Erratum: Estimation of coherent error sources from stabilizer measurements **[Phys. Rev. A **93**, 042303 (2016)]**

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Equation (6) in the original article is as follows:

$$\Delta p_a = \{(n_x^a)^2 + [1 - (n_x^a)^2] \cos \lambda_a\} \\ \times \prod_{b \in \mathcal{N}_a} \{(n_z^b)^2 + [1 - (n_z^b)^2] \cos \lambda_b\},$$

which was supposed to give the difference in probability of obtaining +1 and −1 outcomes when measuring a correlator $K_a = X_a \otimes_{b \in \mathcal{N}_a} Z_b$ on a perturbed graph state. This formula is, in general, incorrect: It only represents one of a series of terms.

In this Erratum, we correct Eq. (6) in the original paper and derive some conditions under which the original expression still holds: In many of the cases considered in the original paper, the extra terms vanish. The high-level conclusions, as outlined in the abstract of the original article, are not affected.

General correction of Eq. (6) in the original paper

We consider a graph state perturbed by coherent noise, which we interpret as a stray field: $|\tilde{G}\rangle := U_{\text{tot}}|G\rangle$, where $U_{\text{tot}} = \bigotimes_{a \in V} U_a = \bigotimes_{a \in V} e^{-i\lambda_a \hat{n}_a S_a/2}$. The quantity we want to evaluate is as follows:

$$\Delta p_a = \langle \tilde{G} | K_a | \tilde{G} \rangle \\ = \text{Tr}[|G\rangle\langle G| U_{\text{tot}}^\dagger K_a U_{\text{tot}}] \\ = \text{Tr}[|G\rangle\langle G| K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}}]. \quad (1)$$

First, we expand $|G\rangle\langle G|$ as a sum of stabilizer operators using the identity derived in Proposition 3 of Ref. [1],

$$|G\rangle\langle G| = \frac{1}{2^N} \sum_{\xi \in \mathcal{S}} \xi = \frac{1}{2^N} \sum_{\substack{(j_1, \dots, j_N) \\ \in \{0,1\}^N}} K_1^{j_1} \cdots K_N^{j_N}, \quad (2)$$

where $K_a^0 = \mathbb{1}$, $K_a^1 = K_a$, and the stabilizer group $\mathcal{S}$ is the set of elements ξ of the multiqubit Pauli group $\mathcal{P}_V$ that stabilizes $|G\rangle$, $\xi|G\rangle = |G\rangle$. That is, $|G\rangle\langle G|$ is equal to the sum over all stabilizers of $|G\rangle$, and each stabilizer can be factorized uniquely as a product of correlators of $|G\rangle$.

Second, we expand $K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}}$ as a linear combination of Pauli operators. The single-qubit unitaries U_c acting on qubits c outside of $\mathcal{N}'_a := \mathcal{N}_a \cup \{a\}$ commute with K_a , and these operators cancel each other out. Consequently, $K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}}$ has support on (i.e., acts nontrivially only on) the qubits in the

 set $\mathcal{N}'_a$,

$$K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}} = K_a^\dagger \left(\bigotimes_{c \in \mathcal{N}'_a} U_c^\dagger \right) K_a \left(\bigotimes_{d \in \mathcal{N}'_a} U_d \right) \quad (3) \\ = (\alpha_{\mathbb{1}}^a \mathbb{1}_a + i\alpha_Y^a Y_a + i\alpha_Z^a Z_a) \\ \times \bigotimes_{b \in \mathcal{N}_a} (\tilde{\alpha}_{\mathbb{1}}^b \mathbb{1}_b + i\tilde{\alpha}_X^b X_b + i\tilde{\alpha}_Y^b Y_b) \bigotimes_{c \in V \setminus \mathcal{N}'_a} \mathbb{1}_c. \quad (4)$$

The coefficients $\alpha_{\mathbb{1}}^a$, α_Y^a , α_Z^a , $\tilde{\alpha}_{\mathbb{1}}^b$, $\tilde{\alpha}_X^b$, and $\tilde{\alpha}_Y^b$ depend on the parameters of the field at vertex a and at its neighbors $b \in \mathcal{N}_a$, and they can be computed expanding the expression in Eq. (3) as performed in Appendix A of the original article,

$$\alpha_{\mathbb{1}}^a = (n_x^a)^2 + [1 - (n_x^a)^2] \cos(\lambda_a), \\ \alpha_Y^a = -n_y^a \sin(\lambda_a) - n_x^a n_z^a [1 - \cos(\lambda_a)], \\ \alpha_Z^a = -n_z^a \sin(\lambda_a) + n_x^a n_y^a [1 - \cos(\lambda_a)], \\ \tilde{\alpha}_{\mathbb{1}}^b = (n_z^b)^2 + [1 - (n_z^b)^2] \cos(\lambda_b), \\ \tilde{\alpha}_X^b = -n_x^b \sin(\lambda_b) - n_y^b n_z^b [1 - \cos(\lambda_b)], \\ \tilde{\alpha}_Y^b = -n_y^b \sin(\lambda_b) + n_x^b n_z^b [1 - \cos(\lambda_b)]. \quad (5)$$

Equation (4) can be compactly written as

$$K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}} = \sum_{\xi \in \mathcal{P}_{\mathcal{N}'_a}} c_\xi^a \xi, \quad (6)$$

where the set $\mathcal{P}_{\mathcal{N}'_a} = \{\xi \in \{\mathbb{1}, X, Y, Z\}^{\otimes N} | \xi_c = \mathbb{1}_c \forall c \notin \mathcal{N}'_a\}$ consists of multiqubit Pauli operators that act as the identity outside of the set $\mathcal{N}'_a$ and $c_\xi^a \in \mathbb{C}$ denotes a product of $\alpha_{\mathbb{1}}^a$, $i\alpha_Y^a$, $i\alpha_Z^a$, $\tilde{\alpha}_{\mathbb{1}}^b$, $i\tilde{\alpha}_X^b$, and $i\tilde{\alpha}_Y^b$.

Using the expressions now given for $|G\rangle\langle G|$ and $K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}}$ and the linearity of the trace, Eq. (1) becomes

$$\Delta p_a = \sum_{\substack{(j_1, \dots, j_N) \\ \in \{0,1\}^N}} \sum_{\xi \in \mathcal{P}_{\mathcal{N}'_a}} \frac{c_\xi^a}{2^N} \text{Tr}[K_1^{j_1} \cdots K_N^{j_N} \xi]. \quad (7)$$

We then use $\text{Tr}[\zeta] = 2^N \delta_{\zeta, \mathbb{1}}$ for $\zeta \in \{\mathbb{1}, X, Y, Z\}^{\otimes N}$. The methods given in Sec. 2 of Ref. [1] can be used to show that, if $K_1^{j_1} \cdots K_N^{j_N} \xi = \mu \mathbb{1}^{\otimes N}$, then the phase is $\mu = \pm 1$ and, therefore, $K_1^{j_1} \cdots K_N^{j_N} = \mu_\xi \xi$, with $\mu_\xi \in \{-1, +1\}$. Hence,

$$\Delta p_a = \sum_{\xi \in \mathcal{S} \cap \mathcal{P}_{\mathcal{N}'_a}} \mu_\xi c_\xi^a. \quad (8)$$

That is, the sum in Eq. (8) runs over multiqubit Pauli operators that satisfy two requirements: First, they are stabilizer operators; second, these operators act nontrivially only on the qubits in the set $\mathcal{N}'_a$. There are always, at least, two operators that satisfy both requirements: $\mathbb{1}^{\otimes N}$ and K_a . If $\mathbb{1}^{\otimes N}$ and K_a are the only stabilizers with support contained in $\mathcal{N}'_a$, then we have $\Delta p_a = c_{\mathbb{1}}^a + c_{K_a}^a = c_{\mathbb{1}}^a$ since Eq. (4) implies $c_{K_a}^a = 0$.

In conclusion, Eq. (6) in the original paper is correct exactly when Eq. (8) consists of only one term,

$$\Delta p_a = c_{\mathbb{1}}^a = \alpha_{\mathbb{1}}^a \prod_{b \in \mathcal{N}_a} \tilde{\alpha}_b^a = \{(n_x^a)^2 + [1 - (n_x^a)^2] \cos \lambda_a\} \times \prod_{b \in \mathcal{N}_a} \{(n_z^b)^2 + [1 - (n_z^b)^2] \cos \lambda_b\}. \quad (9)$$

However, Eq. (9) does not always hold as other nonzero coefficients can be present in Eq. (8). A counterexample is given by the two-qubit graph state $|G\rangle = CZ_{1,2}|+\rangle_1|+\rangle_2$, which has stabilizers $\mathbb{1} \otimes \mathbb{1}$, $X \otimes Z$, $Z \otimes X$, and $Y \otimes Y$ so that we have

$$\begin{aligned} \Delta p_1 &= \langle \tilde{G} | X_1 \otimes Z_2 | \tilde{G} \rangle = c_{\mathbb{1} \otimes \mathbb{1}}^1 + c_{X \otimes Z}^1 + c_{Z \otimes X}^1 + c_{Y \otimes Y}^1, \\ \Delta p_2 &= \langle \tilde{G} | Z_1 \otimes X_2 | \tilde{G} \rangle = c_{\mathbb{1} \otimes \mathbb{1}}^2 + c_{X \otimes Z}^2 + c_{Z \otimes X}^2 + c_{Y \otimes Y}^2. \end{aligned} \quad (10)$$

From Eq. (4), it follows that $c_{X \otimes Z}^1 = c_{Z \otimes X}^2 = 0$, but the other coefficients are, in general, different from zero.

Conditions for correctness of Eq. (6) in the original paper

We now show, following two lines of reasoning, that in many of the cases analyzed in the original article, Eq. (6) in the original paper still holds. First, we give conditions under which $\mathbb{1}$ and K_a are the only stabilizers with support contained in $\mathcal{N}'_a$. Second, we show that, even if other stabilizers ξ have support contained in $\mathcal{N}'_a$, their coefficients c_{ξ}^a can still be zero because of the specific form of Eq. (4).

First, a stabilizer ξ has support contained in $\mathcal{N}'_a$ when its factorization $\xi = K_1^{j_1} \cdots K_N^{j_N}$ consists of a product of correlators K_c each centered at a qubit $c \in \mathcal{N}'_a$. This is the case because a correlator K_c has the form $K_c = X_c \otimes_{d \in \mathcal{N}_c} Z_d$ and all other correlators either act as $\mathbb{1}_c$ or as Z_c on the qubit at the vertex c ; thus, they cannot cancel out the X_c operator appearing in K_c . Therefore, the sum in Eq. (8) contains at most 2^{d_a} terms, where $d_a = |\mathcal{N}'_a|$ is the number of vertices in $\mathcal{N}'_a$. Only the stabilizers that are products of correlators K_c with $c \in \mathcal{N}'_a$ can contribute. For a given subset $M \subseteq \mathcal{N}'_a$, the operator $\prod_{a \in M} K_a$ has support contained in $\mathcal{N}'_a$ if and only if all the Z operators acting on qubits outside of $\mathcal{N}'_a$ cancel each other out. This happens exactly when each vertex $c \notin \mathcal{N}'_a$ is adjacent to an even number of vertices belonging to M . Using this fact, in certain cases we can prove that $\mathbb{1}$ and K_a are the only stabilizers with support contained in $\mathcal{N}'_a$. In particular, this holds (for all $a \in V$) for square lattices with sides of length $m_1 \geq 4$ and $m_2 \geq 2$ for planar geometry and for square lattices with $m_1 \geq 5$ and $m_2 \geq 3$ for toroidal geometry. Hence, using $c_{K_a}^a = 0$, in all these cases, we have $\Delta p_a = c_{\mathbb{1}}^a$, and thus Eq. (6) in the original paper holds.

Second, we use Eq. (4) to show that, when the stray fields are parallel to one of the Cartesian axes, some

simplifications take place, often resulting in Eq. (6) in the original paper being still valid. For instance, for Z fields, we have $\Delta p_a = \cos \lambda_a$ for all graph states. Thus, as stated in the original article, Z fields are always reconstructible. For X and Y fields we have

$$\begin{aligned} \text{X field: } K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}} &= \bigotimes_{b \in \mathcal{N}_a} [\cos(\lambda_b) \mathbb{1}_b - i \sin(\lambda_b) X_b] \bigotimes_{c \in V \setminus \mathcal{N}_a} \mathbb{1}_c, \quad (11) \end{aligned}$$

$$\begin{aligned} \text{Y field: } K_a^\dagger U_{\text{tot}}^\dagger K_a U_{\text{tot}} &= [\cos(\lambda_a) \mathbb{1}_a - i \sin(\lambda_a) Y_a] \\ &\times \bigotimes_{b \in \mathcal{N}_a} [\cos(\lambda_b) \mathbb{1}_b - i \sin(\lambda_b) Y_b] \bigotimes_{c \in V \setminus \mathcal{N}_a} \mathbb{1}_c. \quad (12) \end{aligned}$$

As a consequence, many coefficients c_{ξ}^a vanish

$$\begin{aligned} \text{X field: } c_{\xi}^a &= 0, \\ &\text{if } \xi_a \notin \{\mathbb{1}_a\}, \text{ or if } \xi_b \notin \{\mathbb{1}_b, X_b\} \text{ for some } b \in \mathcal{N}_a, \\ \text{Y field: } c_{\xi}^a &= 0, \\ &\text{if } \xi_a \notin \{\mathbb{1}_a, Y_a\}, \text{ or} \\ &\text{if } \xi_b \notin \{\mathbb{1}_b, Y_b\} \text{ for some } b \in \mathcal{N}_a. \end{aligned}$$

These two sets of simplifications are crucial for linear chains (i.e., one-dimensional cluster states). First, consider closed linear chains with $m \geq 5$ qubits. In this case, we have $\Delta p_a = c_{\mathbb{1}}^a$ for all vertices. In fact, the set of stabilizer operators having support on $\mathcal{N}'_a$ consists only of $\mathbb{1}$ and K_a since the correlators K_{a-1} and K_{a+1} act as Z on the vertices in positions $a-2$ and $a+2$, and, for chains of length greater than 4, these Z operators cannot cancel each other out. Next, consider open-ended linear chains with $m \geq 4$ qubits (numbered in the obvious way): For each vertex a , except for $a=2$ and $a=m-1$, only $\mathbb{1}$ and K_a have support on $\mathcal{N}'_a$. We then have

$$\begin{aligned} \Delta p_a &= c_{\mathbb{1}}^a, & \text{if } a \notin \{2, m-1\}, \\ \Delta p_a &= c_{\mathbb{1}}^a + c_{X_1 \otimes Z_2}^a, & \text{if } a = 2, \\ \Delta p_a &= c_{\mathbb{1}}^a + c_{Z_{m-1} \otimes X_m}^a, & \text{if } a = m-1. \end{aligned} \quad (13)$$

Hence, Eq. (6) in the original paper does not generically hold for open-ended linear chains. But specializing to the X - and Y -field cases, we have $\Delta p_a = c_{\mathbb{1}}^a$ for all the vertices $a \in V$, hence, for all these cases, the analyses in the original article are correct.

Correction of tables and plots

Several cases considered in the original article have to be analyzed on an individual basis. These computations are lengthy but straightforward. Here, we show how the plots and the tables in the original article have to be modified when using the correct formula Eq. (8) instead of Eq. (6) in the original paper. We present here corrected versions of Tables I and III and Figs. 1–3. The main differences with the original tables and plots are described in the captions.

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FIG. 1. The plots are all qualitatively similar to the plots in the original article except for three cases. In panel (X field, ϵ dependence), qubits 1 and 10 have a larger reconstruction error than what was previously displayed. In panel (Z field, ϵ dependence), qubits 2 and 9 have a larger reconstruction error than what was previously displayed (the curves for qubits 2 and 9 before were essentially overlapping with the curves for qubits 3–8). In panel (Z field, ϵ dependence), qubits 2 and 9 have a larger reconstruction error than what was previously displayed (the curves for all qubits were essentially overlapping).

TABLE I. We consider linear chains of length $m \in \mathbb{N}$. In each entry of the table, we give the set of values of m such that $\Delta p_a = c_{\perp}^a$ holds $\forall a$ and, hence, Eq. (6) in the original paper is correct. Z fields can always be reconstructed.

	Open-end linear chain	Closed linear chain
General field	No m	$m \geq 5$
X field	$m \neq 1, 3$	$m \neq 1, 4$
Y field	$m \neq 2$	$m \neq 2, 3$

TABLE III. We consider fully connected graphs and star graphs with N vertices (both representing GHZ states up to local unitaries). In each entry of the table, we give the set of values of $N \in \mathbb{N}$ such that $\Delta p_a = c_{\perp}^a$ holds $\forall a$.

General field	No N	No N
X field	$N \geq 2$	No N
Y field	No N	Even N

FIG. 2. The gray cells correspond to the cases where Eq. (6) in the original paper is not valid, and, thus, the reconstruction methods are not applicable in these cases. All other cases remain unaltered.

FIG. 3. These are the results of the numerical analyses for field reconstruction including the correction of Eq. (6) in the original paper. It seems that an anisotropy in the dispersion of the data points for the reconstructed fields has been removed.