Erratum: Conditions for coherence transformations under incoherent operations [Phys. Rev. A **91**, 052120 (2015)]

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There is an error in the proof of the necessity of Theorem 1. That is, the cross term of the second equation of (21) does not vanish, which is due to $\delta_{j,i(l)}$ being dependent on n . Chitambar and Gour have made this point in [1]. In the following, we correct the error.

Now, we suppose $\dim H = 3$ and there is an incoherent operation Φ that transforms $|\psi\rangle\langle\psi|$ to $|\phi\rangle\langle\phi|$. Let

$$\Phi(|\psi\rangle\langle\psi|) = \sum_n K_n |\psi\rangle\langle\psi| K_n^\dagger = |\phi\rangle\langle\phi|. \quad (1)$$

Hence there exist complex numbers α_n such that $K_n |\psi\rangle = \alpha_n |\phi\rangle$. Let $k_j^{(n)}$ ($j = 1, 2, 3$) be the nonzero element of K_n at the j th column (if there is no nonzero element in the j th column, then $k_j^{(n)} = 0$). Suppose $k_j^{(n)}$ locates the $f_n(j)$ th row. Here, $f_n(j)$ is a function that maps $\{2, 3\}$ to $\{1, 2, 3\}$ with the property that $1 \leq f_n(j) \leq j$. Let

$$\delta_{s,t} = \begin{cases} 1, & s = t \\ 0, & s \neq t \end{cases}.$$

Then there is a permutation π_n such that

$$K_n = P_{\pi_n} \begin{pmatrix} k_1^{(n)} & \delta_{1,f_n(2)} k_2^{(n)} & \delta_{1,f_n(3)} k_3^{(n)} \\ 0 & \delta_{2,f_n(2)} k_2^{(n)} & \delta_{2,f_n(3)} k_3^{(n)} \\ 0 & 0 & \delta_{3,f_n(3)} k_3^{(n)} \end{pmatrix}. \quad (2)$$

From $\sum_n K_n^\dagger K_n = I$, we get that

$$\begin{aligned} \sum_n |k_j^{(n)}|^2 &= 1 \quad (j = 1, 2, 3), \\ \sum_n \overline{k_1^{(n)}} \delta_{1,f_n(2)} k_2^{(n)} &= 0, \\ \sum_n \overline{k_1^{(n)}} \delta_{1,f_n(3)} k_3^{(n)} &= 0, \\ \sum_n (\delta_{1,f_n(2)} \delta_{1,f_n(3)} + \delta_{2,f_n(2)} \delta_{2,f_n(3)}) \overline{k_2^{(n)}} k_3^{(n)} &= 0. \end{aligned} \quad (3)$$

For $|\psi\rangle = (\psi_1, \psi_2, \psi_3)^t$, by a direct computation, one can get

$$K_n |\psi\rangle = P_{\pi_n} \begin{pmatrix} k_1^{(n)} \psi_1 + \delta_{1,f_n(2)} k_2^{(n)} \psi_2 + \delta_{1,f_n(3)} k_3^{(n)} \psi_3 \\ \delta_{2,f_n(2)} k_2^{(n)} \psi_2 + \delta_{2,f_n(3)} k_3^{(n)} \psi_3 \\ \delta_{3,f_n(3)} k_3^{(n)} \psi_3 \end{pmatrix}, \quad (4)$$

and so

$$\begin{aligned} k_1^{(n)} \psi_1 + \delta_{1,f_n(2)} k_2^{(n)} \psi_2 + \delta_{1,f_n(3)} k_3^{(n)} \psi_3 &= \alpha_n \phi_{\pi_n^{-1}(1)}, \\ \delta_{2,f_n(2)} k_2^{(n)} \psi_2 + \delta_{2,f_n(3)} k_3^{(n)} \psi_3 &= \alpha_n \phi_{\pi_n^{-1}(2)}, \\ \delta_{3,f_n(3)} k_3^{(n)} \psi_3 &= \alpha_n \phi_{\pi_n^{-1}(3)}. \end{aligned} \quad (5)$$

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Applying $\sum_n |\cdot|^2$ to the above equations, we have

$$\begin{aligned}
 & \psi_1^2 + \sum_n \delta_{1,f_n(2)} |k_2^{(n)}|^2 \psi_2^2 + \sum_n \delta_{1,f_n(3)} |k_3^{(n)}|^2 \psi_3^2 + \sum_n \delta_{1,f_n(2)} \delta_{1,f_n(3)} \overline{k_2^{(n)}} k_3^{(n)} \psi_2 \psi_3 + \sum_n \delta_{1,f_n(2)} \delta_{1,f_n(3)} \overline{k_3^{(n)}} k_2^{(n)} \psi_3 \psi_2 \\
 &= \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(1)}^2, \\
 & \sum_n \delta_{2,f_n(2)} |k_2^{(n)}|^2 \psi_2^2 + \sum_n \delta_{2,f_n(3)} |k_3^{(n)}|^2 \psi_3^2 + \sum_n \delta_{2,f_n(2)} \delta_{2,f_n(3)} \overline{k_2^{(n)}} k_3^{(n)} \psi_2 \psi_3 + \sum_n \delta_{2,f_n(2)} \delta_{2,f_n(3)} \overline{k_3^{(n)}} k_2^{(n)} \psi_3 \psi_2 \\
 &= \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(2)}^2, \\
 & \sum_n \delta_{3,f_n(3)} |k_3^{(n)}|^2 \psi_3^2 = \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(3)}^2.
 \end{aligned} \tag{6}$$

Note that, for $s = 1, 2, 3$,

$$\sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(s)}^2 = \sum_{n, \pi_n^{-1}(s)=1} |\alpha_n|^2 \phi_1^2 + \sum_{n, \pi_n^{-1}(s)=2} |\alpha_n|^2 \phi_2^2 + \sum_{n, \pi_n^{-1}(s)=3} |\alpha_n|^2 \phi_3^2. \tag{7}$$

Let $d_{ij} = \sum_{n, \pi_n^{-1}(i)=j} |\alpha_n|^2$, $1 \leq i, j \leq 3$, then the matrix $D = (d_{ij})$ is a doubly stochastic matrix, since $\sum_n |\alpha_n|^2 = 1$. Furthermore,

$$D(\phi_1^2, \phi_2^2, \phi_3^2)^t = \left(\sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(1)}^2, \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(2)}^2, \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(3)}^2 \right)^t. \tag{8}$$

This implies that

$$\left(\sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(1)}^2, \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(2)}^2, \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(3)}^2 \right)^t < (\phi_1^2, \phi_2^2, \phi_3^2)^t. \tag{9}$$

On the other hand, note that in the first equation of (6),

$$\begin{aligned}
 & \sum_n \delta_{1,f_n(2)} |k_2^{(n)}|^2 \psi_2^2 + \sum_n \delta_{1,f_n(3)} |k_3^{(n)}|^2 \psi_3^2 + \sum_n \delta_{1,f_n(2)} \delta_{1,f_n(3)} \overline{k_2^{(n)}} k_3^{(n)} \psi_2 \psi_3 + \sum_n \delta_{1,f_n(2)} \delta_{1,f_n(3)} \overline{k_3^{(n)}} k_2^{(n)} \psi_3 \psi_2 \\
 &= \sum_n |\delta_{1,f_n(2)} k_2^{(n)} \psi_2 + \delta_{1,f_n(3)} k_3^{(n)} \psi_3|^2.
 \end{aligned} \tag{10}$$

This tells us that

$$\psi_1^2 \leq \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(1)}^2.$$

Adding the first and the second equations of (6) together, by (3), we have

$$\psi_1^2 + \psi_2^2 \leq \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(1)}^2 + \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(2)}^2.$$

From the definition of majorization, one can check that

$$(\psi_1^2, \psi_2^2, \psi_3^2)^t < \left(\sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(1)}^2, \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(2)}^2, \sum_n |\alpha_n|^2 \phi_{\pi_n^{-1}(3)}^2 \right)^t. \tag{11}$$

Combining (9) with (11), we have

$$(\psi_1^2, \psi_2^2, \psi_3^2)^t < (\phi_1^2, \phi_2^2, \phi_3^2)^t.$$

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[1] E. Chitambar and G. Gour, [Phys. Rev. A **94**, 052336 \(2016\)](#).