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Phys. Rev. Lett. **110**, 040501 — Published 23 January 2013

DOI: 10.1103/PhysRevLett.110.040501

The classical capacity of quantum thermal noise channels to within 1.45 bits

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We find a tight upper bound for the classical capacity of quantum thermal noise channels that is within $1/\ln 2$ bits of Holevo's lower bound. This lower bound is achievable using unentangled, classical signal states, namely displaced coherent states. Thus, we find that while quantum tricks might offer benefits, when it comes to classical communication they can only help a bit.

Thermal noise affects almost all communication systems. Even optical systems, where thermal photons are very unlikely at room temperature, are effectively subjected to thermal noise by the noise inherent in amplification [1]. The Additive White Gaussian Noise (AWGN) channel describes classical systems subjected to thermal noise. The communication capacity of this channel, found by Shannon, is a central tool in classical information theory [2]. Finding the capacity of thermal noise channels with quantum effects taken into account has long been recognized as an important question [3]. The central issue is whether using special signal states, e.g., entangled or non-classical states, can boost capacity beyond strategies involving only unentangled classical states. Holevo has computed an achievable rate for thermal channels using displaced coherent states [4]. The purpose of this note is to show that the ultimate capacity of thermal channels differs from Holevo's rate by no more than $1/\ln 2 \approx 1.45$ bits.

The input to a thermal noise channel is a bosonic mode with field quadratures (Q, P). This input interacts with a thermal state, resulting in the channel's output. We can model this interaction as a beam splitter with transmissivity λ, so that letting (q, p) be the thermal environment's quadratures, the output's quadratures are $(\sqrt{\lambda}Q + \sqrt{1-\lambda}q, \sqrt{\lambda}P + \sqrt{1-\lambda}p)$. When the thermal state has mean photon number N_E, we denote the channel by $\mathcal{E}_{\lambda,N_E}$ (see [5] for more on gaussian channels).

The classical capacity of a channel is the maximum rate at which information can be transmitted from sender to receiver with errors vanishing in the limit of many uses. It is measured in bits per channel use. Typically, there is a mean photon number (or power) constraint on the signal states used. Holevo has shown, by a random coding argument, that the capacity of a thermal noise channel with transmissivity λ, environment photon number N_E with signal photon-number constraint N satisfies

$$C_N(\mathcal{E}_{\lambda,N_E}) \geq (g(\lambda N + (1-\lambda)N_E) - g((1-\lambda)N_E))\frac{1}{\ln 2}, \tag{1}$$

where $g(x) = (x+1)\ln(x+1) - x\ln x$. This means that there exist good communication schemes with rates approaching the right hand side. Indeed, Holevo's coding scheme is remarkably simple and requires only displaced coherent states as signals [6].

In contrast to classical information theory [2, 7], for most quantum channels we do not know a simple expression for classical capacity. This is because of the superadditivity of Holevo information [8] and intimately related to the potential of using entangled signal states to boost capacity. There are, however, a few channels for which classical capacity can be evaluated [9–11]. The pure loss channel, $\mathcal{E}_{\lambda,0}$, the thermal noise channel with zero environment photon-number, is one such example, with capacity given by $C_N(\mathcal{E}_{\lambda,0}) = g(\lambda N)/\ln 2$ [12].

We do not know the classical capacity of the general thermal noise channel, $\mathcal{E}_{\lambda,N_E}$. There are, however, some upper and lower bounds known. Two upper bounds are the entanglement assisted capacity [13] and maximum output entropy, both computed in [6]. The gap between both of these bounds and the lower bound, Eq. (1), can be arbitrarily large since it grows with N [6]. Very recently, we have found stronger bounds based on the quantum entropy power inequality [14, 15]. For $\lambda = \frac{1}{2}$ these bounds differ from Eq. (1) by at most 0.06 bits, but for $\lambda \neq \frac{1}{2}$ they are looser. There has been some hope that the capacity is simply Eq. (1), a possibility explored in [16], but no proof has been found. Because that work focused primarily on the single-letter minimum output entropy [17], unfortunately it does not lead to bounds on the capacity.

The capacity satisfies a pipelining property, $C_N(\mathcal{E}_1 \circ \mathcal{E}_2) \leq C_N(\mathcal{E}_2)$, where $\mathcal{E}_1 \circ \mathcal{E}_2$ is the concatenation of $\mathcal{E}_1$ and $\mathcal{E}_2$. This can be seen operationally by noting that one potential strategy for communicating via $\mathcal{E}_2$ is for the receiver to immediately apply $\mathcal{E}_1$, so that any rate achievable over the concatenated channel is also achievable over $\mathcal{E}_2$ alone using the same signal states. This property leads to the method of additive extensions [18], where one finds upper bounds for the capacity of a channel $\mathcal{E}$ by decomposing it as $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, where $\mathcal{E}_2$ has an easily computed capacity. We apply this to the thermal noise channel, which can be decomposed as

$$\mathcal{E}_{\lambda,N_E} = \mathcal{A}_G \circ \mathcal{E}_{\tilde{\lambda},0} \ , \tag{2}$$

where $\mathcal{A}_G$ is an amplification channel with gain $G = (1-\lambda)N_E + 1$, and $\mathcal{E}_{\tilde{\lambda},0}$ is the pure loss channel with transmissivity $\tilde{\lambda} = \lambda/G$ [21]. Because $C_N(\mathcal{E}_{\tilde{\lambda},0}) = g(\tilde{\lambda}N)/\ln 2$ is known [12], this leads to the bound

$$C_N(\mathcal{E}_{\lambda,N_E}) \leq g\left(\frac{\lambda N}{(1-\lambda)N_E+1}\right)\cdot\frac{1}{\ln 2} \ . \tag{3}$$

The upper bound in Eq. (3) is remarkably tight. Indeed, we find the following theorem.

Theorem 1. *Let $\mathcal{E}_{\lambda,N_E}$ be the thermal noise channel with transmissivity λ and environment photon number N_E. Consider its classical capacity $C_N(\mathcal{E}_{\lambda,N_E})$ subject to the signal photon-number constraint N. Let*

$$\gamma(\lambda,N_E,N) := \left(g(\lambda N + (1-\lambda)N_E) - g((1-\lambda)N_E)\right)\frac{1}{\ln 2}$$

denote the rate achievable by coding with displaced coherent states. Then

$$\gamma(\lambda,N_E,N) \leq C_N(\mathcal{E}_{\lambda,N_E}) \leq \gamma(\lambda,N_E,N) + 1/\ln 2 \ .$$

To prove this theorem, we show that the difference between the upper bound Eq. (3) and Holevo's lower bound (1) does not exceed $1/\ln 2$. This is an immediate consequence of Eq. (4) of the following lemma, applied with $X = \lambda N$ and $Y = (1-\lambda)N_E$. In fact, get the slightly stronger result that the gap is no more than $(1-\lambda)N_E \ln\left(1 + ((1-\lambda)N_E)^{-1}\right)/\ln 2$.

Lemma 1. *Let $g(x) = (x+1)\ln(x+1) - x\ln x$. For $Y > 0$, define the function*

$$\Delta_Y(X) = g(X(Y+1)^{-1}) - g(X+Y) + g(Y) \ .$$

Then

(i) $\lim_{X\to\infty}\Delta_Y(X) = Y\ln(1+Y^{-1}) < 1$.

(ii) $\Delta'_Y(X) > 0$ *for all* $X > 0$.

In particular,

$$\Delta_Y(X) < Y\ln(1+Y^{-1}) < 1 \qquad \textit{for all } X, Y > 0 \ . \tag{4}$$

Proof. Using $g(x) = \ln(x+1) + 1 + O(1/x)$, one immediately gets

$$\begin{aligned}\Delta_Y(X) &= \ln\left(X(Y+1)^{-1}+1\right) - \ln(X+Y+1) + g(Y) + O(1/X)\\ &\to -\ln(Y+1) + g(Y) = Y\ln(1+Y^{-1}) \qquad \text{for } X\to\infty \ .\end{aligned}$$

Statement (i) then follows from the fact that $\ln(1+\epsilon) \leq |\epsilon|$. Similarly, with $g'(x) = \ln(1+x^{-1})$, we obtain

$$\Delta'_Y(X) = (Y+1)^{-1}\ln(1+(Y+1)X^{-1}) - \ln(1+(X+Y)^{-1}) \to 0 \qquad \text{for } X\to\infty \ . \tag{5}$$

Finally, we compute the second derivative of $\Delta_Y(\cdot)$ using $g''(x) = -(x(x+1))^{-1}$. Simple algebra gives

$$\Delta''_Y(X) = \frac{1}{X+Y+1}\left(\frac{1}{X+Y} - \frac{1}{X}\right) < 0 \qquad \text{for all } X > 0 \ ,$$

which shows that $\Delta'_Y(\cdot)$ is decreasing. With Eq. (5), this implies (ii). □

Note added. After appearance of our work on the arXiv, Giovanetti et al. [20] obtained new bounds on the classical capacity of bosonic Gaussian channels by generalizing the results of [12] to the multi-mode case.

Acknowledgments— We are grateful to Mark Wilde for comments on the manuscript. We were both supported by DARPA QUEST program under contract no.HR0011-09-C-0047.

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