Erratum: Minimal distances for certain quantum product codes and tensor products of chain complexes [Phys. Rev. A **102, 062402 (2020)]**

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Lemma 1 in our paper is not valid as originally stated. Here we give a formulation with a condition added. The condition is satisfied by all code families considered in the paper; hence it does not affect other results. We restate the lemma and its proof as follows.

Lemma 1 (Z-puncturing bound). Consider a stabilizer code $Q = \text{CSS}(H_X, H_Z)$ with the parameters $[[n, k, (d_X, d_Z)]]_q$ and a qudit index set $V = \{1, 2, \dots, n\}$. Given a partition into complementary sets $I \subset V$ and $J = V \setminus I$, suppose a logical generator matrix L_X can be chosen so that none of its k rows is supported both in I and in J . Let $Q' = \text{CSS}((H_X)_I, H_Z[I])$ and $Q'' = \text{CSS}((H_X)_J, H_Z[J])$ be the codes whose X generator matrices are shortened and Z generator matrices punctured to I and J , respectively. Then the Z distances of the three codes satisfy the inequality $d_Z \geq \min(d'_Z, d''_Z)$.

Proof. The case $k = 0$ is trivial since it gives infinite d_Z . Assume $k > 0$. The distance d_Z of the code is the minimum weight in the set $Q_Z = C_{H_X}^\perp \setminus C_{H_Z}$ of all nontrivial Z -like codewords and their equivalent vectors. For any $c \in Q_Z$, the punctured vectors $c[I]$ and $c[J]$ are orthogonal to the rows of $(H_X)_I$ and $(H_X)_J$, respectively; the corresponding Pauli errors are undetectable. Further, since $L_X c^T \neq 0$, it is impossible that $c[I]$ is orthogonal to the rows of $(L_X)_I = L_X[I]$ and at the same time $c[J]$ is orthogonal to the rows of $(L_X)_J = L_X[J]$. Therefore, at most one of the vectors $c[I]$ and $c[J]$ can be trivial in the corresponding code.

Now consider the identity $\text{wgt } c[I] + \text{wgt } c[J] = \text{wgt } c > 0$. The punctured pieces $c[I]$ and $c[J]$ contribute to the distances d'_Z and d''_Z , respectively, only if the corresponding vectors are nontrivial. Let $d(c)$ equal infinity if c is trivial in Q and $\text{wgt } c \geq 1$ otherwise; define similar functions $d'(c)$ and $d''(c)$ for vectors corresponding to undetectable errors in Q' and Q'' , respectively. Then $d'_Z \leq \min_{c \in Q_Z} d'(c[I])$ and $d''_Z \leq \min_{c \in Q_Z} d''(c[J])$. The stated result is obtained by minimizing the inequality $\min(d'(c[I]), d''(c[J])) \leq d(c)$ over all $c \in Q_Z$. ■

The additional condition is needed to exclude the case where a codeword in Q becomes trivial both in Q' and in Q'' after puncturing. For example, consider a qubit code $[[4, 1, 2]]_2$ with

$$H_X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad H_Z = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then, with index sets $I = \{1, 2\}$ and $J = \{3, 4\}$, we get two trivial codes with no logical qubits and infinite distances, $d'_Z = d''_Z = \infty$, larger than d_Z . Indeed, it is easy to check by explicit enumeration that an X codeword of the original code cannot be entirely supported on either I or J . In comparison, with index sets $I = \{1, 2, 3\}$ and $J = \{4\}$, we may take $L_X = (0 \ 1 \ 1 \ 0)$ and $L_Z = (1 \ 1 \ 0 \ 0)$ to satisfy the conditions of the lemma; in this case we get $d'_Z = 2$ and a trivial code with $d''_Z = \infty$.

In our paper Lemma 1 is only used in Sec. IV C to get lower bounds on the homological distance of a product of chain complexes. Here decomposition is only done along the block boundaries; the condition in Lemma 1 can be verified by explicitly constructing bases of the product chain and product cochain complexes and using the Künneth formula to make sure that no vectors are lost.

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