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Detecting Incapacity

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Using unreliable or noisy components for reliable communication requires error correction. But which noise processes can support information transmission, and which are too destructive? For classical systems any channel whose output depends on its input has the capacity for communication, but the situation is substantially more complicated in the quantum setting. We find a generic test for incapacity based on any suitable forbidden transformation—a protocol for communication with a channel passing our test would also allow one to implement the associated forbidden transformation. Our approach includes both known quantum incapacity tests—positive partial transposition (PPT) and antidegradability (no cloning)—as special cases, putting them both on the same footing.

Understanding how to use communication resources optimally is a major part of information theory [1, 2]. Usually, a sender and receiver have access to repeated uses of a noisy communication channel and want to use error correction to introduce redundancy into their messages, making them less susceptible to noise. The capacity of a channel, measured in bits per channel use, is the best rate that can be achieved with vanishing probability of transmission error in the limit of many channel uses.

Quantum information theory expands the notions of both information and noise to include quantum effects [3]. Indeed, a quantum channel has many different capacities, depending on what kind of information is to be transmitted, and what other resources are available. For example, the classical capacity of a quantum channel is the best rate for transmitting classical information, the private capacity relates to quantum cryptography, and the quantum capacity is the best rate for coherent communication of quantum systems. This last, which establishes the fundamental limits on quantum error correcting codes, will be our main concern here.

In contrast to the simple capacity formula of classical information theory, finding the quantum capacity of a quantum channel seems intractable in general. The best known formula for the quantum capacity [4–6] of a channel involves a maximization over states on tensor products of an arbitrary number of uses of the channel. No bound on the size of this maximization is yet known. Even the *classical* capacity of a quantum channel has this problem [7].

We will consider the simpler incapacity question: which channels have nonzero quantum capacity, and which don't? This is still very hard to answer, but there are two known criteria for showing a channel has no capacity. The first, a mathematical condition called the positive partial transpose (PPT) criterion, is due to [8, 9]. The second, due to [10, 11], gives a condition for when quantum capacity would imply a violation of the no-cloning principle [12]. Adding to the complexity of the zero-quantum-capacity channels is the existence of the superactivation phenomenon, where two zero-capacity

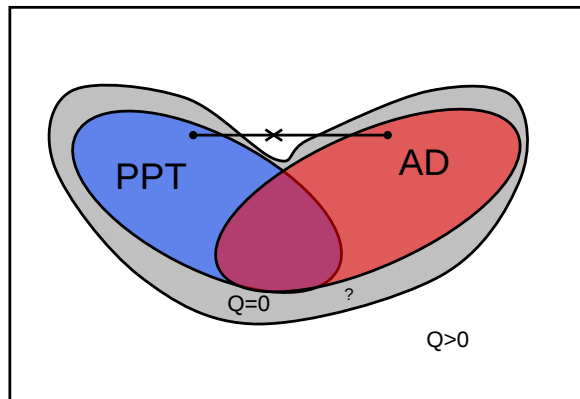


FIG. 1: Zero quantum capacity channels. There are two known incapacity tests, and two associated known sets of zero quantum capacity channels. Antidegradable channels have no capacity because otherwise they could be used to violate no cloning. Positive partial transpose channels satisfy the mathematical criterion that their Choi matrices remain positive under partial transposition. It is possible to find pairs of zero quantum capacity channels, one PPT and the other Antidegradable, that have positive joint quantum capacity. This can be used to show that the set of zero quantum capacity channels is non convex. It is unknown whether there are zero quantum capacity channels that are neither antidegradable nor PPT. In this paper we find that the reason PPT channels have no quantum capacity is that any capacity would allow them to implement the forbidden transpose operation.

channels interact synergistically to jointly generate capacity [13]. A particular consequence of this phenomenon is that the set of zero-capacity quantum channels is not convex. Unfortunately, there is no simple test to determine, given its parameters, whether a channel can transmit quantum information.

We will address the incapacity question from an abstract point of view. Our arguments require only the existence of physical states and maps, as well as the existence of a suitable forbidden (or unphysical) map on the states. One motivation of this work is to understand incapacity of quantum channels, but our approach is sufficiently general to include such things as generalized

probabilistic theories [14] as well as the discrete quantum mechanics of [15]. Our findings will also apply to classical systems with proscribed operations.

Preliminaries— The theories we will consider have minimal structure. We assume a physical state space B and a set of allowed physical operations \mathbf{P} from $B \rightarrow B$ that is closed under composition. For quantum mechanics, B will be the set of density matrices and \mathbf{P} will be the set of trace-preserving completely positive maps. We will also require a nonphysical operation $R : B \rightarrow B$ with $R \notin \mathbf{P}$. This R will need the following crucial property:

Definition 1: (\mathbf{P} -commutation) An unphysical map R is \mathbf{P} -commutative if for every $\mathcal{D} \in \mathbf{P}$ there is a $\mathcal{D}^* \in \mathbf{P}$ such that $R \circ \mathcal{D} = \mathcal{D}^* \circ R$. See Fig. 2.

Note that for any unphysical R there must be a set of states $S \subset B$ such that no $\mathcal{N} \in \mathbf{P}$ has for all $\phi \in S$ $R(\phi) = \mathcal{N}(\phi)$. We say that R is unphysical on S . The following simple lemma will be remarkably useful (see Figure 2).

Lemma 1: If R is unphysical on S and is \mathbf{P} -commutative, then any \mathcal{N} with $R \circ \mathcal{N} \in \mathbf{P}$ cannot reliably transmit the states in S .

Proof: Suppose there were physical encoding and decoding operations that allowed transmission of states in S . In other words, there are physical \mathcal{E} and \mathcal{D} such that

$$\psi = (\mathcal{D} \circ \mathcal{N} \circ \mathcal{E})(\psi) \quad \forall \psi \in S. \quad (1)$$

Then

$$R(\psi) = R \circ \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\psi) = \mathcal{D}^* \circ R \circ \mathcal{N} \circ \mathcal{E}(\psi) \quad (2)$$

which, recalling that $\mathcal{M} = R \circ \mathcal{N}$ is physical, gives a prescription for physically implementing R on S as $\mathcal{D}^* \circ \mathcal{M} \circ \mathcal{E}$, which is impossible by hypothesis. \square

Note that this argument implies that if R is continuous, and there is no physical operation that approximates R to high precision on S then \mathcal{N} cannot transmit the states in S to high precision.

Application to quantum mechanics— The rest of the paper is dedicated to extending and exploring the consequences of this line of reasoning, focusing on its application to quantum mechanics. We will show all of the following:

- The nonphysicality of the transpose map T , often thought of as time-reversal, implies that PPT channels have no quantum capacity.
- T is the *only* linear positive map that can be used with Lemma 1 to find channels with zero quantum capacity.
- The nonphysicality of cloning quantum states (*a nonlinear map*) implies that antidegradable channels have no quantum capacity.

While a quantum channel \mathcal{N} implicitly acts only on some particular part of B of dimension d , our unphysical

maps will be defined for any input dimension. So, the unphysical map can be thought of as a family of maps acting on different input dimensions. To streamline notation we will suppress labels indicating this dimension whenever possible.

Our usual notion of capacity is defined over many uses of a channel, $\mathcal{N}^{\otimes n}$, and we will also need to consider nonphysical maps acting on d^n , which we call $R^{(n)}$. The parentheses indicate that this map may not be a tensor product of maps R on the individual systems, but can act jointly on all n systems. For linear maps, the tensor product is well defined and in this case we will typically take $R^{(n)} = R^{\otimes n}$.

We are now ready to consider a simple example of Lemma 1 applied to the time-reversal, or transpose, operation T . Recall that the transpose operation preserves inner products between states, but is anti-unitary rather than unitary. It is also the canonical example of a positive map that is not completely positive [21]. As such, the transpose maps states to states, but is not a physically implementable operation. In fact, letting S be the set of all qubit states, transpose is unphysical on S . Now, given a channel $\mathcal{D}(\rho) = \sum_i A_i \rho A_i^\dagger$ letting $\mathcal{D}^*(\rho) = \sum_i A_i^* \rho (A_i^*)^\dagger$ (with $*$ denoting complex conjugation), it is easy to check that \mathcal{D}^* is also a channel and furthermore $T \circ \mathcal{D} = \mathcal{D}^* \circ T$. Having demonstrated \mathbf{P} -commutativity, we can now apply Lemma 1. In fact, it is easy to show that if $T \circ \mathcal{N}$ is a quantum channel, so is $T \circ \mathcal{N}^{\otimes n}$ for any n , so we see from Lemma 1 that any channel with $T \circ \mathcal{N}$ physical must have zero capacity.

Translating this to a statement about the channel's Choi matrix [22], we find that any channel \mathcal{N} such that $(I \otimes T)((I \otimes \mathcal{N})(|\phi_d\rangle\langle\phi_d|)) \geq 0$ (in other words a PPT Choi matrix) must have zero quantum capacity. The *reason* is that any capacity for such a channel would lead to a protocol for implementing the (unphysical) transpose operation. A similar argument shows that any protocol for distilling pure entanglement from a PPT state could also be used to implement the transpose (this is shown in the appendix).

There being no reason to restrict to the transpose operation, we can easily see that given any continuous \mathbf{P} -commuting $R^{(n)}$ which cannot be approximated by a physical map on qubit states, then any \mathcal{N} with $R^{(n)} \circ \mathcal{N}^{\otimes n} \in \mathbf{P}$ has zero quantum capacity.

Positive linear maps— If we take R to be a linear map that preserves system dimension, it is possible to completely characterize the zero-capacity channels detected by our method. This is achieved through the following theorem originally communicated to us by Choi [16]. We present here an alternative proof that makes an explicit connection to the theory of group representations.

Theorem 1: Take states and channels to be the usual quantum mechanics with d -level systems. If R is linear, invertible, preserves system dimension and trace, and is \mathbf{P} -commutative it is either of the form $R(\rho) = (1-p)\rho^T +$

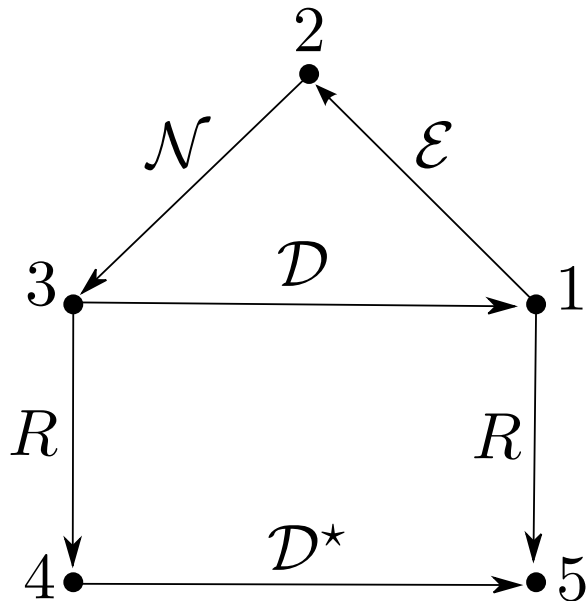


FIG. 2: Sketch of Lemma 1: The assumptions that go into the lemma are illustrated by the connections on this graph. \mathbf{P} -commutation is represented by $3 \rightarrow 1 \rightarrow 5$ leading to the same result as $3 \rightarrow 4 \rightarrow 5$. The existence of encoder and decoder that transmit states successfully though \mathcal{N} is indicated by the path $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ which is the identity. We also assume that the path $R \circ \mathcal{N}$ or $2 \rightarrow 3 \rightarrow 4$ is physical even though R is not. Then the nonphysical operation R ($1 \rightarrow 5$) is implemented by the physical path $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$.

pI/d or $R(\rho) = (1-p)\rho + pI/d$.

Proof: Let R be such a map, and consider the requirements of \mathbf{P} -commutation. Defining $\mathcal{N}_U(\rho) = U\rho U^\dagger$, we have $\mathcal{N}_U^* = R \circ \mathcal{N}_U \circ R^{-1}$ and see immediately that since \mathcal{N}_U is invertible, so is \mathcal{N}_U^* . Since it preserves dimension and is a physical channel, \mathcal{N}_U^* must be simply conjugation by a unitary V_U . Furthermore, $\mathcal{N}_{U_1 U_2}^* = R \circ \mathcal{N}_{U_1 U_2}^* \circ R^{-1} = R \circ \mathcal{N}_{U_1}^* \circ R^{-1} \circ R \circ \mathcal{N}_{U_2}^* \circ R^{-1} = \mathcal{N}_{U_1}^* \circ \mathcal{N}_{U_2}^*$, so that V_U must be a d -dimensional representation of $\mathbb{U}(d)$. Since $\mathcal{N}_U = R^{-1} \circ \mathcal{N}_U^* \circ R$, this representation must be faithful (*i.e.*, invertible) up to an overall phase which will cancel under conjugation. There are only two such representations: the fundamental and complex conjugate representations.

Suppose we have $\mathcal{N}_U^* = \mathcal{N}_U$. Then, for all ρ , $U^\dagger R(U\rho U^\dagger)U = R(\rho)$, so that letting $|\phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle|i\rangle$ and $|\phi_d\rangle\langle\phi_d|^\Gamma$ denote its partial transpose, we have

$$U^\dagger \otimes U^\dagger (I \otimes R)(|\phi_d\rangle\langle\phi_d|^\Gamma)U \otimes U = (I \otimes R)(|\phi_d\rangle\langle\phi_d|^\Gamma).$$

As a result, we have

$$(I \otimes R)(|\phi_d\rangle\langle\phi_d|^\Gamma) = aI + bF \text{ where } F = \sum_{i,j} |i\rangle\langle j|j\rangle\langle i|$$

so that $(I \otimes R)(|\phi_d\rangle\langle\phi_d|) = aI + db|\phi_d\rangle\langle\phi_d|$. Requiring R to preserve trace, we thus find $R(\rho) = pI/d + (1-p)\rho$.

We now consider $\mathcal{N}_U^* = \mathcal{N}_{U^*}$, which works similarly. Specifically, we now have $(I \otimes R)(|\phi_d\rangle\langle\phi_d|) = aI + bF$ so that, requiring R to preserve trace, we find $R(\rho) = pI/d + (1-p)\rho^T$. \square

Decoder-dependent R and no cloning—The above theorem shows that the only nonphysical \mathbf{P} -commutative linear maps are of the form $R(\rho) = (1-p)\rho^T + pI/d$. In order to make statements about capacities, $R^{\otimes n}$ must also be \mathbf{P} -commutative. The only such nonphysical map is $R(\rho) = \rho^T$. As a result detecting incapacity in non-PPT channels using this method requires R to be nonlinear.

In order to accommodate nonlinear maps, we must generalize our ideas about \mathbf{P} -commutation. Specifically, we introduce the following definition of \mathbf{P} -commutation for a class of unphysical operations.

Definition 2: (\mathbf{P} -commutation) An unphysical map $R \notin \mathbf{P}$ together with a set of unphysical maps $\{R_{\mathcal{D}}\}_{\mathcal{D} \in \mathbf{P}}$ with $R_{\mathcal{D}} \notin \mathbf{P}$ is \mathbf{P} -commutative if for all maps $\mathcal{D} \in \mathbf{P}$ there is a $\mathcal{D}^* \in \mathbf{P}$ with $R_{\mathcal{D}} \circ \mathcal{D} = \mathcal{D}^* \circ R$.

In Eq. (2), we were able to commute R and \mathcal{D} by replacing \mathcal{D} with \mathcal{D}^* . The motivation behind our new definition is to allow more freedom in finding decoders that remain physical after commutation with R , at the cost of having channel-dependent nonphysical maps $R_{\mathcal{D}}$. Note that if R is invertible, $\mathcal{D}^* = R_{\mathcal{D}} \circ \mathcal{D} \circ R^{-1}$ will define \star .

Now, given a \mathbf{P} -commuting family of maps $\{R_{\mathcal{D}}\}$ that are unphysical on a qubit with associated \star and R , any channel \mathcal{N} for which $R \circ \mathcal{N}$ is physical cannot transmit all states in some set S reliably. Within quantum mechanics, given $R_{\mathcal{D}}^{(n)}$, $R^{(n)}$, and \star then if $R^{(n)} \circ \mathcal{N}^{\otimes n}$ is physical, \mathcal{N} has no quantum capacity.

In quantum mechanics, any decoder can be written in terms of a unitary U as $\mathcal{D}(\psi) = \text{Tr}_E U(\psi \otimes |0\rangle\langle 0|_E)U^\dagger \approx \psi$ which implies $U(\psi \otimes |0\rangle\langle 0|_E)U^\dagger \approx \psi \otimes \sigma$ for some σ independent of ψ . We therefore only need consider the simpler set of nonphysical maps R_U with $U^* R(\psi \otimes |0\rangle\langle 0|_E)(U^*)^\dagger = R_U(U\psi \otimes |0\rangle\langle 0|_E)U^\dagger$ defining \star to detect the incapacity of any \mathcal{N} with $R \circ (\mathcal{N} \otimes |0\rangle\langle 0|_E)$ physical.

Every quantum channel has an isometric extension to an environment. An antidegradable channel is a channel for which the environment, by further processing, can simulate the original channel [17, 18]. As a result of the no cloning theorem, such channels can be shown to have zero quantum capacity [10, 11]. We now use the non-physical cloning operation to give a simple proof that antidegradable channels have zero quantum capacity.

If \mathcal{M} is antidegradable with input A and output B , then there is an extension of \mathcal{M} , \mathcal{M}_{12} from A to $B_1 B_2$, such that for all ρ , $\text{Tr}_{B_2} \mathcal{M}_{12}(\rho) = \text{Tr}_{B_1} \mathcal{M}_{12}(\rho) = \mathcal{M}(\rho)$. Any ψ on B has a unique decomposition $\psi = \tilde{\psi} + \sigma$, with $\tilde{\psi}$ in the range of \mathcal{M} and σ in its orthogonal complement. Now define $\tilde{R}(\psi) = \mathcal{M}_{12} \circ \mathcal{M}^{-1}(\psi) + \sigma \otimes \sigma$ where the pseudo-inverse \mathcal{M}^{-1} maps $\tilde{\psi}$ to its unique preimage in the orthogonal complement of the kernel of \mathcal{M} . This \tilde{R} is continuous and $\text{Tr}_1 \tilde{R}(\psi) = \text{Tr}_2 \tilde{R}(\psi) = \psi$.

Furthermore, $\tilde{R} \circ \mathcal{M} = \mathcal{M}_{12}$ is physical. We can similarly extend \tilde{R} to a continuous R from BE to $B_1 B_2 E_1 E_2$ with $R \circ (\mathcal{M} \otimes |0\rangle\langle 0|_E) = \mathcal{M}_{12} \otimes |0\rangle\langle 0|_{E_1} \otimes |0\rangle\langle 0|_{E_2}$ and $\text{Tr}_1 R(\psi) = \text{Tr}_2 R(\psi) = \psi$.

Now, let $U^* = U \otimes U$. Choosing $R_U(\rho_{BE}) = U \otimes UR(U^\dagger \rho_{BE} U)(U^\dagger \otimes U^\dagger)$ and $\mathcal{N}_U(\rho) = U \rho U^\dagger$, we have $R_U \circ \mathcal{N}_U = \mathcal{N}_{U^*} \circ R$ and $R \circ (\mathcal{M} \otimes |0\rangle\langle 0|_E) = \mathcal{M}_{12} \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$. $\text{Tr}_1 R_U(\psi \otimes |0\rangle\langle 0|) = \text{Tr}_2 R_U(\psi \otimes |0\rangle\langle 0|) = \psi \otimes |0\rangle\langle 0|$, so R_U can clone an arbitrary state and is therefore unphysical.

Having demonstrated a nonphysical R_U , a \star and physical $R \circ (\mathcal{M} \otimes |0\rangle\langle 0|_E)$, we have so far shown that a single use of an anti-degradable channel can't be used to transmit a quantum state with high fidelity. However, since the tensor product of two anti-degradable channels is again anti-degradable, this also shows that many copies cannot transmit quantum information either. As a result, the capacity must be zero.

Discussion— We have presented a general approach for detecting the incapacity for quantum communication using unphysical transformations, and shown that both known incapacity tests fall into this framework. Furthermore we have discovered a connection to the theory of representations of the unitary group, and both positive partial transposition and antidegradability correspond to simple representations. This paves the way for the discovery of new incapacity tests, and Theorem 1 suggests a fruitful direction, to seek forbidden transformations that are not linear. This is no simple task: Positive maps (even linear ones!) are notoriously difficult to contrive and we have been unable, as of yet, to find any new channels with zero quantum capacity. Perhaps there aren't any.

We have focused primarily on the standard (one-way) quantum capacity, but these ideas can also be extended to the two-way capacity and questions of entanglement distillation. For example, in the appendix we demonstrate non-distillability of PPT states by showing that any successful distillation protocol could be used to implement the unphysical transpose operation. Our argument there makes crucial use of the linearity of transposition, which in light of Theorem 1 severely restricts the detection of two-way capacity and distillability. Finding an argument relating two-way capacities and nonlinear forbidden transformations is an important challenge.

We end on a speculative note. We have shown that both known incapacity tests are derived from fundamentally unphysical transformations on state space: transpose and cloning. Could it be that *any* zero quantum capacity channel has such a “reason” for its incapacity? Formally, of course, the answer is “yes”—the forbidden transformation could just be a successful encoding or decoding operation for the channel. However, we would be much more satisfied with something less tautological. A good starting point might be to identify a minimal set of primitive forbidden operations that includes cloning and

the transpose as examples.

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- [20] M. Choi, *Linear Algebra and Its Applications* pp. 285–290 (1975).
- [21] Recall that a positive map γ has the property that $\gamma(\rho) \geq 0$, for all $\rho \geq 0$ but not necessarily $(\mathbb{I} \otimes \gamma)(\rho') \geq 0$ for some $\rho' \geq 0$ on a larger space. A completely positive map η has $(\mathbb{I} \otimes \eta)(\rho') \geq 0$ for all $\rho' \geq 0$ [20]
- [22] The Choi matrix of a channel \mathcal{N} completely characterizes the channel and is defined as $(I \otimes \mathcal{N})(|\phi_d\rangle\langle\phi_d|)$, where $|\phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle|i\rangle$.

Appendix: Two-way capacities and distillable entanglement— We now argue that if the Choi matrix of a channel is PPT, it cannot be distillable [19] via local op-

eration and classical communication (LOCC). Since the Choi matrix of \mathcal{N} is PPT iff $T \circ \mathcal{N}$ is a physical channel, showing this will also prove that such a channel has no Q_2 , or quantum capacity assisted by LOCC.

By teleporting through the Choi matrix of a channel \mathcal{N} , an LOCC protocol with Kraus operators $A_i \otimes B_i$ can be used to prepare the state

$$\frac{d_{\text{out}}}{d_{\text{in}}} \sum_i \frac{1}{d_{\text{in}}^2} \sum_u B_i (\mathcal{N}(A_i^T \sigma_u \psi \sigma_u^\dagger A_i^*)) B_i^\dagger \otimes |u\rangle\langle u|. \quad (3)$$

where the σ_u s are generalized Pauli matrices.

Furthermore, if the Choi matrix can be distilled to a maximally entangled state, there is an LOCC protocol such that

$$\psi = \frac{2}{d} \sum_i B_i \mathcal{N}(A_i^T \psi A_i^*) B_i^\dagger \quad (4)$$

for an arbitrary qubit state ψ . Letting $T(\rho) = \rho^T$, this

implies that

$$T(\psi) = \frac{2}{d} \sum_i B_i^* T \circ \mathcal{N}(A_i^T \psi A_i^*) B_i^T, \quad (5)$$

and

$$T(\sigma_u \psi \sigma_u^\dagger) = \frac{2}{d} \sum_i B_i^* T \circ \mathcal{N}(A_i^T \sigma_u \psi \sigma_u^\dagger A_i^*) B_i^T. \quad (6)$$

This latter can be used to show that

$$T(\psi) = \frac{2}{d} \sum_i \sigma_u^* B_i^* T \circ \mathcal{N}(A_i^T \sigma_u \psi \sigma_u^\dagger A_i^*) B_i^T \sigma_u^T \quad (7)$$

the right hand of which, using the fact that $T \circ \mathcal{N}$ is physical combined with Eq. (3), gives a recipe for physically implementing T using the LOCC operation with Kraus operators $A_i \otimes B_i^*$. As a result, the Choi matrix of \mathcal{N} must not be distillable.