h and e on a and b near the point of $\min h$.

dependencies of A , B , a , and b on α were plotted, an example is in Fig. 18. Very often these dependences appeared to be not smooth, in these cases a more thorough search for the minimum was needed.

In the search for the minimum h we can easily come to the boundary where the limiting value of e and α are reached. To move along the “bottom” of the ravine we can solve the system of equations:

$$\begin{aligned}
 &\frac{\partial h}{\partial A} \cdot \Delta A + \frac{\partial h}{\partial B} \cdot \Delta B + \frac{\partial h}{\partial a} \cdot \Delta a \\
 &+ \frac{\partial h}{\partial b} \cdot \Delta b = \Delta h, \\
 &\frac{\partial e}{\partial A} \cdot \Delta A + \frac{\partial e}{\partial B} \cdot \Delta B + \frac{\partial e}{\partial a} \cdot \Delta a \\
 &+ \frac{\partial e}{\partial b} \cdot \Delta b = \Delta e = 0, \\
 &\frac{\partial \alpha}{\partial A} \cdot \Delta A + \frac{\partial \alpha}{\partial B} \cdot \Delta B + \frac{\partial \alpha}{\partial a} \cdot \Delta a \\
 &+ \frac{\partial \alpha}{\partial b} \cdot \Delta b = \Delta \alpha = 0.
 \end{aligned} \quad (3)$$

Here, Δh is a negative value to be added to h to decrease it.

Now, we have three equations and four unknowns: ΔA , ΔB , Δa , Δb . The fourth equation can be added if we want to go to the minimum by the shortest way, so that the length of the vector of increments is minimal:

$$\Delta A^2 + \Delta B^2 + \Delta a^2 + \Delta b^2 = s^2. \quad (4)$$

For any Δh we can choose s , preferably small and solve this system of equations. So, we can consider this approach as an implementation of the gradient method in the case of the ravine function and the function under some constraints.

We can solve the system of equations (3) and (4) for any negative value of Δh and find a lower point than the point

we started of. It looks like we can decrease h indefinitely long time. Of course, this is possible only as long as the absolute value of Δh is small enough so that the partial derivatives do not change significantly.

Nevertheless, it is not clear from this system that the minimum is reached. It becomes clear if we consider the dependencies of h or e on A , B , a , and b . In Fig. 7 the dependencies of h and e on a and b are shown for the case of $\beta = 1$, $R_a = 35$ mm, $e = 2$, $\alpha = 100^\circ$ (see this point in Table II and Fig. 17). The minimum h appears near the point of break of this dependence. When the derivative is considered for the left side of the point of break ($\Delta a < 0$), the result of solution of the system (3)–(4) gives $\Delta a > 0$, and vice versa, considering $\Delta a > 0$, the solution gives $\Delta a < 0$. This contradiction shows that further decrease of h is impossible, so the minimum is reached.

The same consideration can be done for all other partial derivatives in (3). However, in practice, only derivatives of h and e with respect to a and b can change their behavior near the point of minimum h .

The cause of this break is the following: there are two local maxima on the curves of H vs L/L_0 , Fig. 5; one maximum is at $L = 0$, another one is in the right part of each curve, like the maximum near the A point in the RE cavity. There can be also two local maxima on the flat part of the curve E vs L/L_0 , Fig. 4. The second local maximum E at $L/L_0 = 1$ appears by thorough optimization and practically cannot be seen on the flat part of the curve 2, Fig. 4. These maxima are not very well pronounced on these graphs because the curves become very flat between them in the process of optimization. When changing, for example, the half-axis a , trying to minimize H_{pk} (and so the h), the bigger maximum of H decreases but the smaller one grows and can overtake the other one. At the point of break (Fig. 7), these maxima become equal, and if we continue to change a , the value of H_{pk} is defined now by the other hump and its dependence on a changes. So, the minimum h is reached when the humps on the curve H vs L/L_0 or on the curve E vs L/L_0 become equal by height. In the last case the decrease of $h = H_{pk}/E_{acc}$ due to equalization of local maxima E becomes not because of minimization of H_{pk} but because of increase of E_{acc} when the force lines are reaching their maximum uniformly around the iris tip.

V. MORE CONSTRAINTS TO THE SHAPE OF THE ELLIPTIC CAVITY

The proposed optimization can lead to shapes that are not realizable as an inner cell of a multicell cavity. So, for the reentrant cavity, and in most of cases for the non-reentrant cavity (except of the case of very big half-axes B), the value of A cannot be bigger than the cavity length. Moreover, taking into account the thickness of the wall, usually 3 mm, the restriction $A + 3 < L$ should be applied. If we consider

the necessary gap between cells needed for welding them together, we should make this restriction even more severe. However, this restriction is valid for the frequency of 1300 MHz. If the same thickness 3 mm is used for a cavity, say, 650 MHz, the restriction due to the touch of the walls becomes $A + 1.5 < L$. Here A and L are parameters of cavity at 1300 MHz (not at 650 MHz). So, the scaling factor should be taken into account. The absolute restriction is $A < L$.

Nowadays, the cavities are mainly stamped from niobium sheets. To guarantee accuracy in the process of stamping the half-cells, the curvature radius of the iris cannot be too small even if it does not increase the E_{pk} . The reasonable minimal value of this radius is twice the thickness of the niobium sheet used in fabrication (usually 3 mm): $r_c > 6$ mm. In case of the frequency different from 1300 MHz, the scaling should be also taken into account.

Even if the curvature is bigger than 6 mm ($f = 1300$ MHz), the value of a cannot be smaller than 3 mm because the outer surface of the iris should be reachable. This gap should be possibly several millimeters wide to make possible welding of the cells.

For example the TRASCO-ASH cavity for $\beta = 0.47$ and frequency $f = 704.4$ MHz [16] has the radius of curvature at the iris tip $r_c = 6.1$ mm that corresponds to $r_c = 3.31$ mm at $f = 1300$ MHz. The niobium sheet thickness for the cavities described in this paper was ranging from 3.3 to 4.3 mm.

VI. AN EXAMPLE OF OPTIMIZATION FOR THE TESLA CAVITY

The TESLA cavity [4,9] is thoroughly studied because it was proposed to be used in the International Linear Collider. In spite of “round” values of the elliptical half-axes ($A = B = 42$, $a = 12$, $b = 19$ mm) it is very close to the optimized one in the way of optimization proposed here. In Fig. 8 are presented the values of normalized parameters h and e , and the wall slope angle α for fixed values of $a = 12$ mm and $b = 19$ mm and for different values of A and B around the nominal point. The pink area for e and the blue area for α are “forbidden” areas, i.e., areas beyond the limiting values of $e = 0.997$ and $\alpha = 103.17^\circ$. (We call these areas “forbidden” in the sense described in Sec. IV: to find the minimal h for e less than or equal to a given one and α bigger than or equal to a given one, so the values e and α violating these inequalities are “forbidden.” Also see the second paragraph in this section). On the lower right picture all three parameters are presented, “twice forbidden” area is colored green. One can see that there is a possible improvement of the value of h from $h = 0.99$, being in the white area.

After further optimization we will come to the Fig. 9. We have chosen the angle $\alpha = 100^\circ$ that is a little bit less than in the original cavity (103.17°) and have chosen $e = 1$ (we also prefer “round” values.) Both of these changes

FIG. 8. Map for the original TESLA cavity. $a = 12$ mm, $b = 19$ mm, $\beta = 1$.

improve the value of h from 0.99 to 0.97, but this is not the main result.

We believe that the main result consists in the fact that moving along the colored line and staying in the white area, we can change both A and B in a wide range changing the value of h not more than 0.5%. As will be shown later, this gives a possibility to create a geometry of a cavity, practically optimal, but free of multipactor or tune off the dangerous higher order modes.

FIG. 9. Map for the optimized TESLA cavity. $a = 12.13$ mm, $b = 19.70$ mm, $\beta = 1$.

One can understand that this fact of a very flat minimum where some values can be changed in a broad range is a reverse side of the difficulty to find an exact minimum as described in the section “Method of optimization.”

VII. AN EXAMPLE OF OPTIMIZATION FOR THE SNS ELLIPTIC CAVITY WITH $\beta_{\text{geo}} = 0.81$

As an example of an elliptic cavity with $\beta = 0.81$ let us analyze the shape and fields of the Spallation Neutron Source (SNS) cavity [16]. According to the data from this paper the inner half-cell of the cavity for frequency $f = 805$ MHz has the wall distance $d = 15$ mm, wall angle $\gamma = 7^\circ$, length $L = 75.5$ mm, aperture radius $R_a = 48.8$ mm, $\beta_{\text{geo}} = 0.81$ and $\beta_{\text{eff}} = 0.83$. The values of normalized peak fields are reported as $E_{\text{pk}}/E_{\text{acc}} = 2.19$ and $B_{\text{pk}}/E_{\text{acc}} = 4.72$ mT/(MV/m).

From these data, using formulas (1) and (2) we calculated the half-axes of ellipses and the normalized peak fields: $A = B = 52.376$ mm, $a = 18.677$, $b = 33.619$ mm, $E_{\text{pk}}/E_{\text{acc}} = 2.145$, and $B_{\text{pk}}/E_{\text{acc}} = 4.572$. The equatorial radius was calculated by tuning the cavity to the given frequency.

Our results are 2.3% lower for the $E_{\text{pk}}/E_{\text{acc}}$ and 3.3% lower for the $B_{\text{pk}}/E_{\text{acc}}$ value. These values were calculated for $\beta_{\text{geo}} = 0.81$. If $\beta_{\text{eff}} = 0.83$ is used, the deviation of $E_{\text{pk}}/E_{\text{acc}}$ and $B_{\text{pk}}/E_{\text{acc}}$ from the values presented in the paper is even bigger: $E_{\text{pk}}/E_{\text{acc}} = 2.1228$ and $B_{\text{pk}}/E_{\text{acc}} = 4.519$, or $e = 1.0614$ and $h = 1.076$.

This discrepancy can be explained by rounded values of geometrical parameters given in the paper (we do not have direct values of half-axes A , B , a , and b), we also believe that the software SUPERLANS used in our calculations has better accuracy [12] than SUPERFISH used in the cited work.

We will perform the optimization for the effective value of beta, because the used beam has this value of speed. For further analysis we will scale the dimensions of this cavity from 805 MHz to 1300 MHz. Let us analyze first the original geometry after scaling. In Fig. 10 are presented the values of normalized parameters h and e , and the wall slope angle α for fixed values of a and b and different values of A and B . The pink area for e and the blue area for α are “forbidden” areas, i.e., areas beyond the limiting values of $e = 1.0614$ and $\alpha = 97^\circ$. On the lower right picture all three parameters are presented, “twice forbidden” area is colored green. One can see that there is a possible improvement of the value of h from $h = 1.076$, being in the white area.

Figure 11 shows the same parameters after optimization using the same limiting values of e and α . The value of h is improved to 1.069, though not so much: 0.6%. But the more interesting feature of this result, as well as in the previous section, is the fact that moving from the point near the lower left corner to the point near the upper right corner

FIG. 10. Map for the original SNS cavity. $a = 11.565$ mm, $b = 20.818$ mm, $\beta = 0.81$.

FIG. 11. Map for the optimized SNS cavity. $a = 11.387$ mm, $b = 21.009$ mm.

and remaining in the noncolored area we will have the value of h practically the same as at the point of optimum (it is in the center) but changing significantly the values of A and B . These changes give us freedom to avoid multipactor or/and to tune the most dangerous higher order modes to a nondangerous position being near the lowest possible value of h , i.e., lowest magnetic field and lowest losses.

VIII. MULTIPACTOR CONSIDERATION

Multipactor is a parasitic rf phenomenon limiting the accelerating rate of a cavity. There are several varieties of

FIG. 12. Equatorial cross-field multipacting zone of the first order.

multipactor depending on the geometry of an rf device. For a superconducting elliptical cavity the 2-point multipactor on the cavity equator occurs most often. We have a method of predicting the appearance of this type of multipactor without time consuming simulations but immediately from the field calculations done with a 2D program [17].

Zone of existence of multipactor on the cavity equator, Fig. 12, is defined by two coefficients: the geometrical coefficient $p = \beta/(\alpha + \beta)$ and the field coefficient $M = eB_0/m\omega$, where e/m is the specific charge of the electron, $\omega = 2\pi f$ is the angular frequency of oscillations and α and β are factors of proportionality for the longitudinal and radial components of the electric field near the equator, $E_z = \alpha d$ and $E_r = \beta d$, respectively. The points of definition of these factors are at a distance $d = 1$ mm, Fig. 13, this is a characteristic size for the orbit of an electron in this kind of multipactor at any frequency as follows from [17]. For a frequency lower than $f = 1300$ MHz this value of d can be proportionally increased to have a better accuracy of the calculated fields. Actually, p weakly depends on d because the area of linearity for E_z and E_r is bigger than 1 mm at $f = 1300$ MHz.

The value of B_0 in the expression for the field coefficient M is the amplitude of the magnetic field at the equator where the multipactor takes place. B_0 is proportional to the accelerating field E_{acc} in the cavity and usually is maximal or close to the maximal magnetic field in the cavity, B_{pk} .

FIG. 13. Definition of the geometrical parameter p .

FIG. 14. Electric force lines for the entire half-cell of the TESLA cavity and near the equator. Force lines near the equator are elliptic arcs with ratio of half-axes $b_{fl}/a_{fl} = 0.633 < 0.655$, see text.

The value of B_{pk}/E_{acc} is always known when the cavity is being designed. So, knowing B_0 when the multipactor starts, we can easily find E_{acc} for this condition.

Each cavity has its own geometrical coefficient p . Moving along the line $p = \text{const}$, as shown in Fig. 12, we can increase the magnetic field at the point of the equator, i.e., the field coefficient M . When the energy of the primary electrons E_p (see the graph) becomes high enough for the secondary emission yield > 1 , the multipactor starts. This happens usually near the point $M = 2$ as can be seen from the figure. The results with different cavities analyzed in the paper [17] and presented in Fig. 12 show that the critical value for multipactor is about $p = 0.3$, so that we have a very weak and usually easily processed multipactor in the TESLA cavity ($p = 0.286$), a well pronounced multipactor in the Mark I cavity fabricated specially for studying this phenomenon ($p = 0.303$), and practically no multipactor at the equator of the Cornell ERL cavity ($p = 0.276$). More details and references can be found in the cited paper.

The force lines of the electric field near the equator have a shape of elliptic arcs (Fig. 14) as follows from

$$\frac{E_r}{E_z} = \frac{dr}{dz} = \frac{-\beta z}{\alpha r}, \quad (5)$$

here r is measured from the equator to the axis.

FIG. 15. Parameter p for the TESLA cavity, original (left) and optimized, in the same coordinates as in Figs. 8 and 9.

FIG. 16. Map for one of the points from Fig. 21 for $\beta = 0.7$. Wide choice of A and B keeping the optimal h . A possibility to get rid of multipactor choosing $p < 0.3$.

The solution of this differential equation is

$$\frac{z^2}{\alpha} + \frac{r^2}{\beta} = \text{const.} \quad (6)$$

So, the ratio of the half-axes of the elliptic force lines near the equator is $b_{fl}/a_{fl} = \sqrt{\beta/\alpha} = \sqrt{p/(1-p)}$ and the condition $p > 0.3$ corresponds to $b_{fl}/a_{fl} > \sqrt{3/7} \approx 0.655$.

As it was stated above in the section “An example of optimization of the TESLA cavity,” we can change the dimensions A and B so that the value of normalized magnetic field h will only slightly change in a wide range of A and B . For the example presented in Figs. 8 and 9, let us present values of p in the same coordinates, Fig. 15. We will not overlap this additional graph with the previous ones but it is easily seen that (i) the optimized cavity has significantly decreased p , so that there will be no multipacting, and (ii) this parameter can be further decreased increasing the value of h not more than 0.5%.

IX. HIGHER ORDER MODES OF THE MODIFIED SHAPES

To avoid the higher order modes (HOM) cavity frequencies falling on dangerous frequency spectrum lines of beam is an important design task for the ERL cavity designer. The presented approach to change the cavity shape keeping the optimal value of H_{pk}/E_{acc} for given E_{pk}/E_{acc} , R_a , and α will also change the spectrum of the cavity HOMs. We calculated the lowest monopole and dipole modes of the TESLA cavity when the bigger half-axes of it are changed from 42 to 41.1 mm for A and from 42 to 38 mm for the B half-axis. According to Fig. 8 the value of h will not change compared to the original shape. The data are presented in Table I. These data are calculated for an infinitely long multicell cavity consisting of identical cells.

One can see that the change in frequencies is present though not so big as wished it to be. However, the range of

changes of these half-axes can be increased, so will be increased the shift of the HOMs.

X. TABLES AND GRAPHS FOR DIFFERENT VALUES OF β

A. Elliptic cavities with $\beta = 1$

Results of optimization for $\beta = 1$ are presented in Table II and in Figs. 17 and 18. Curves in Fig. 18 are plotted in accordance to data in Table II.

B. Elliptic cavities with $\beta = 0.9$

Results of optimization for $\beta = 0.9$ are presented in Fig. 19 and Table III.

TABLE I. Frequencies of higher order modes for geometries specified in Fig. 8: original TESLA cavity and its modification without increasing magnetic field. Smaller elliptic half-axes are kept same: $a = 12$ mm, $b = 19$ mm. Bigger half-axes A and B are seen in the Table.

Geometry, dimensions in mm	$A = 42$ mm, $B = 42$ mm	$A = 41.1$ mm, $B = 38$ mm
Lowest monopoles, MHz	1275.7	1275.7
	1300	1300
	2672.0	2675.5
	2774.4	2778.9
	3757.2	3758.0
	3857.8	3865.5
Lowest dipoles, MHz	1792.4	1792.6
	1888.3	1888.5
	2471.9	2472.5
	3124.6	3125.3
	3343.9	3353.5
	3361.6	3371.3

TABLE II. Values of H_{pk}/E_{acc} , GR/Q and dimensions of optimized inner cell (two elliptic arcs and a straight segment) vs the slope angle α . Optimization for min H_{pk}/E_{acc} (or min h).

Optimization for min h , $\beta = 1$								
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2$ ($e = 1$)								
Normalized peak magnetic field (also norm. to 42)		Geometrical dimensions in mm						Loss parameter (also normalized to TESLA)
α°	H_{pk}/E_{acc} , Oe/(MV/m); (h)	A	B	a	b	l	R_{eq}	GR/Q , Ohm ² ; (gr/q)
82.70	39.90 (0.9500)	45.36	36.19	12.81	21.35	0	100.683	32 077 (1.041)
85	39.91 (0.9502)	45.33	36.03	12.78	21.57	2.45	100.693	32 068 (1.041)
90	40.00 (0.9523)	44.94	35.65	12.71	21.48	8.71	100.833	32 015 (1.039)
95	40.23 (0.9579)	43.99	35.06	12.53	20.95	15.73	101.205	31 838 (1.034)
100	40.74 (0.9701)	42.49	35.50	12.13	19.70	22.76	102.024	31 420 (1.020)
105	41.78 (0.9942)	39.94	34.81	11.57	18.02	31.22	103.116	30 599 (0.993)
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2.2$ ($e = 1.1$)								
74.01	38.60 (0.9191)	49.02	35.65	10.49	15.46	0	99.398	33 296 (1.081)
80	38.64 (0.9201)	48.81	35.59	10.50	15.25	5.43	99.469	33 292 (1.081)
85	38.75 (0.9226)	48.30	34.96	10.45	14.95	10.70	99.594	33 232 (1.079)
90	38.95 (0.9274)	47.41	33.59	10.24	14.73	16.50	99.810	33 067 (1.074)
95	39.32 (0.9363)	45.87	31.95	9.94	14.18	22.92	100.252	32 748 (1.063)
100	39.96 (0.9515)	43.63	31.08	9.56	13.35	29.22	101.118	32 181 (1.045)
105	40.95 (0.9751)	41.14	33.38	8.99	11.84	34.66	102.673	31 247 (1.015)
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2.4$ ($e = 1.2$)								
67.79	37.79 (0.8997)	51.54	36.24	9.18	11.90	0	98.711	34 064 (1.106)
70	37.80 (0.8997)	51.46	36.21	9.22	11.85	2.68	98.722	34 064 (1.106)
75	37.83 (0.9006)	51.37	36.10	9.10	11.84	5.47	98.801	34 043 (1.105)
80	37.92 (0.9028)	51.11	35.11	8.88	11.96	9.94	98.838	33 950 (1.102)
85	38.09 (0.9069)	50.34	34.07	8.78	11.60	15.06	99.010	33 842 (1.099)
90	38.37 (0.9135)	49.10	32.36	8.55	11.22	20.71	99.285	33 603 (1.091)
95	38.84 (0.9248)	47.17	30.52	8.29	10.62	26.66	99.797	33 199 (1.078)
100	39.54 (0.9413)	44.50	28.78	7.91	9.78	32.90	100.622	32 568 (1.057)
105	40.59 (0.9664)	41.36	29.36	7.48	9.00	38.29	102.098	31 579 (1.025)
$R_a = 30$ mm, $E_{pk}/E_{acc} = 2.0$ ($e = 1.0$)								
77.08	36.62 (0.8720)	48.73	38.05	10.39	17.59	0	98.684	37 081 (1.204)
80	36.63 (0.8722)	48.70	38.05	10.34	17.63	2.83	98.753	37 072 (1.204)
85	36.71 (0.8740)	48.36	37.57	10.27	17.46	8.74	98.845	37 011 (1.202)
90	36.90 (0.8787)	47.53	36.55	10.12	16.97	15.56	99.073	36 829 (1.196)
95	37.32 (0.8885)	46.01	35.48	9.86	16.14	22.74	99.599	36 395 (1.182)
100	38.02 (0.9052)	43.87	35.54	9.39	14.67	29.77	100.604	35 605 (1.156)
105	39.32 (0.9361)	40.90	35.82	8.68	12.60	37.62	102.012	34 327 (1.115)
$R_a = 30$ mm, $E_{pk}/E_{acc} = 2.2$ ($e = 1.1$)								
70.02	35.64 (0.8486)	51.75	38.48	8.76	13.10	0	97.874	38 237 (1.241)
80	35.75 (0.8512)	51.39	37.71	8.57	13.04	9.14	97.982	38 129 (1.238)
90	36.22 (0.8623)	49.39	35.00	8.26	12.07	21.36	98.427	37 648 (1.222)
100	37.54 (0.8939)	44.71	32.55	7.53	10.06	34.39	100.027	36 160 (1.174)
105	38.79 (0.9236)	41.83	35.17	6.90	8.60	39.81	101.771	34 742 (1.128)

C. Elliptic cavities with $\beta = 0.8$

Results of optimization for $\beta = 0.8$ are presented in Fig. 20 and in Table IV.

D. Elliptic cavities with $\beta = 0.7$

Results of optimization for $\beta = 0.7$ are presented in Fig. 21 and in Table V.

Figure 16 shows for one of the points from Fig. 21 that the choice of half-axes A and B can also be done in a very large range with a deviation from the optimal value not more than a fraction of a percent. Here, it is also shown the geometrical parameter p . It is clearly to be seen, also as it was earlier (Fig. 15), that the lowest p corresponds to the maximal radius of curvature at the equator. However,

FIG. 17. Normalized magnetic peak field and normalized loss parameter for different angles of the wall slope for $\beta = 1$.

making the equator flat can lead to the one-point multipactor.

E. Elliptic cavities with $\beta = 0.6$

Results of optimization for $\beta = 0.6$ are presented in Fig. 22 and in Table VI. Here, for comparison, data from [16] are also shown for the SNS cavity with $\beta = 0.61$. After scaling from 805 MHz to 1300 MHz the aperture radius R_a

changed from 43 to 26.627 mm. Point a for the normalized magnetic field is taken from this paper: $B_{peak}/E_{acc} = 5.73$ mT/(MV/m) that corresponds to $h = 1.364$. Our calculations, however, give for this point $h = 1.294$ (point b), and after optimization this shape was slightly improved—to $h = 1.277$ (point c).

The values of R_a and E_{pk}/E_{acc} are chosen so that the data for the SNS cavity are between these values, and the difference in β is not large.

FIG. 18. Elliptic half-axes for the previous figure. Designation of points are the same. See details in Sec. X A.

FIG. 19. Normalized magnetic peak field for different angles of the wall slope for $\beta = 0.9$.

TABLE III. Values of $h = H_{pk}/E_{acc}/42$, and dimensions of optimized inner cell (two elliptic arcs and a straight segment) vs the slope angle α . Optimization for min H_{pk}/E_{acc} (or min h).

Optimization for min h , $\beta = 0.9$							
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2$ ($e = 1$)							
Geometrical dimensions in mm							
α°	Normalized peak magnetic field h	A	B	a	b	l	R_{eq}
90.5	1.096	36.02	35.96	15.87	32.40	0.60	103.012
95	1.099	35.61	34.77	15.80	32.10	9.78	103.039
100	1.111	34.16	33.38	15.55	31.08	20.50	103.649
105	1.139	31.42	32.01	15.00	29.02	31.89	105.053
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2.2$ ($e = 1.1$)							
81.65	1.028	40.51	37.01	12.07	19.74	0	101.282
85	1.029	40.39	37.64	12.04	19.69	3.32	101.474
90	1.033	39.93	37.92	11.95	19.46	9.47	101.802
95	1.042	38.94	37.65	11.76	19.05	16.54	102.335
100	1.062	37.28	37.97	11.39	17.90	24.25	103.345
105	1.097	34.61	38.12	10.86	16.56	33.37	104.856
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2.4$ ($e = 1.2$)							
76.53	0.988	43.35	38.50	10.03	14.39	0	100.579
80	0.989	43.28	38.53	10.00	14.40	3.24	100.636
85	0.992	42.89	38.41	9.95	14.14	8.56	100.823
90	0.999	42.10	38.16	9.78	13.80	14.44	101.231
95	1.013	40.77	37.57	9.51	13.28	20.73	101.861
100	1.037	38.70	36.88	9.14	12.60	27.88	102.848
105	1.075	35.60	36.16	8.64	11.48	36.30	104.357

FIG. 20. Normalized magnetic peak field for different angles of the wall slope for $\beta = 0.8$.

TABLE IV. Values of $h = H_{pk}/E_{acc}/42$, and dimensions of optimized inner cell (two elliptic arcs and a straight segment) vs the slope angle α . Optimization for min H_{pk}/E_{acc} (or min h).

Optimization for min h , $\beta = 0.8$							
$R_a = 30$ mm, $E_{pk}/E_{acc} = 2$ ($e = 1$)							
Geometrical dimensions in mm							
α°	Normalized peak magnetic field h	A	B	a	b	l	R_{eq}
89.78	1.149	30.54	33.86	15.58	37.88	0	102.248
95	1.152	30.05	34.63	15.53	37.7	11.78	102.754
100	1.172	28.18	32.92	15.28	36.45	25.59	103.798
105	1.224	23.8	27.41	14.68	33.73	42.05	105.991
$R_a = 30$ mm, $E_{pk}/E_{acc} = 2.2$ ($e = 1.1$)							
81.91	1.048	35.74	38.8	11.17	20.59	0	100.529
85	1.049	35.64	38.8	11.17	20.5	4.46	100.591
90	1.056	35.05	37.4	11.07	20.22	13.13	100.739
95	1.075	33.51	36.1	10.98	20.04	22.00	101.505
100	1.112	30.8	34.9	10.89	19.98	31.79	103.043
105	1.174	26.81	34.06	10.75	19.89	42.93	105.605
$R_a = 30$ mm, $E_{pk}/E_{acc} = 2.4$ ($e = 1.2$)							
74.89	0.997	39	38.5	9.2	14.80	0	99.109
80	0.998	38.57	40.67	9.10	14.59	3.20	99.813
85	1.004	38.31	37.79	9.09	14.36	12.11	99.43
90	1.018	37.08	37.05	9.05	13.96	19.00	100.011
95	1.041	35.11	34.87	8.80	13.00	27.93	100.882
100	1.075	32.97	38.55	8.27	12.11	33.54	102.880
105	1.138	29.73	41.98	7.73	10.91	42.82	105.535
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2.4$ ($e = 1.2$)							
90	1.160	34.1	38.2	12.0	19.9	9.98	102.810
95	1.175	32.83	35.07	11.84	19.42	19.71	103.104
100	1.207	30.23	31.16	11.54	18.72	30.40	104.100

FIG. 21. Normalized magnetic peak field for different angles of the wall slope for $\beta = 0.7$.

F. Elliptic cavities with $\beta = 0.5$

Results of optimization for $\beta = 0.5$ are presented in Fig. 23 and in Table VII. Here are also shown for

TABLE V. Values of $h = H_{pk}/E_{acc}/42$, and dimensions of optimized inner cell (two elliptic arcs and a straight segment) vs the slope angle α . Optimization for min H_{pk}/E_{acc} (or min h).

Optimization for min h , $\beta = 0.7$							
$R_a = 35$ mm, $E_{pk}/E_{acc} = 2.4$ ($e = 1.2$)							
		Geometrical dimensions in mm					
α°	Normalized peak magnetic field h	A	B	a	b	l	R_{eq}
94.86	1.639	19.85	32.68	21.19	58.62	0.85	109.205
95	1.640	19.84	32.68	21.19	58.62	1.39	109.208
100	1.647	19.22	35.59	20.80	56.40	19.08	110.410
105	1.729	17.31	51.51	20.50	54.77	39.23	115.180
$R_a = 30$ mm, $E_{pk}/E_{acc} = 2.4$ ($e = 1.2$)							
90	1.207	28.93	37.5	11.43	21.74	12.57	101.804
95	1.236	26.99	29.66	11.24	20.99	27.51	101.834
100	1.288	23.95	30.15	11.04	20.43	37.29	104.428
105	1.385	19.10	30.26	10.27	17.60	51.95	108.866
$R_a = 25$ mm, $E_{pk}/E_{acc} = 2.4$ ($e = 1.2$)							
90	1.064	30.88	31.54	9.48	21.77	20.31	98.623
95	1.086	28.99	36.34	9.43	21.74	26.24	100.962
100	1.144	25.27	30.98	9.27	21.11	40.69	102.770
$R_a = 25$ mm, $E_{pk}/E_{acc} = 3$ ($e = 1.5$)							
85	0.938	37.01	44.09	5.13	6.63	17.84	98.783
90	0.963	35.45	40.71	4.90	6.14	27.25	99.094
95	1.009	32.50	34.40	4.65	5.54	38.64	99.695
100	1.082	26.72	19.00	4.28	4.91	55.53	100.256

FIG. 22. Normalized magnetic peak field for different angles of the wall slope for $\beta = 0.6$. Points a, b, and c are shown for comparison with data from [16], see text.

comparison data from [16] for the TRASCO-ASH cavity, for 704.4 MHz (on the left), and RIA cavity, for 805 MHz (on the right), both with $\beta = 0.47$. After scaling to 1300 MHz their aperture radii R_a changed to 21.674 mm and 23.902 mm, respectively. Points a and d for the normalized magnetic field are taken from this paper. Our calculations, however, give for these points somewhat lower values (points b and e), and after optimization this shape was even slightly more improved (point c and f).

The values of R_a and E_{pk}/E_{acc} for the curves are chosen so that the data for the TRASCO-ASH and RIA cavity are

TABLE VI. Values of $h = H_{pk}/E_{acc}/42$, and dimensions of optimized inner cell (two elliptic arcs and a straight segment) vs the slope angle α . Optimization for min H_{pk}/E_{acc} (or min h).

Optimization for min h , $\beta = 0.6$							
$R_a = 30$ mm, $E_{pk}/E_{acc} = 2.8$ ($e = 1.4$)							
		Geometrical dimensions in mm					
α°	Normalized peak magnetic field h	A	B	a	b	l	R_{eq}
95	1.465	23.29	55.28	10.50	19.00	16.30	106.266
97.5	1.500	22.34	55.48	10.40	18.63	24.83	107.201
100	1.554	20.59	53.72	10.22	18.06	35.26	108.671
$R_a = 25$ mm, $E_{pk}/E_{acc} = 2.6$ ($e = 1.3$)							
95	1.222	23.07	20.38	7.82	13.26	44.28	99.234
97.5	1.261	21.07	21.86	7.63	12.63	47.96	101.392
100	1.316	18.75	23.79	7.43	11.88	52.64	104.097

FIG. 23. Normalized magnetic peak field for different angles of the wall slope for $\beta = 0.5$. Points a, b, and c, also as d, e, f are shown for comparison with data from [16], see text.

TABLE VII. Values of $h = H_{pk}/E_{acc}/42$, and dimensions of optimized inner cell (two elliptic arcs and a straight segment) vs the slope angle α . Optimization for min H_{pk}/E_{acc} (or min h).

Optimization for min h , $\beta = 0.5$							
$R_a = 25$ mm, $E_{pk}/E_{acc} = 3$ ($e = 1.5$)							
Geometrical dimensions in mm							
α°	Normalized peak magnetic field h	A	B	a	b	l	R_{eq}
90	1.473	19.59	42.95	9.24	19.80	14.62	102.381
95	1.514	18.22	42.82	8.27	15.33	32.34	104.287
97.5	1.582	16.51	42.54	7.88	13.74	42.22	106.396
100	1.702	13.62	41.60	7.59	12.68	55.00	110.137
$R_a = 25$ mm, $E_{pk}/E_{acc} = 4$ ($e = 2$)							
90	1.302	23.68	50.58	5.15	8.81	54.69	102.196
95	1.382	21.23	44.34	4.93	8.09	35.18	103.364
97.5	1.453	19.12	41.32	4.74	7.49	44.38	104.970
100	1.557	16.27	39.5	4.53	6.84	54.64	107.854
$R_a = 20$ mm, $E_{pk}/E_{acc} = 3$ ($e = 1.5$)							
90	1.079	24.12	39.35	4.70	6.76	32.35	98.457
95	1.182	19.98	24.53	4.45	6.15	52.17	99.298
97.5	1.255	16.83	19.10	4.32	5.84	60.74	101.322
100	1.360	12.84	14.46	4.16	5.48	70.17	104.991
$R_a = 20$ mm, $E_{pk}/E_{acc} = 4$ ($e = 2$)							
90	1.056	25.71	15.24	3.12	2.72	55.38	93.328
95	1.146	20.79	15.11	2.76	2.49	61.15	97.358
97.5	1.215	17.71	14.11	2.62	2.29	65.89	99.993
100	1.311	14.27	15.16	2.50	2.08	70.99	104.055

FIG. 24. Normalized magnetic peak field for different angles of the wall slope for $\beta = 0.4$.

TABLE VIII. Values of $h = H_{pk}/E_{acc}/42$, and dimensions of optimized inner cell (two elliptic arcs and a straight segment) vs the slope angle α . Optimization for min H_{pk}/E_{acc} (or min h) and $r_c \geq 3$ mm.

Optimization for min h , $\beta = 0.4$							
$R_a = 20$ mm, $E_{pk}/E_{acc} = 3.5$ ($e = 1.75$)							
Geometrical dimensions in mm							
α°	Normalized peak magnetic field h	A	B	a	b	l	R_{eq}
90	1.318	18.34	30.00	4.72	6.99	40.21	97.197
92.5	1.393	16.36	25.14	4.61	6.64	48.88	98.508
95	1.497	13.42	16.76	4.51	6.37	60.14	100.435
97.5	1.655	9.42	9.89	4.41	6.07	71.94	104.848
100	1.965	2.86	4.14	4.28	5.78	92.89	119.033
$R_a = 20$ mm, $E_{pk}/E_{acc} = 4$ ($e = 2$)							
90	1.268	19.43	34.86	3.63	4.39	38.70	97.952
92.5	1.348	17.18	20.06	3.54	4.20	54.33	97.310
95	1.447	14.03	12.14	3.46	3.98	64.62	99.180
97.5	1.595	9.89	5.78	3.4	3.76	75.35	103.261
100	1.860	4.80	7.55	3.25	3.52	87.79	114.847
$R_a = 17.5$ mm, $E_{pk}/E_{acc} = 3$ ($e = 1.5$)							
92.5	1.272	15.45	3.98	4.76	7.56	65.47	93.882
95	1.369	12.44	5.57	4.64	7.16	69.23	98.025
97.5	1.522	8.6	4.84	4.5	6.72	77.13	103.869
100	1.841	1.98	7.96	4.35	6.3	99.26	123.349
$R_a = 17.5$ mm, $E_{pk}/E_{acc} = 3.5$ ($e = 1.75$)							
92.5	1.219	16.65	6.29	3.52	4.13	66.43	93.979
95	1.313	13.46	5.57	3.41	3.88	71.33	97.420
97.5	1.454	9.89	7.71	3.29	3.61	76.33	103.192
100	1.721	4.67	11.54	3.16	3.33	90.29	116.067

between these values, and the difference in β is not large. Values of surface overvoltage are also in between the parameters used for the curves: $E_{pk}/E_{acc} = 3.57$ for the TRASCO-ASH and 3.41 for the RIA cavity. One can see that the shapes are very close to optimal ones even in spite of the chosen ratio $R = 1$ for the RIA cavity (on the right). For the TRASCO-ASH cavity this ratio is 1.6 and the presented shape is practically optimal.

G. Elliptic cavities with $\beta = 0.4$

Results of optimization for $\beta = 0.4$ are presented in Fig. 24 and in Table VIII.

Small values of the elliptic half axes for $\beta = 0.4$ lead to small curvature radius at $f = 1300$ MHz which as previously explained poses difficulties in fabrication of the cavity. This suggests to consider elliptic cavities for low β applications only at lower frequencies.

XI. CONCLUSION

Tables and graphs for dependencies of the minimal values of H_{pk}/E_{acc} on the cavity wall slope angle are presented for different values of β from 1 to 0.4 with a step of 0.1, different aperture radii, and peak surface electric fields. Geometrical data of the cavities are included in the tables. These data can be used for a preliminary design of elliptic SRF cavities for different accelerating structures. The data are given for the frequency of 1300 MHz but they are valid for a wide range of frequencies because most of the important parameters and figures of merit of the SRF cavities, depend on the cavity shape only, not on the size.

A new property of the elliptic cavity shape is revealed: it is possible to maintain the minimal H_{pk}/E_{acc} but to change the shape of the cavity. This makes it possible to avoid multipactor keeping the optimal shape. Examples of this solution are presented. This change of the shape can be also used for detuning the most adverse higher order modes or/and for decreasing the Lorentz force detuning.

Geometrical constraints for the shape of elliptic cavities are analyzed. Small values of the elliptic half-axes for the cavities with smallest β suggest to conclude that such cavities can be fabricated for the frequencies at about two times lower than the “most popular” frequency of 1300 MHz. As practice shows, this is true for the existing examples of elliptic cavities. We can confirm the earlier statement that the value of $\beta = 0.4$ should be the lowest limit for this class of cavities.

Comparison of the presented optimal shapes with some existing cavities is done and a reasonable agreement in their geometry is demonstrated.

ACKNOWLEDGMENTS

This research was supported by Deutsche Forschungsgemeinschaft (DFG) Grant No. RI 814/25-1, Bundesministerium für Bildung und Forschung (BMBF) Grant No. 05K13HR1 and Future Circular Collider (FCC) study at CERN.

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