Erratum: Comparison of incoherent operations and measures of coherence **[Phys. Rev. A **94**, 052336 (2016)]**

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Theorem 21 is not correct. The described algorithm for generating IO Kraus operators does not always succeed. Namely, when repeating the previous steps in step (5), the established condition $\langle 0|\tilde{M}_j|x\rangle\langle 1|\tilde{M}_j|x\rangle = 0$ may no longer hold. We thank Xiao-Dong Yu for bringing this to our attention as well as for providing an explicit counterexample,

$$\begin{aligned} M_0 &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |+\rangle\langle 1|), \\ M_1 &= \frac{1}{\sqrt{2}}(|1\rangle\langle 0| + |-\rangle\langle 1|), \end{aligned} \quad (1)$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. These are Kraus operators for a MIO channel which is not IO.

Theorem 21 plays a crucial role in our proof for Theorem 24, a main result of the paper. Nevertheless, Theorem 24 still holds true as we will now show by a different proof.

Theorem 24. For qubit states ρ and σ , the transformation $\rho \rightarrow \sigma$ is possible by either SIO, DIO, IO, or MIO if and only if both $C_R(\rho) \geq C_R(\sigma)$ and $C_{R,\Delta}(\rho) \geq C_{R,\Delta}(\sigma)$.

Proof. By Lemma 23, these conditions are sufficient for the transformation $\rho \rightarrow \sigma$ by SIO (and therefore also by DIO, IO, MIO). Conversely, as shown in Sec. II A 2, C_R is a MIO monotone. What remains to be shown is that $C_{R,\Delta}$ is also a MIO monotone.

To prove monotonicity of $C_{R,\Delta}$ for general transformations, we only need to consider special transformations of the form

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \rightarrow \sigma = \begin{pmatrix} q & t \\ t & 1-q \end{pmatrix}. \quad (2)$$

Indeed, suppose that $\rho' = \begin{pmatrix} p & s \\ s & 1-p \end{pmatrix}$ is an arbitrary standard-form qubit state that can be converted into σ by MIO. Then the state

$\rho = \frac{1}{2} \begin{pmatrix} 1 & s/\sqrt{p(1-p)} \\ s/\sqrt{p(1-p)} & 1 \end{pmatrix}$ satisfies $C_R(\rho) \geq C_R(\rho')$ and $C_{R,\Delta}(\rho) = C_{R,\Delta}(\rho')$. Hence the transformation $\rho \rightarrow \rho'$ is possible by MIO, and therefore also the transformation $\rho \rightarrow \sigma$. Since ρ and ρ' have the same values of $C_{R,\Delta}$, we see that if $C_{R,\Delta}$ is a monotone under a MIO when the initial state has uniform diagonal elements, then it must also be a monotone for general initial states.

To prove monotonicity for transformation (2), we study the Choi matrix $\Omega^{AB} = \mathcal{E}^A \otimes \mathbb{I}^B(|\Phi\rangle\langle\Phi|^{AB})$ for a general MIO map $\mathcal{E}$, where $|\Phi\rangle = |00\rangle + |11\rangle$. The MIO conditions on $\mathcal{E}$ translate into the following conditions on Ω^{AB} :

$$\text{Tr}_B[\mathbb{I} \otimes |y\rangle\langle y| \Omega^{AB}] - \Delta(\text{Tr}_B[\mathbb{I} \otimes |y\rangle\langle y| \Omega^{AB}]) = 0, \quad y = 0, 1, \quad (3)$$

$$\text{Tr}_A \Omega^{AB} = \mathbb{I}, \quad (4)$$

$$\Omega^{AB} \geq 0. \quad (5)$$

Represent Ω^{AB} in the Pauli basis as

$$\Omega^{AB} = \frac{1}{2} \left(\mathbb{I} \otimes \mathbb{I} + \vec{a} \cdot \vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{b} \cdot \vec{\sigma} + \sum_{ij} c_{ij} \sigma_i \otimes \sigma_j \right). \quad (6)$$

Then conditions (3) and (4) require that

$$\vec{b} = 0, \quad (7)$$

$$a_x = a_y = c_{xz} = c_{yz} = 0. \quad (8)$$

Since we consider transformation of real density matrices, we can assume without loss of generality that Ω^{AB} is real [i.e., that $(\Omega^{AB})^T = \Omega^{AB}$]. Thus,

$$c_{yx} = c_{xy} = c_{zy} = 0. \quad (9)$$

Additionally, since the initial state in transformation (2) is invariant under conjugation by σ_x , we can also assume that Ω^{AB} is invariant under conjugation by $\mathbb{I} \otimes \sigma_x$. This further yields the condition

$$c_{yy} = 0. \quad (10)$$

The remaining free parameters are $\{a_z, c_{xx}, c_{zx}, c_{zz}\}$, and the smallest eigenvalue of Ω^{AB} can be explicitly computed as

$$\begin{aligned} 0 \leq \lambda_{\min}(\Omega^{AB}) &= \frac{1}{2} \left(1 - \sqrt{c_{xx}^2 + c_{zx}^2 + c_{zz}^2 + a_z^2 + 2\sqrt{c_{xx}^2 c_{zz}^2 + a_z^2 (c_{zx}^2 + c_{zz}^2)}} \right) \\ &\leq \frac{1}{2} \left(1 - \sqrt{c_{xx}^2 + c_{zx}^2 + a_z^2 + 2|a_z c_{zx}|} \right), \end{aligned} \quad (11)$$

where the second inequality is obtained by setting $c_{zz} = 0$ in the first line. Therefore, $\Omega^{AB} \geq 0$ requires that

$$\begin{aligned} c_{xx}^2 &\leq 1 - (a_z^2 + c_{zx}^2 + 2|a_z c_{zx}|) \\ &\leq 1 - (a_z^2 + c_{zx}^2 r^2 + 2|a_z c_{zx}|r) \\ &\leq 1 - (a_z + r c_{zx})^2, \end{aligned} \quad (12)$$

for $0 \leq r \leq 1$.

Now, since $\rho = \frac{1}{2}(\mathbb{I} + r\sigma_x)$, we have

$$\sigma = \text{Tr}_B[\mathbb{I} \otimes \rho \Omega^{AB}] = \frac{1}{2}[\mathbb{I} + (a_z + r c_{zx})\sigma_z + r c_{xx}\sigma_x]. \quad (13)$$

The Δ robustness of σ therefore satisfies

$$C_{R,\Delta}(\sigma) = \frac{r c_{xx}}{\sqrt{[1 + (a_z + r c_{zx})][1 - (a_z + r c_{zx})]}} = r \frac{c_{xx}}{\sqrt{1 - (a_z + r c_{zx})^2}} \leq r = C_{R,\Delta}(\rho), \quad (14)$$

where the last inequality follows from Eq. (12). This completes the proof. $\blacksquare$

Notice that $C_{R,\Delta}$ remains invariant under MIO if and only if $c_{zz} = c_{zx} = 0$ and $c_{xx} = \sqrt{1 - a_z^2}$. For example, for the transformation $\rho = \frac{1}{2} \begin{pmatrix} 1 & s/\sqrt{p(1-p)} \\ s/\sqrt{p(1-p)} & 1 \end{pmatrix} \rightarrow \rho' = \begin{pmatrix} p & s \\ s & p \end{pmatrix}$ described above, one takes $a_z = 2p - 1$, $c_{xx} = 2\sqrt{p(1-p)}$, and $c_{zz} = c_{zx} = 0$.