Tunable polarization plasma channel undulator for narrow bandwidth photon emission

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The theory of a plasma undulator excited by a short intense laser pulse in a parabolic plasma channel is presented. The undulator fields are generated either by the laser pulse incident off-axis and/or under the angle with respect to the channel axis. Linear plasma theory is used to derive the wakefield structure. It is shown that the electrons injected into the plasma wakefields experience betatron motion and undulator oscillations. Optimal electron beam injection conditions are derived for minimizing the amplitude of the betatron motion, producing narrow-bandwidth undulator radiation. Polarization control is readily achieved by varying the laser pulse injection conditions.

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I. INTRODUCTION

Synchrotron radiation generated by electrons traveling in bending magnets or insertion devices is essential for our understanding of the microcosm. X-ray pulses produced by the synchrotron facilities are widely used in, for example, chemistry, biology and material science [1–3]. The brightest x-ray pulses are generated using free-electron lasers (FELs) [4]. FEL facilities typically require hundreds of meters to kilometers for the linear accelerator (to reach multi-GeV electron beam energies) and tens of meters of undulator. For example, the soft x-ray FEL facility at DESY (FLASH [5]) uses a linear accelerator with a length of roughly 200 m reaching electron energies of ~ 1 GeV. Using an undulator with a period of 2.7 cm and a total length of 27 m, it provides bright radiation at 4 nm, reaching the water window. Hard x-ray FEL facilities require kilometer-scale distances. Reducing the size and cost of such facilities is highly beneficial as it will greatly increase the user accessibility of synchrotrons and FELs. There are two paths to make these light sources more compact. First, one can replace the conventional accelerator with a compact laser-plasma electron accelerator (LPA) and, second, one can decrease the period of the undulator.

Laser-driven plasma accelerators [6,7] (LPAs) provide ultrahigh gradients, enabling extremely compact accelerators. LPA technology has been showing steady progress in

the recent decades. Notably, monoenergetic electron bunches [8–10] have been demonstrated, and the energy of such bunches has recently reached the multi-GeV level [11]. The accelerating gradients in the LPA can be several orders of magnitude higher than those in conventional linacs, and the compactness of the accelerator itself makes it attractive as a source of secondary radiation, ranging from THz to gamma rays [12–15]. Significant effort is now devoted to combining the LPAs with the traditional magnetic undulators to demonstrate free-electron lasing [16–18]. Novel injection techniques [19–21] might help to reduce the energy spread of the LPA electron beams to the values needed to produce an FEL instability.

Another path to reduce the size of the FEL facility is to decrease the undulator period. The wavelength of the on-axis radiation produced by an electron with the Lorentz factor γ undergoing oscillations inside a linearly polarized undulator with a period λ_u and strength a_u is given by

$$\lambda = \frac{\lambda_u}{2\gamma^2} (1 + a_u^2/2). \quad (1)$$

Decreasing the period of the undulator leads to a twofold benefit. The total length of the undulator will be smaller, and the electron energy required to reach a desired x-ray wavelength will decrease, leading to a more compact accelerator. The lower limit for the period of the magnetic undulators is currently at the 1 mm level [2]. Undulators with periods of 1 mm or less (often referred to as microundulators) that are capable of reaching high strengths $a_u \sim 1$ would, therefore, be highly beneficial. Several microundulator concepts have been proposed including crystalline [22–24], electrostatic [25,26], microwave [27], plasma [28,29], nanowire [30] and laser-based [31–39] undulators. Realizing such techniques is

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of increasing importance in the context of LPA progress because the size of conventional undulators (much larger than LPAs) is becoming a constraint on the footprint of compact radiation sources.

It is also worth discussing the difference between plasma-undulator-based and inverse Compton scattering (ICS) sources. Recent experiments on the nonlinear ICS using LPA electron beams [40–43] have demonstrated bright photon spectra reaching gamma-ray energies. The measured ICS spectrum is quite broad. The broad spectrum is not only due to the electron beam parameters, but also due to the large and varying laser pulse amplitudes $a_0 > 1$. For laser pulses with varying temporal envelope, additional photon spectrum broadening due to the varying ponderomotive force appears [44–49]. Hence, the main advantage of the plasma-undulator-based approach is the possibility to generate narrow-band photon spectra while reaching high undulator strength ($a_u \sim 1$), and, thus, providing a possible solution for a tabletop FEL, whereas the main advantage of ICS is the possibility to generate harder photons, albeit with a broadband spectrum.

Recently, a novel type of laser-plasma-based undulator for narrow-band photon emission was proposed [50]. This undulator is based on laser pulse centroid oscillations inside a plasma channel. Electrons injected into the wakefield, created by the laser pulse undergoing centroid oscillations inside a parabolic plasma channel, wiggle with the characteristic wavelength $\lambda_u = 2\pi Z_R$, where Z_R is the Rayleigh length of the laser pulse. High undulator strength $a_u \sim 1$ is achievable for undulator periods of 1 mm or less. This makes such a plasma undulator or plasma wiggler (PIGGLER) a promising path towards a “tabletop” bright incoherent soft x-ray source, or, with sufficient LPA beam quality, a compact FEL. This laser-plasma undulator has great flexibility in polarization tunability, determined by the initial position and direction of the laser pulse at the entrance of the plasma channel.

In this paper a comprehensive theoretical study of the laser wakefield undulator excited by the laser pulse traveling inside a plasma channel is provided. The paper is organized as follows. In Sec. II propagation of a Gaussian laser pulse inside a parabolic plasma channel is discussed. Expressions for the evolution of the centroid and spotsize of the laser pulse are derived. In Sec. III, the wakefield structure of the plasma undulator created by the laser pulse undergoing centroid oscillations is derived using linear plasma theory. Analytical expressions for the wakefield structure are used to derive the trajectory of an electron beam injected into the undulator. The results of analytical and numerical trajectory calculations are presented in Secs. IV and V for a single electron and an electron beam, respectively. Radiation calculations and polarization properties are presented in Secs. VI and VII. Section VIII contains a summary of the results and discussion.

II. LASER PULSE PROPAGATION IN A PLASMA CHANNEL

Laser pulse guiding is important for LPAs to avoid diffraction and to maintain the high accelerating gradients for many Rayleigh lengths [7,51]. Typically, and quite routinely, parabolic plasma channels created by capillary discharges are used for this purpose [52–54]. Consider the dimensionless variables: $\tau = \omega_p t'$, $x = k_p x'$, $y = k_p y'$, $z = k_p z'$, with the plasma wave number $k_p = \omega_p/c = (4\pi e^2 n_0/mc^2)^{1/2}$, where c is the speed of light in vacuum, and $-e$ and m are the electron charge and mass, respectively. Now consider a plasma channel with the transverse electron density profile given by

$$n(r) = n_0 \left(1 + \frac{\Delta n}{n_0} \frac{r^2}{r_m^2} \right), \quad (2)$$

where $r = \sqrt{x^2 + y^2}$ is the radial coordinate, x and y are the transverse coordinates, n_0 is the on-axis plasma density, $r_m = k_p / \sqrt{\pi r_{cl} \Delta n}$ is the dimensionless matched laser spotsize, with $r_{cl} = e^2/mc^2$ the classical electron radius, and Δn is the channel depth.

The dynamics of the laser pulse in a plasma channel can be studied by solving the wave equation. In the comoving coordinate $\zeta = k_p(z' - ct')$, and dimensionless variables x and y , and τ , the wave equation is

$$\left(\nabla_\perp^2 + 2 \frac{\partial^2}{\partial \zeta \partial \tau} - \frac{\partial^2}{\partial \tau^2} \right) a_\perp = \frac{n(\rho)}{n_0} a_\perp, \quad (3)$$

where $a_\perp = eA_\perp/mc^2$ is the perpendicular vector potential component and $\rho^2 = x^2 + y^2$. Equation (3) assumes $a_0 < 1$, and that self-focusing is not important, which is valid if the power of the laser pulse satisfies $P < P_c$, where $P_c [GW] \approx 17(k_L/k_p)^2$ and k_L is the laser pulse wave number. Under the slow-varying envelope approximation, the vector potential $a_\perp = (\tilde{a}_\perp/2) \exp(iM_p \zeta) + \text{c.c}$ satisfies $|\partial_\zeta \tilde{a}_\perp| \ll |M_p \tilde{a}_\perp|$, and the equation for the laser envelope reads

$$\left(\nabla_\perp^2 + 2 \frac{\partial^2}{\partial \zeta \partial \tau} + 2iM_p \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \tau^2} \right) \tilde{a}_\perp = \left(1 + \frac{\rho^2}{R^2} \right) \tilde{a}_\perp, \quad (4)$$

where $R = r_m^2/2$ is the dimensionless channel radius, and $M_p = k_L/k_p$ is the ratio of the laser and plasma wave numbers. For a typical on-axis plasma density $n_0 = 10^{18} \text{ cm}^{-3}$, the plasma wavelength is equal to $\lambda_p \approx 33 \text{ } \mu\text{m}$. Assuming, for simplicity, a laser wavelength of $\lambda_L = 1 \text{ } \mu\text{m}$, yields $M_p \approx 33$. In the paraxial approximation, the cross derivative and the second order derivative in τ are neglected and one recovers the simple Schrödinger equation for a harmonic oscillator

$$i \frac{\partial \tilde{a}_\perp}{\partial \tau} = \left[\frac{\hat{p}^2}{2M_p} + V(\rho) \right] \tilde{a}_\perp = \hat{H} \tilde{a}_\perp, \quad (5)$$

where $\hat{H}$ is the Hamiltonian of the oscillator, $\hat{p} = -i\nabla_\perp$ is the momentum operator and $V(\rho) = (1/M_p^2 + \Omega^2 \rho^2) M_p/2$ is the harmonic oscillator potential with the eigenfrequency $\Omega = (M_p R)^{-1}$.

It is convenient to use the apparatus of quantum mechanics to study the behavior of the laser pulse centroid and spotsizes in the plasma channel. The definition of the expectation value of some operator $\hat{O}$ is the following:

$$\langle \hat{O} \rangle = \langle \tilde{a}_\perp | \hat{O} | \tilde{a}_\perp \rangle = \frac{\int_{-\infty}^{\infty} \tilde{a}_\perp^* \hat{O} \tilde{a}_\perp dx dy}{\int_{-\infty}^{\infty} \tilde{a}_\perp^* \tilde{a}_\perp dx dy}. \quad (6)$$

Here, we have used the bra-ket notation and asterisk superscript denotes complex conjugation. The behavior of the expectation value of any operator can be studied using Ehrenfest's theorem. For example, for the expectation value of the operator $\hat{x} = x$, which corresponds to the laser pulse centroid position in the x -direction, one obtains the following equation:

$$\frac{d^2 \langle x \rangle}{d\tau^2} + \Omega^2 \langle x \rangle = 0, \quad (7)$$

which has known solutions

$$\langle x \rangle = \langle x_0 \rangle \cos \Omega \tau + \frac{\langle \hat{p}_{x0} \rangle}{M_p \Omega} \sin \Omega \tau. \quad (8)$$

Here, $\langle x_0 \rangle$ is the initial centroid displacement with respect to the channel axis, and $\langle \hat{p}_{x0} \rangle$ is the initial expectation value of the momentum operator $\hat{p}_x = -i\partial_x$, which is related to the angle of injection of the laser pulse into the channel. One can see that the laser pulse centroid oscillates with the characteristic frequency of Ω .

One can also easily study the behavior of the expectation value of x^2 , and, hence, find the evolution of the transverse laser pulse spotsizes, given by $w = 2(\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$.

The general solution for w is

$$w^2 = \frac{w_0^2}{2} + \frac{2\sigma_{p_x,0}^2}{M_p^2 \Omega^2} + \left(\frac{w_0^2}{2} - \frac{2\sigma_{p_x,0}^2}{M_p^2 \Omega^2} \right) \cos 2\Omega \tau + \frac{2}{M_p \Omega} (\langle x \hat{p}_x + \hat{p}_x x \rangle_0 - 2\langle x \rangle_0 \langle \hat{p}_x \rangle_0) \sin 2\Omega \tau, \quad (9)$$

where again subscript 0 denotes the expectation value of an operator taken at the initial time $\tau = 0$, and $\sigma_{p,0}^2 = \langle \hat{p}_x^2 \rangle_0 - \langle \hat{p}_x \rangle_0^2$. Knowing the initial laser pulse state, i.e., the function for the pulse entering the channel, one can use Eqs. (8) and (9) to find out how the centroid and the spotsizes of the laser pulse evolve while propagating along the channel. (The same equations can be also written for the

FIG. 1. Laser pulse centroid $\langle x \rangle (M_p/2\pi)$ (dashed line) and laser spotsizes $w(M_p/2\pi)$ (dotted and solid lines) as functions of the propagation distance z'/λ_L . Initial centroid displacement $\langle x \rangle_0 (M_p/2\pi) = 1$. The channel parameters are chosen in such a way that the matched laser pulse spotsizes is equal to $r_m (M_p/2\pi) = 7$. Matched (solid line), with $w_0 = r_m$, and mismatched (dotted line), with $w_0 = 1.5r_m$ cases are shown.

laser pulse vector potential, centroid position and spotsizes in the y direction).

As an example, consider a Gaussian pulse entering the parabolic plasma channel with some initial offset x_0 with respect to the channel axis (initial centroid displacement), and some initial angle θ_x ,

$$\tilde{a}_\perp(\tau = 0, x) = a_0 e^{[-(x-x_0)^2/w_0^2 + iM_p \theta_x x]}. \quad (10)$$

Again, for simplicity, only x components are considered. One can immediately calculate, that $\langle x \rangle_0 = x_0$ and $\langle \hat{p}_x \rangle_0 = M_p \theta_x$, and the centroid evolution is given by Eq. (8). Equation (9) for the laser pulse spotsizes has the following solution:

$$w^2 = \frac{w_0^2}{2} + \frac{2R^2}{w_0^2} + \left(\frac{w_0^2}{2} - \frac{2R^2}{w_0^2} \right) \cos 2\Omega \tau. \quad (11)$$

One can also calculate the matched case, i.e., the case when the laser spotsizes w is constant. This is true when the second term on the right-hand side (rhs) of Eq. (11) is zero, i.e., when $w_0 = r_m$. In the matched case, independent of the behavior of the centroid, the laser pulse spotsizes remains constant during the propagation. The frequency of the centroid oscillations in the matched case is $\Omega_m = 2/(M_p r_m^2)$, and is the inverse dimensionless Rayleigh length of the laser pulse, where the dimensionless Rayleigh length is $Z = k_p Z_R = M_p R = M_p r_m^2/2$. Examples of the evolution of the centroid and the spotsizes for a Gaussian laser pulse propagating along the plasma channel with the matched spotsizes $(M_p/2\pi)r_m = 7$ (the coefficient $M_p/2\pi$ transforms dimensionless space and time into the units measured in the laser wavelength and period, respectively) are presented in Fig. 1. The dashed line demonstrates the

laser pulse centroid $\langle x \rangle (M_p/2\pi)$ as a function of z'/λ_L , according to Eq. (8) for the case $\langle x \rangle_0 (M_p/2\pi) = 1$. Here, λ_L is the wavelength of the laser pulse. Note that, because the paraxial approximation was applied, we can interchange the dimensionless propagation time τ and propagation distance z . Using z for visualization provides a better insight for the propagation of short laser pulses in the plasma channel. The laser pulse spotsize $w(M_p/2\pi)$ as a function of z'/λ_L according to Eq. (11) is presented for two cases: (1) matched case, $w_0 = r_m$ (solid line), and (2) unmatched case with $w_0 = 1.5r_m$ (dotted line). In the matched case, the spotsize remains constant during the propagation. In the unmatched case, the laser pulse spotsize oscillates between the maximum value $w_{\max} = w_0$ and minimum value $w_{\min} = r_m^2/w_0$. (The same results are obtained for the y component).

It is also possible to inject the laser pulse with some displacement and under some angle so that the laser pulse centroid exhibits an ellipsoidal trajectory while the laser pulse propagates along the channel. For example, if the laser pulse is matched in both x and y , and if $\langle x \rangle_0 = x_0$, $\theta_x = 0$, $\langle y \rangle_0 = 0$ and $\theta_y = x_0/(M_p R) = x_0/Z$, then the laser pulse centroid will move along a circular trajectory while keeping constant spotsize. Here, analogously, $\langle y \rangle_0$ and θ_y are initial dimensionless centroid displacement and angle of injection in the y direction, respectively. A more elaborate theory of the propagation of the laser pulse with nonsymmetrical capillary coupling conditions can be found in Refs. [55,56]. As has been theoretically shown in [50], in the case of the matched laser pulse propagation with initial centroid displacement, one can create a novel type of compact plasma undulator. Polarization control of such an undulator can be achieved by injecting the laser pulse not only of axis, but also under a certain angle. The derivation of the field structure and electron trajectories in such an undulator is presented in Sec. III.

III. UNDULATOR FIELDS

The structure of the laser-excited plasma wakefield can be calculated from the laser pulse propagation. Here, the following assumptions are employed: (1) the laser pulse intensity is nonrelativistic, i.e., $a_0 < 1$ and linear plasma theory is valid; (2) the laser pulse propagates near the speed of light in vacuum; (3) the plasma channel is broad, $r_m \gg 1$, and the effect of the channel curvature on the wakefield is neglected (the wakefield curvature effect is briefly discussed below and is shown to have little effect on the radiation from plasma undulator). Within these assumptions, the equation for the dimensionless scalar potential $\phi = e\Phi/mc^2$ is [7]

$$\frac{\partial^2 \phi}{\partial \xi^2} + \phi = \frac{a_{\perp} a_{\perp}^*}{2}, \quad (12)$$

and has the following solution [7]:

$$\phi(\zeta, x, y) = \int_{-\infty}^{\zeta} d\zeta' \sin(\zeta - \zeta') \frac{\tilde{a}_{\perp} \tilde{a}_{\perp}^*}{2}, \quad (13)$$

which depends on the laser pulse shape.

The evolution of the matched laser pulse with centroid displacement inside a parabolic plasma channel has been discussed in Sec. II. Assuming a Gaussian laser shape in any direction and circular polarization, the intensity of the matched laser pulse $I = \tilde{a}_{\perp} \tilde{a}_{\perp}^*$ takes the form

$$I = a_0^2 e^{-2\zeta^2/w_z^2} e^{-2[r - r_c(\tau)]^2/r_m^2}, \quad (14)$$

where w_z is the dimensionless longitudinal size of the laser pulse (or dimensionless duration), $\mathbf{r} = (x, y)$ is a two-dimensional vector representing transverse coordinates, and $\mathbf{r}_c = (x_c(\tau), y_c(\tau))$ is a two-dimensional vector representing the laser pulse centroid trajectory. The solution for $\phi(\zeta, x, y)$ is

$$\phi(\zeta, x, y) = -\frac{a_0^2}{2} \sqrt{\frac{\pi}{2}} w_z e^{-w_z^2/8} e^{-2[r - r_c(\tau)]^2/r_m^2} \sin(\zeta). \quad (15)$$

The amplitude of the wakefield has a maximum at $w_z = 2$, and in this case,

$$\phi(\zeta, x, y) = -a_0^2 C e^{-2[r - r_c(\tau)]^2/r_m^2} \sin(\zeta), \quad (16)$$

where $C = \sqrt{e^{-1}\pi/2} \approx 0.76$ (note that $C \approx 0.38$ for a linearly polarized laser pulse). Magnetic fields, that are proportional to a_0^4 [57], are neglected in the linear case $a_0 < 1$, and the electric fields are given by $\mathbf{E}(\zeta, x, y) = -\nabla \phi(\zeta, x, y)$, where gradient is taken in (ζ, x, y) coordinates, and the electric field is normalized to $mc^2 k_p/e$. Assuming that the centroid oscillation amplitude is small $|\mathbf{r}_c| < 1$, one obtains

$$E_z = a_0^2 C e^{-2[r - r_c(\tau)]^2/r_m^2} \cos(\zeta), \quad (17)$$

$$\mathbf{E}_{\perp} = -\frac{4a_0^2 C}{r_m^2} (\mathbf{r} - \mathbf{r}_c(\tau)) e^{-2[r - r_c(\tau)]^2/r_m^2} \sin(\zeta), \quad (18)$$

where $\mathbf{E}_{\perp} = (E_x, E_y)$ is a vector containing transverse field components. In the case when centroid oscillation amplitude is much smaller than the matched laser pulse spotsize, the exponential terms can be neglected. The laser pulse centroid trajectory in the x -direction is given by Eq. (8), with a similar expression for the centroid motion in the y -direction. From Eq. (18), one can see that the transverse focusing field consists of two parts. The first part is proportional to $\mathbf{r}$ and is constant in time for the particle comoving with the wakefield structure, such that $\sin(\zeta) = \text{const}$. This term is responsible for the so-called betatron oscillations. The second part is proportional to

$\mathbf{r}_c(\tau)$, and is responsible for the induced electron undulator oscillations, as will be further discussed in detail in Sec. IV.

IV. ELECTRON TRAJECTORIES

The trajectory of an electron, or an electron beam, injected in the plasma undulator can be calculated from the structure of the wakefields. In order to generate narrow-bandwidth radiation, highly energetic electrons should be injected in such a phase ζ_i , where the accelerating field E_z is zero, to avoid evolution of the electron energy; hence, according to Eqs. (17) and (18), at a phase such that $\cos(\zeta_i) = 0$. Moreover, the fields E_x and E_y should be focusing. Hence, an electron should be injected in the phase $\zeta_i = 3\pi/2$. In this case, the equations of motion for the electron, injected at the initial phase $\zeta_i = 3\pi/2$ are

$$\frac{d\mathbf{u}_\perp}{d\tau} = -\mathbf{E}_\perp|_{\zeta=3\pi/2}, \quad (19)$$

$$\frac{d\mathbf{r}_e}{d\tau} = \frac{\mathbf{u}_\perp}{\gamma_0}, \quad (20)$$

where $\mathbf{u}_\perp = \mathbf{p}_\perp/mc$ is the dimensionless electron perpendicular momentum and γ_0 the initial electron Lorentz gamma factor. It is assumed that $\gamma_0 \gg 1$, so that its change during the electron motion may be neglected.

Using Eq. (18) for the transverse electric fields in the case when the amplitude of the centroid displacement is much smaller than r_m (so that the exponential term can be neglected), together with Eqs. (19) and (20), one finds the following equation for the transverse electron coordinates $\mathbf{r}_e$:

$$\frac{d^2\mathbf{r}_e}{d\tau^2} + \Omega_\beta^2\mathbf{r}_e = \Omega_\beta^2\mathbf{r}_c(\tau), \quad (21)$$

where

$$\Omega_\beta = \left(\frac{4a_0^2 C}{\gamma_0 r_m^2} \right)^{\frac{1}{2}} \quad (22)$$

is the frequency of the betatron oscillations. The laser pulse centroid trajectory $\mathbf{r}_c(\tau)$, using Eq. (8), is

$$\mathbf{r}_c(\tau) = \frac{1}{2}r_{c,0}\boldsymbol{\epsilon}e^{i(\Omega\tau)} + \text{c.c.}, \quad (23)$$

where $r_{c,0}$ is the amplitude of the laser pulse centroid oscillation and $\boldsymbol{\epsilon}$ is the vector defining the ellipticity of laser pulse centroid oscillations. Without loss of generality, we assume that the absolute value of this vector ranges from 1 (linear polarization) to 2 (circular polarization), and the amplitude of the first component of the ellipticity vector is equal to 1, and the second component is purely imaginary with its absolute value ranging from 0 to 1. In this way, we

are consistent with the general definitions of the strength of a circularly polarized undulator. For example, $\boldsymbol{\epsilon} = (1, 0)$ means linear polarization of the centroid oscillations in the x direction, and $\boldsymbol{\epsilon} = (1, \pm i)$ describes left- and right-circular polarization of the centroid oscillations, respectively. The solution for the transverse momentum components of the electron is

$$\begin{aligned} \mathbf{u}_\perp = & -\left[\frac{a_u \Omega_\beta}{\Omega} \left(\frac{\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^*}{2} \right) + \gamma_0 \Omega_\beta \mathbf{r}_{e,0} \right] \sin \Omega_\beta \tau \\ & + \left[\mathbf{u}_{\perp,0} + a_u \left(\frac{i\boldsymbol{\epsilon} - i\boldsymbol{\epsilon}^*}{2} \right) \right] \cos \Omega_\beta \tau \\ & - a_u \left[\frac{i\boldsymbol{\epsilon} e^{i\Omega\tau} - i\boldsymbol{\epsilon}^* e^{-i\Omega\tau}}{2} \right], \end{aligned} \quad (24)$$

where $\mathbf{r}_{e,0} = (x_{e,0}, y_{e,0})$ and $\mathbf{u}_{\perp,0} = (u_{x,0}, u_{y,0})$ are the vectors of initial electron coordinates and momenta components, respectively, and

$$a_u = \frac{\gamma_0 \Omega \Omega_\beta^2 r_{c,0}}{\Omega^2 - \Omega_\beta^2} \quad (25)$$

is the undulator strength (the amplitude of the normalized transverse momentum for the induced electron oscillations). Note that the equation for the undulator strength assumes $\Omega > \Omega_\beta$, which is generally the case. In the case when $\Omega \gg \Omega_\beta$, the undulator strength is approximately given by

$$a_u \approx 2a_0^2 C M_p r_{c,0} = \frac{4\pi a_0^2 C r'_{c,0}}{\lambda_L}, \quad (26)$$

where $r'_{c,0}$ is the amplitude of the laser pulse centroid oscillations in centimeter-gram-second (cgs) units. From Eq. (24), an electron traveling in the wakefield created by the laser pulse undergoing centroid oscillations exhibits two oscillations: (1) betatron oscillations with the frequency Ω_β , and (2) undulator oscillations with the frequency Ω .

The strength (or the amplitude) of the betatron oscillations is

$$\begin{aligned} a_{\beta,i}^2 = & \left(\frac{a_u \Omega_\beta}{\Omega} \frac{\epsilon_i + \epsilon_i^*}{2} + \gamma_0 \Omega_\beta [\mathbf{r}_{e,0}]_i \right)^2 \\ & + \left(u_{i,0} + a_u \frac{i\epsilon_i - i\epsilon_i^*}{2} \right)^2, \end{aligned} \quad (27)$$

where i in the subscript denotes x or y component and $[\mathbf{r}_{e,0}]_i$ denotes x or y component of the initial electron coordinates. For the generation of narrow-bandwidth radiation, it is beneficial to keep the betatron strength low, $a_{\beta,i} \ll 1$. This means that there are optimal injection conditions (laser and electron) such that Eq. (27) is minimized. For a single electron injected into the plasma undulator, both terms on

the rhs of Eq. (27) can be made to vanish. Consider, for example, circularly polarized undulator with the “cosine” mode in the x -direction, and “sine” mode in the y -direction, represented by the ellipticity parameter $\epsilon = (1, -i)$. For initial conditions $x_{e,0} = -a_u/(\gamma_0\Omega)$ and $u_{y,0} = -a_u$, the betatron strengths for a single electron injected into the plasma undulator are $a_{\beta,x} = u_{x,0}$ and $a_{\beta,y} = \gamma_0\Omega\beta y_{e,0}$. Hence for these initial conditions injecting with zero $u_{x,0}$ and zero $y_{e,0}$ corresponds to “optimal” injection, where betatron motion is absent for an electron or, for a electron beam, is minimized.

The importance of initial conditions of the electron injection is illustrated on Fig. 2(a), where u_x and u_y are plotted as functions of propagation distance z'/λ_L for two cases: (1) “optimal” electron injection as defined above to minimize betatron motion (blue curve), given by $(x_{e,0}, y_{e,0}) = (-a_u/(\gamma_0\Omega), 0)$ and $(u_{x,0}, u_{y,0}) = (0, -a_u)$, and (2) “incorrect” electron injection (red curve), with $(x_{e,0}, y_{e,0}) = (0, 0)$ and $(u_{x,0}, u_{y,0}) = (0, 0)$. In both cases, the matched laser pulse spotsize is $(M_p/2\pi)r_m = 7\lambda_L$, the initial centroid displacement is $(M_p/2\pi)r_{c,0} = 2.5\lambda_L$, the exponential term in the expression for the focusing field given by Eq. (18) is neglected, and the undulator is circularly polarized. In the first case, the trajectory is perfectly circular with $a_{\beta,x} = 0$, $a_{\beta,y} = 0$, and $a_u = 1$. In the case of the incorrect injection, the trajectory is not circular and significant betatron motion occurs with amplitude of the transverse momentum u_y reaching values approximately 2 [red curve on the rear panel of the plot in Fig. 2(a)]. This betatron motion leads to significant radiation spectrum broadening, as will be shown in Sec. VI. Note that in the case of an electron beam with nonzero emittance, some electrons will experience betatron motion due to their distribution about the ideal trajectory. For typical conditions, the amplitude of the betatron oscillations can be significantly smaller than the amplitude of the undulator oscillations, driven by the ratio of beam radius to initial displacement. This is true provided that the electron beam is injected with optimal initial conditions and the focusing forces are sufficiently weak. The latter condition is valid for the linear wakefields. Figure 2(b) shows the plots of u_x and u_y as functions of propagation distance z'/λ_L for the case of correct electron injection, $(x_{e,0}, y_{e,0}) = (-a_u/(\gamma_0\Omega), 0)$ and $(u_{x,0}, u_{y,0}) = (0, -a_u)$, for two cases: (1) the exponential term in the expression for the focusing field E_r given by Eq. (18) is neglected (blue curve), and (2) the exponential term is taken into account (green curve). The first case is the same as the case of correct injection presented on Fig. 2(a). For the case when the exponential term in the expression of the focusing field is taken into account, the amplitude of the transverse momentum $u_\perp$ is slightly smaller (0.9 versus 1), and, as will be shown in Sec. VI, the produced radiation has a narrow bandwidth. In all cases presented on Fig. 2, the wavelength of undulator

FIG. 2. Electron dimensionless transverse momenta components u_x and u_y as functions of the propagation distance z'/λ_L . The blue curve corresponds to the case of injection of an electron with $(x_{e,0}, y_{e,0}) = (-a_u/(\gamma_0\Omega), 0)$ and $(u_{x,0}, u_{y,0}) = (0, -a_u)$ into the circular undulator with $a_u = 1$. (a) The red curve is the case of the electron injection with $(x_{e,0}, y_{e,0}) = (0, 0)$ and $(u_{x,0}, u_{y,0}) = (0, 0)$. (b) The green line corresponds to the case of the electron injection conditions with $(x_{e,0}, y_{e,0}) = (-a_u/(\gamma_0\Omega), 0)$ and $(u_{x,0}, u_{y,0}) = (0, -a_u)$, for the case when the exponential term in Eq. (18) is included.

oscillations (in cgs units) is equal to $\lambda_u \approx 1000\lambda_L$. For the laser with $\lambda_L = 0.8 \mu\text{m}$, the undulator wavelength is, hence, below millimeter.

V. ELECTRON BEAM EVOLUTION IN A PLASMA UNDULATOR

In many respects the behavior of the electron beam inside the laser wakefield undulator is analogous to the behavior of the laser pulse inside the plasma channel. Namely, if the electron beam is not matched to the focusing fields, its envelope will oscillate. The centroid of the electron beam will behave like a single electron and its

trajectory can be found using Eq. (21). Assuming no energy spread of the electron beam, and taking only terms linear with respect to the transverse coordinate $\mathbf{r}$ in Eq. (18), the normalized emittance of the electron beam is constant throughout the propagation (a more general case, including the electron beam energy spread and emittance growth can be found in Refs. [58,59]). With these assumptions, the electron beam rms spotsize $\sigma_{e,i}$ and rms divergence $\sigma_{\theta,i}$ are

$$\sigma_{e,i}^2 = \sigma_{e,i,0}^2 \cos^2(\Omega_\beta \tau) + \frac{r_{b,i,m}^4}{\sigma_{e,i,0}^2} \sin^2(\Omega_\beta \tau), \quad (28)$$

$$\sigma_{\theta,i}^2 = \frac{\sigma_{\theta,i,m}^4}{\sigma_{\theta,i,0}^2} \sin^2(\Omega_\beta \tau) + \sigma_{\theta,i,0}^2 \cos^2(\Omega_\beta \tau), \quad (29)$$

where $\sigma_{e,i,0}$ is initial electron rms beam radius (where the i subscript denotes x or y), $\sigma_{\theta,i,0}$ is initial electron beam rms divergence, $r_{b,i,m}$ is the dimensionless matched electron rms beam size and $\sigma_{\theta,i,m} = \gamma_0 \Omega_\beta r_{b,i,m}$ is the matched rms divergence. As discussed in Ref. [60], the dimensionless matched electron beam size $r_{b,i,m}$ is

$$r_{b,i,m} = \left(\frac{\epsilon_{n,i}}{\gamma_0 \Omega_\beta} \right)^{\frac{1}{2}}, \quad (30)$$

where $\epsilon_{n,i}$ is the dimensionless rms normalized beam emittance. (The normalized rms beam emittance in cgs units is $\epsilon_{n,i}/k_p$.) In the case when the electron beam is not matched, its divergence will change while it is propagating through the plasma undulator and this can lead to additional broadening of the generated radiation spectrum, as discussed in Sec. VI. In the case when the electron beam is matched to the plasma focusing forces, both its size and divergence stay constant during the propagation.

Using Eq. (27), the average betatron strength for the matched electron beam is

$$\begin{aligned} \langle a_{\beta,i}^2 \rangle &= \gamma_0^2 \Omega_\beta^2 \sigma_{e,i,0}^2 + \sigma_{\theta,i,0}^2 \\ &+ \left[\gamma_0 \Omega_\beta \langle [\mathbf{r}_{e,0}]_i \rangle + \frac{a_u \Omega_\beta}{\Omega} \left(\frac{\epsilon_i + \epsilon_i^*}{2} \right) \right]^2 \\ &+ \left[\langle u_{i,0} \rangle + a_u \left(\frac{i\epsilon_i - i\epsilon_i^*}{2} \right) \right]^2. \end{aligned} \quad (31)$$

Here, $\langle [\mathbf{r}_{e,0}]_i \rangle$ is initial electron beam centroid position and $\langle u_{i,0} \rangle$ is the average initial electron beam momentum, related to the angle of injection. For a beam injected matched, $\sigma_{e,i,0} = r_{b,i,m}$ and $\sigma_{\theta,i,0} = \sigma_{\theta,i,m} = \gamma_0 \Omega_\beta r_{b,i,m}$. Similar to the case of a single particle presented in Sec. IV, for narrow bandwidth radiation generation, the matched electron beam should be injected with initial conditions so that the average betatron strength is minimized, i.e., the last two terms on the rhs of Eq. (31) vanish. Optimized injection conditions for the electron beam are

the same as for the single electron. Note that, for the case of a beam, the betatron strength cannot vanish, due to the finite beam emittance (spread in transverse position and momentum distributions of the beam).

Four different cases illustrate the importance of both correct matching of the beam and of the exponential term in the expression for transverse field as given by Eq. (18). Beam evolution while propagating in the plasma undulator are presented for each case in Fig. 3. In all four cases, the plasma undulator is created by a Gaussian laser pulse with amplitude $a_0 = 0.2$, matched spotsize $r'_m = 7\lambda_L$ and initial centroid displacement $r'_{c,0} = 2.5\lambda_L$, where r'_m and $r'_{c,0}$ are the matched spotsize and centroid displacement in cgs units, respectively. The laser pulse parameters are chosen in such a way that the undulator wavelength is on the order of a millimeter and the undulator strength $a_u = 1.02$ is on the order of unity. The normalized emittance of the cylindrically symmetric electron beam is $0.1 \mu\text{m}$, the Lorentz gamma factor is $\gamma_0 = 1000$, energy spread is neglected, and the beam is assumed to be infinitely small in propagation direction z , for purely illustrative purposes. For these parameters, the matched electron beam radius given by Eq. (30) is equal to $r'_{b,m} = 0.25\lambda_L$, where $r'_{b,m}$ is given in cgs units. Linear undulator polarization in the y direction was chosen. Numerical simulations were performed for each case using the particle tracking code VDSR [61]. Figure 3(a) shows the evolution of the matched electron beam propagating inside the undulator with the exponential term in the focusing force neglected, and with optimal electron beam initial conditions ($\langle x_{e,0} \rangle, \langle y_{e,0} \rangle \rangle = (-a_u/(\gamma_0 \Omega), 0)$ and $\langle u_{x,0} \rangle, \langle u_{y,0} \rangle \rangle = (0, -a_u)$ for minimizing the betatron amplitude. The color image shows the distribution of the x' positions of electrons within the beam for each z' position. The electron beam spotsize stayed constant (within 1% accuracy). The white solid line presents the analytical solution Eq. (21) and agrees well with the numerical simulations. The electron beam as a whole performs oscillations with undulator frequency that is inversely proportional to the laser pulse Rayleigh length. Figure 3(b) shows the dynamics of the matched electron beam injected such that $\langle x_{e,0} \rangle, \langle y_{e,0} \rangle \rangle = (0, 0)$ and $\langle u_{x,0} \rangle, \langle u_{y,0} \rangle \rangle = (0, 0)$, i.e., along the channel axis and with no initial angle. The undulator structure is the same as in Fig. 3(a), i.e., the exponential term in the focusing force is neglected. The white solid line shows the analytical solution, Eq. (21), and agrees well. The electron beam spotsize stays constant, but the beam undergoes large betatron oscillations. This effect is detrimental for narrow-bandwidth photon sources; however this effect can also be beneficial for enhancing the photon output of betatron x rays. Figure 3(c) shows the dynamics of the electron beam with optimal injection, but unmatched to the focusing force [undulator structure is the same as in Figs. 3(a) and 3(b)]. In this case the electron beam radius and divergence oscillate, and this leads to additional

FIG. 3. Electron beam evolution inside the plasma undulator. The color image at each position z' represents the distribution of the x' positions of the beam electrons, and the white solid line is the analytical solution for the single electron obtained from Eq. (24). (a) The case of a matched electron beam injected into the undulator with optimal conditions for minimizing the betatron amplitude: $(\langle x_{e,0} \rangle, \langle y_{e,0} \rangle) = (-a_u/(\gamma_0 \Omega), 0)$ and $(\langle u_{x,0} \rangle, \langle u_{y,0} \rangle) = (0, -a_u)$. The exponential term is neglected. (b) Undulator structure is the same as in case (a), but a matched electron beam is injected with $(\langle x_{e,0} \rangle, \langle y_{e,0} \rangle) = (0, 0)$ and $(\langle u_{x,0} \rangle, \langle u_{y,0} \rangle) = (0, 0)$. (c) The case of an unmatched beam with optimal injection for the undulator structure the same as in case (a). (d) The case of a matched electron beam with optimal injection into the undulator [same as in (a)], with the exponential term included in the expressions for the transverse field as given by Eq. (18).

broadening of the emitted radiation. The centroid of the electron beam still oscillates as expected (the analytical solution is shown with the white solid line). Finally, Fig. 3(d) demonstrates the dynamics of the matched electron beam injected with the optimal centroid condition into the wakefield structure given by Eq. (18) with the exponential term included. The electron beam oscillates as a whole and the difference from the case presented in Fig. 3(a) mainly manifests itself in the shift of the central frequency of emitted radiation due to the decreased undulator strength a_u . This will be seen in Sec. VI. It is important to mention that in experiments it might be easier to adjust the injection of the laser pulse into the capillary, rather than the injection of the electron beam, but the principle remains the same.

VI. UNDULATOR RADIATION

Radiation emitted by a single relativistic electron undergoing harmonic oscillations in an undulator with constant strength a_u has been extensively studied and can be applied to the present case [1–3,62,63]. For example, the near-axis photon energy-angular spectrum of the fundamental emission line is given by

$$\frac{d^2 N_{ph}}{d\omega' d(\gamma_0^2 \theta^2)} = \pi \alpha_f \frac{a_u^2 |JJ|^2}{1 + a_u^2/2} \frac{S(\omega')}{\omega'}, \quad (\text{linear}) \quad (32)$$

$$\frac{d^2 N_{ph}}{d\omega' d(\gamma_0^2 \theta^2)} = 2\pi \alpha_f \frac{a_u^2}{1 + a_u^2} \frac{S(\omega')}{\omega'}, \quad (\text{circular}), \quad (33)$$

where $\alpha_f = e^2/(\hbar c)$ is the fine structure constant, ω' is the generated frequency (in cgs units), θ is the polar angle in radians,

$$JJ = J_0(\chi) - J_1(\chi), \quad (34)$$

where J_n denotes the Bessel function of the n th order, $\chi = a_u^2/[4(1 + a_u^2/2)]$, and the emission shape $S(\omega')$ is given by

$$S(\omega') = \frac{\sin^2(\pi N_u \frac{\omega' - \omega'_1}{\omega'_1})}{\sin^2(\pi \frac{\omega' - \omega'_1}{\omega'_1})}, \quad (35)$$

with N_u the number of oscillations (number of undulator periods) and

$$\omega'_1 = \frac{2\gamma_0^2 \omega'_u}{1 + a_u^2/2}, \quad (\text{linear}) \quad (36)$$

$$\omega'_1 = \frac{2\gamma_0^2 \omega'_u}{1 + a_u^2}, \quad (\text{circular}) \quad (37)$$

is the central frequency of the emitted photons. Here, ω'_u is the undulator frequency (in cgs units). Using Eqs. (32) and (33) one finds that the natural bandwidth is $\Delta\omega/\omega = 1/N_u$ and the total number of photons emitted by a single electron traveling through the undulator is

$$N_{\text{ph, natural}} = \pi\alpha_f \frac{a_u^2}{1 + a_u^2/2} |JJ|^2, \quad (\text{linear}) \quad (38)$$

$$N_{\text{ph, natural}} = 2\pi\alpha_f \frac{a_u^2}{1 + a_u^2}. \quad (\text{circular}). \quad (39)$$

Achieving the natural bandwidth requires collimation of the emitted radiation up to the angle θ_{natural} , given by

$$\theta_{\text{natural}} = \frac{1}{\gamma_0} \sqrt{\frac{1 + a_u^2/2}{N_u}}, \quad (\text{linear}) \quad (40)$$

$$\theta_{\text{natural}} = \frac{1}{\gamma_0} \sqrt{\frac{1 + a_u^2}{N_u}}. \quad (\text{circular}). \quad (41)$$

The on-axis photon energy spectrum generated by an electron propagating inside the circularly-polarized plasma undulator with total number of periods $N_u = 30$ created by the matched Gaussian pulse is shown in Fig. 4. Figure 4 was calculated using the code VDSR [61]. The plasma undulator parameters are the same as in Fig. 3. Three different cases were calculated: (1) optimally injected electron, with the exponential term in the expression for transverse fields from Eq. (18) neglected (blue curve, corresponds to the trajectory plotted in blue shown in Fig. 2); (2) electron injected with zero initial offset and zero transverse momentum, exponential term included [red curve, corresponds to the trajectory plotted in green in Fig. 2(a)]; (3) optimally injected electron with exponential term included [green curve, corresponds to the trajectory plotted in green in Fig. 2(b)]. The on-axis spectrum from the undulator theory Eq. (33) is plotted (dashed black curve); the analytical results fit the numerical calculations. For convenience, the dimensionless photon energy (or dimensionless emitted frequency),

$$\kappa = \frac{\omega'}{2\gamma_0^2 \omega'_u}, \quad (42)$$

was introduced.

The spectrum for the case when the exponential term is taken into account has a similar form to the case when the

FIG. 4. Normalized on-axis single electron radiation spectrum (in arbitrary units) as a function of normalized photon energy κ for the case of a plasma undulator with $N_u = 30$. The blue curve corresponds to the case of optimal injection with the exponential term in Eq. (18) neglected. The black dashed line shows the analytical expression Eq. (33). The red curve corresponds to the case of on axis injection with no initial electron transverse momentum, with the exponential term in Eq. (18) neglected. The green curve is the on-axis radiation spectrum of the electron injected optimally into the plasma undulator with the exponential term in Eq. (18) included.

exponential term is neglected, but blueshifted. This frequency shift is due to the reduced undulator strength in the case when the exponential term is included.

Finally, for the case of an electron injected on-axis, with zero initial offset and zero transverse momentum, one observes that both undulator and betatron strengths are high; in this case $a_u \approx 1$ and $a_\beta \approx 2$. This leads to the shift of the main undulator line and frequency mixing, and hence to the appearance of additional spectral lines. This effect can in principle be used for diagnostics of injection for a staged LPA [64]. It is important to add that the circular trajectory of the particle produces on-axis radiation that is circularly polarized. Polarization of the radiation can be controlled by the injection conditions of the laser pulse into the plasma channel, as discussed in Sec. VII. Such polarization control is a key feature of this plasma undulator.

In the case of an electron beam the radiation spectrum will be broadened compared to the case of a single particle. The main contributions to the spectral broadening are the electron beam transverse angular and energy spread. These effects have been studied for conventional undulators and Thomson scattering x-ray sources [14,63,65], and the results apply to the case of the plasma undulator. The approximate full-width-at-half-maximum (FWHM) bandwidth of the undulator source is

$$\left[\frac{\Delta\kappa}{\kappa} \right]_{\text{FWHM}} \approx \sqrt{\left(\frac{1}{N_u} \right)^2 + \left(\frac{2\Delta\gamma}{\gamma_0} \right)^2 + \frac{(\gamma_0 \Delta\theta)^4}{16}}, \quad (43)$$

FIG. 5. Photon energy-angular spectrum as a function of both normalized photon energy κ (horizontal axis) and normalized angle $\gamma_0\theta$ (vertical axis) for four cases: (a) electron beam is matched and optimally injected into the plasma undulator [with the exponential term in Eq. (18) neglected]; (b) electron beam is matched, but injected on-axis into the same undulator; (c) electron beam is injected optimally, but unmatched to the same undulator; (d) electron beam is injected optimally and matched to the undulator, with exponential term in Eq. (18) included. Undulator parameters and beam parameters are the same as Figs. 3(a)–3(d), with $N_u = 30$.

where $\Delta\gamma \approx 2.35\sigma_\gamma$ is the FWHM electron beam energy spread, with σ_γ the rms electron beam energy spread, and $\Delta\theta \approx 2.35\sigma_\theta$ is the FWHM electron beam angular spread, with σ_θ the rms angular spread. Additional spectral broadening can appear due to injection errors and matching of the electron beam.

Figure 5 shows the radiation generated by an electron beam traveling through the plasma undulator, calculated using VDSR [61], for four different cases of beam injection and matching. Parameters of the undulator and electron beam injection and matching are the same as in Fig. 3, such that Figs. 5(a)–5(d) correspond to the cases presented in Figs. 3(a)–3(d). Here we considered a 1% rms energy spread and $1.7 \mu\text{m}$ rms bunch length. Figure 5 shows the photon energy-angular spectrum $d^2 N_{ph} / [N_e d\kappa d(\gamma_0\theta)]$ as a function of normalized photon energy κ and normalized angle $\gamma_0\theta$. Note that the photon spectrum is normalized with the number of electrons in a bunch N_e , so that the results can be applied to electron beams with different charge. The cyan curve shows the lineout of the on-axis spectrum in arbitrary units.

As can be seen in Figs. 5(a), 5(c), and 5(d), in the case of optimal injection of the electron beam into the plasma undulator, a narrow-bandwidth undulator line is generated. In contrast, in the case of simple on-axis injection of the electron beam into the undulator without angle, presented

in Fig. 5(b), the radiation spectrum is broader and the peak of the spectrum is lower. On all four subfigures, one can observe the betatron spectral line as well, which is well separated, and, in the case of optimal injection is much weaker than the undulator spectral line.

Figure 6 presents the photon energy spectrum integrated in angle from $\theta = 0$ to $\theta = \theta_{\text{natural}}$, given by Eq. (41), for different beam matching and injection cases. The blue curve is the case of optimal injection of the beam with $0.1 \mu\text{m}$ normalized emittance, matched to the plasma undulator with the structure the same as in Fig. 3(a). The bandwidth of the source is dominated by the contribution from the beam angular spread, which in this case is equal to $\Delta\theta \approx 4.7 \text{ mrad}$. The bandwidth of the source in the case of the optimally injected and matched beam with normalized emittance $0.1 \mu\text{m}$ and $\Delta\gamma/\gamma_0 = 2.35\%$ energy spread is approximately 17%, in good agreement with Eq. (43). The total number of photons in this bandwidth per electron is approximately 0.019. The green curve shows a case of lower normalized emittance: $0.025 \mu\text{m}$, also with the beam optimally injected and matched. This produces narrower-band, more intense radiation, with 10% FWHM bandwidth and 0.021 photons per electron. The red curve shows the case where the electron beam is optimally injected to minimize the betatron motion, but not matched, and hence the bandwidth of the source is slightly broader

FIG. 6. Photon energy spectrum, obtained by integrating the energy-angular photon spectrum in angle from $\theta = 0$ to $\theta = \theta_{\text{natural}}$, given by Eq. (41), as a function of normalized photon energy κ for the following cases: (blue curve) optimally injected to minimize the betatron motion, matched beam, with $0.1 \mu\text{m}$ normalized emittance; (green curve) optimally injected, matched beam, with $0.025 \mu\text{m}$ emittance; (red curve) optimally injected, unmatched beam, with $0.1 \mu\text{m}$ normalized emittance; (cyan curve) on-axis injection, matched beam, with $0.1 \mu\text{m}$ normalized emittance. The black curve shows the case of an ideal beam with no transverse emittance and no energy spread. (Note that, for plotting purposes, the curve representing the ideal beam case was multiplied by a factor of 0.5.) The undulator parameters are the same as in Fig. 3(a), i.e., the exponential term in the focusing force is neglected.

and the peak of the spectrum is lower. The cyan curve shows the case of a matched beam (with $0.1 \mu\text{m}$ emittance) injected on axis. For this on-axis injected beam, the spectrum is broadened and the number of photons per electron is reduced to 0.016. Each electron beam case can be compared to the black curve which corresponds to the case of a single particle, i.e., an ideal beam with zero emittance and energy spread, optimally injected into a linearly polarized undulator. Integrating the photon energy spectrum over the normalized energy κ , one can obtain the total number of photons. This case shows good agreement with Eq. (39); the total number of photons per electron is approximately equal to $\pi\alpha_f \approx 0.02$. The natural FWHM bandwidth of the source is in this case equal to 3%.

VII. POLARIZATION CONTROL OF THE PLASMA UNDULATOR

Polarization control of the undulator is readily achieved by controlling the offset ($\langle x \rangle_0, \langle y \rangle_0$) and the angle (θ_x, θ_y) of the laser pulse injection into the plasma channel. Using Eq. (8), and assuming for simplicity that $\theta_x = 0$ and $\langle y \rangle_0 = 0$, the laser pulse centroid position in x and y directions is

$$\langle x \rangle = \langle x \rangle_0 \cos(\Omega\tau) \quad (44)$$

$$\langle y \rangle = \frac{\theta_y}{\Omega} \sin(\Omega\tau). \quad (45)$$

FIG. 7. Photon energy-angular spectrum in arbitrary units as a function of normalized photon energy κ (longitudinal axis) and normalized angle $\gamma_0\theta$ (vertical axis) for the case of optimally injected electron beam. The colored lines show the corresponding contours of the radiation ellipticity (values indicated in colorbar) as functions of normalized photon energy κ and angle $\gamma_0\theta$.

Given an initial centroid displacement in the x direction $\langle x \rangle_0$, one can control the ellipticity of the laser pulse centroid motion by tuning the angle with respect to the y axis θ_y . The electron trajectories undergoing undulator oscillations [i.e., the last term on the rhs of Eq. (24)], and, hence, the generated radiation, will have the same ellipticity as the laser centroid motion. Assuming, also for simplicity, that $|\theta_y/\Omega| \leq |\langle x \rangle_0|$, then the ellipticity η of the radiation can be expressed as

$$\eta = \theta_y Z_R / \langle x \rangle_0. \quad (46)$$

For example, a perfect circularly polarized undulator is obtained for $\theta_y \approx 16 \text{ mrad}$ for the plasma undulator parameters of Fig. 3. Changing the angle of laser injection also provides control over the handedness of the circular or elliptical polarization.

Figure 7 presents an example of the emitted photon spectrum for a single electron injected, optimally to minimize the betatron motion, into a plasma undulator. The density plot shows the normalized photon energy-angular spectrum, with the longitudinal axis representing the normalized photon energy κ and the vertical axis the normalized observation angle $\gamma_0\theta$. Parameters of the plasma undulator are the same as presented in Fig. 4 (the lineout of the photon spectrum at $\theta = 0$ in Fig. 7 corresponds to the blue curve in Fig. 4). Colored lines correspond to different values of ellipticity (with values indicated by the colorbar of Fig. 7). On-axis radiation is perfectly circularly polarized, whereas the radiation at an angle with respect to the z -axis is elliptically polarized, as is also the case for conventional magnetic undulators. In the case of on-axis electron injection, such that the trajectory of the electron is given by the green line in Fig. 2(a), the

FIG. 8. On-axis photon spectrum (red curve and left-vertical axis) in arbitrary units for an electron injected on axis (not optimal). Blue lines show the ellipticity of the on-axis radiation (value on right-vertical axis). (For illustrative reasons, ellipticity is not shown for the spectral regions with intensity less than 5% of the maximum intensity.)

polarization qualities of the undulator are complicated by the betatron motion, shown in Fig. 8. Figure 8 shows the normalized on-axis photon spectrum (red curve) as a function of normalized photon energy κ , and the blue line shows the ellipticity of the radiation (right vertical axis). The polarization properties are due to the large value of a_β , such that the trajectory is no longer circular [as seen in Fig. 2(a)].

VIII. CONCLUSIONS AND DISCUSSION

In this paper we have provided a detailed theoretical study of the plasma undulator based on the wakefield excitation in the plasma channel. It has been demonstrated that a Gaussian laser pulse injected into the channel off axis or under some angle will exhibit oscillations. The wavelength of the centroid oscillations sets the plasma undulator wavelength, $\lambda_u = 2\pi Z_R$, which can be submillimeter.

Using linear plasma theory, the field structure of such a plasma wakefield undulator was derived, as well as the electron trajectories. It was shown that, in general, an electron injected in the plasma undulator will undergo undulator motion and betatron oscillations. Different cases of electron injection into the plasma wakefield were discussed. Injection can be optimized to minimize the amplitude of the betatron motion. In the case of optimal injection conditions, the electron oscillates with the wavelength of λ_u . Electron beam evolution in the plasma undulator was investigated and the radiation generated calculated. The electron injection conditions may be chosen to generate narrow-bandwidth undulator radiation or to increase the betatron emission yield.

The relative yield of betatron radiation and undulator radiation, determined by the beam initial conditions with respect to the laser centroid motion, may be of interest as a diagnostic of the relative laser and beam positions injected

into a plasma channel. This is particularly critical for LPA staging or deceleration experiments, where the electron beam from one LPA stage must be injected into another stage, driven by the fresh laser pulse [64].

Control of the particle trajectory ellipticity and, hence, the polarization of the undulator radiation, is readily achieved by simply adjusting the initial laser centroid and angle with respect to the plasma channel axis. This ease of polarization control is a key feature of the proposed plasma undulator.

In the cases presented in Sec. VI, the amount of photons generated per electron in a bandwidth equal to the on-axis bandwidth is approximately $N_{ph}/N_e = \pi\alpha_f$. The undulator line relative FWHM bandwidth is approximately given by Eq. (43), where the main contribution to the bandwidth is the electron beam angular spread. This is due to the strong focusing forces of the wakefield driven by a Gaussian laser, yielding small matched beam radii and large angular spread. This effect can be mitigated by either decreasing the electron beam emittance and/or decreasing the focusing forces. The latter can be achieved by using higher order Hermite-Gaussian modes for driving the plasma undulator [66]. Using higher-order laser modes also provides a method to reduce the betatron amplitude spread of the beam, theoretically enabling a free-electron laser using the short-period plasma undulator [50].

From the experimental point of view, the proposed plasma undulator concept would require: (1) very good laser beam pointing stability (assuming LPA electron beam to have similar pointing stability), a fraction of the laser spotsize, and (2) timing stability such that the electron injection is in the correct region of the plasma wakefield. According to Ref. [67], laser beam pointing stability can be as good as $2.6 \mu\text{rad}$, which, assuming a 30 cm focusing length, would give the transverse jitter less than $0.8 \mu\text{m}$. Inaccuracy of the laser pulse injection of this order would lead to shot-to-shot variation in the undulator strength and generated wavelength for the values of the laser pulse spot sizes discussed in the paper. Using larger spot sizes or further isolation of the laser system from various types of external noise may increase the stability of the proposed concept. As for the electron beam injection into the plasma wakefield, for the typical values of the plasma density and plasma wavelength on the order of $30 \mu\text{m}$, timing stability on the order of 10 fs would suffice. This value is within experimental capabilities.

Several effects have been neglected in this publication to allow analytical treatment. It was assumed that the laser pulse is propagating at the vacuum speed of light. The lower group velocity of a physical laser pulse in the plasma will result in dephasing of the electron beam with respect to the structure over the structure's length. Mitigation of dephasing effects using channel tapering has been investigated in Ref. [68]. Beam loading [69] was also neglected and can play an important role in operation of the plasma

undulator. Beam loading will limit the amount of charge that can be injected. These topics will be addressed in future work.

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