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Laura Bernard,[1,*] Luc Blanchet,[1,†] Alejandro Bohé,[2,‡] Guillaume Faye,[1,§] and Sylvain Marsat[3,4,¶]

[1] *$\mathcal{GR}\varepsilon\mathbb{C}\mathcal{O}$, Institut d'Astrophysique de Paris, UMR 7095, CNRS, Sorbonne Universités & UPMC Univ Paris 6, 98bis boulevard Arago, 75014 Paris, France*

[2] *Albert Einstein Institut, Am Muehlenberg 1, 14476 Potsdam-Golm, Germany*

[3] *Maryland Center for Fundamental Physics & Joint Space-Science Center, Department of Physics, University of Maryland, College Park, MD 20742, USA*

[4] *Gravitational Astrophysics Laboratory, NASA Goddard Space Flight Center, Greenbelt, MD 20771, USA*

Abstract

The Fokker action governing the motion of compact binary systems without spins is derived in harmonic coordinates at the fourth post-Newtonian approximation (4PN) of general relativity. Dimensional regularization is used for treating the local UV divergences associated with point particles, followed by a renormalization of the poles into a redefinition of the trajectories of the point masses. Effects at the 4PN order associated with wave tails propagating at infinity are included consistently at the level of the action. A finite part procedure based on analytic continuation deals with the IR divergencies at spatial infinity, which are shown to be fully consistent with the presence of near zone tails. Our end result at 4PN order is Lorentz invariant and has the correct self-force limit for the energy of circular orbits. However, we find that it differs from the recently published result derived within the ADM Hamiltonian formulation of general relativity [T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **89**, 064058 (2014)]. More work is needed to understand this discrepancy.

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*Electronic address: bernard@iap.fr

†Electronic address: blanchet@iap.fr

[‡]Electronic address: alejandro.bohe@aei.mpg.de
[§]Electronic address: faye@iap.fr
[¶]Electronic address: sylvain.marsat@aei.mpg.de

I. INTRODUCTION

Gravitational waves emitted by inspiraling and merging compact (neutron stars and/or black holes) binary systems are likely to be routinely detected by the ground-based network of advanced laser interferometric detectors [1]. Banks of extremely accurate replica of theoretical predictions (templates) are a compulsory ingredient of a successful data analysis for these detectors — both on-line and off-line. In the early inspiral phase the post-Newtonian (PN) approximation of general relativity should be pushed to extremely high order [2]. Furthermore high accuracy comparison and matching of PN results are performed with numerical relativity computations appropriate for the final merger and ringdown phases [3].

With these motivations in mind we tackle the problem of the equations of motion of compact binaries (without spin) at the fourth post-Newtonian (4PN) order.[1] Solving this problem is also important for various applications (numerical/analytical self-force comparisons, last stable circular orbit, effective-one-body calculations [4, 5]) and paves the way to the problem of radiation and orbital phase evolution at the 4PN order beyond the Einstein quadrupole formalism — whose solution is needed for building 4PN accurate templates.

Historical works on the PN equations of motion of compact binaries include Lorentz & Droste [6], Einstein, Infeld & Hoffmann [7], Fock [8, 9], Chandrasekhar and collaborators [10–12], as well as Otha *et al.* [13–15]. These works culminated in the 1980s with the derivation of the equations of motion up to 2.5PN order, where radiation reaction effects appear [16–18] (see also [19–24] for alternative derivations), and led to the successful analysis of the time of arrival of the radio pulses from the Hulse-Taylor binary pulsar [25, 26].

In the early 2000s the equations of motion were derived at the 3PN order using different methods: the ADM Hamiltonian formalism of general relativity [27–32], the PN iteration of the equations of motion in harmonic coordinates [33–38], some surface-integral method [39–43], and effective field theory schemes [44]. Furthermore, radiation reaction effects at 3.5PN order were added [45–49], and spin contributions have been extensively investigated [50–60].

Works in the early 2010s partially obtained the equations of motion at the 4PN order using the ADM Hamiltonian formalism [61–63] and the effective field theory [64]. More recently, the important effect of gravitational wave tails at 4PN order [65, 66] was included in the

[1] As usual the nPN order refers to the terms of order $1/c^{2n}$ in the equations of motion beyond the Newtonian acceleration.

ADM Hamiltonian. This permitted to understand the IR divergencies in this calculation and to complete the 4PN dynamics [67] (see also [68]). Notice however that the latter work [67] did not perform a full consistent PN analysis but resorted to an auxiliary self-force calculation [69–71] to fix a last coefficient.

In the present paper we derive the Fokker Lagrangian [72] at the 4PN order in harmonic coordinates. We combine a dimensional regularization of the UV divergencies associated with point particles with a finite part regularization based on analytic continuation dealing with IR divergencies. We show that the IR divergencies are perfectly consistent with the presence of the tail effect at 4PN order, which is incorporated consistently into the Fokker action. However, like in [67], we are obliged to introduce an arbitrary coefficient relating the IR cut-off scale to the *a priori* different scale present in the tail integral. This coefficient is determined by using a self-force calculation (both numerical [69, 70] and analytical [71]), so that our end result for the energy of circular orbits at 4PN order has the correct self-force limit. We also checked that it is manifestly Lorentz-Poincaré invariant. In a companion paper [73] we shall study the conserved integrals of the motion, the reduction to the center-of-mass frame and the dynamics of quasi-circular orbits.

Up to quadratic order in Newton's constant, our Lagrangian is equivalent to the Lagrangian derived by means of effective field theory techniques [64]. However, trying to relate our result to the result obtained from the ADM Hamiltonian approach [61–63, 67], we find a difference with the latter works, occuring at orders G^4 and G^5 in the Hamiltonian. Part of the difference is due to the fact that we disagree with the treatment of the tail part of the Hamiltonian for circular orbits in Ref. [67]. However, even when using our own treatment of tails in their results, there still remains a discrepancy with the works [61–63, 67] that we cannot resolve. More work is needed to understand the origin of this remaining difference and resolve it.

In Sec. II we show how to use the Fokker action in the context of PN approximations. In particular we split the action into a term depending on the PN field in the near zone and a term depending on the field in the far zone. The latter term is crucial to control the tails which are then computed consistently in the action at 4PN order in Sec. III. We explain our method for iterating the PN approximation of the Fokker action in Sec. IV. In Sec. V, we present the full-fledged Lagrangian of compact binaries at the 4PN order, both in harmonic and ADM like coordinates, and we compare with the results [67] obtained for the 4PN ADM

Hamiltonian in Sec. V C. Finally we explain in Sec. V D our disagreement with Ref. [67] regarding the treatment of the tail term. The paper ends with several technical Appendices.

II. THE FOKKER ACTION

A. General statements

We consider the complete Einstein-Hilbert gravitation-plus-matter action $S = S_g + S_m$, where the gravitational piece S_g takes the Landau-Lifshitz form with the usual harmonic gauge-fixing term,[2]

$$S_g = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left[g^{\mu\nu}\left(\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}-\Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda}\right)-\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu\right], \tag{2.1}$$

with $\Gamma^\mu \equiv g^{\rho\sigma}\Gamma^\mu_{\rho\sigma}$, and where S_m denotes the matter piece appropriate for two point particles ($A = 1, 2$) without spin nor internal structure,

$$S_m = -\sum_A m_A c^2 \int \mathrm{d}t \sqrt{-(g_{\mu\nu})_A\, v^\mu_A v^\nu_A/c^2}\,. \tag{2.2}$$

Here m_A is the mass of the particles, $v^\mu_A = \mathrm{d}y^\mu_A/\mathrm{d}t = (c, \boldsymbol{v}_A)$ is the usual coordinate velocity, $y^\mu_A = (ct, \boldsymbol{y}_A)$ the usual trajectory, and $(g_{\mu\nu})_A$ stands for the metric evaluated at the location of the particle A following the dimensional regularization scheme.

A closed-form expression for the gravitational action can be written with the help of the gothic metric $\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and its inverse $\mathfrak{g}_{\mu\nu} = g_{\mu\nu}/\sqrt{-g}$ as

$$\begin{aligned} S_g = \frac{c^3}{32\pi G}\int \mathrm{d}^4x \Bigg[&-\frac{1}{2}\left(\mathfrak{g}_{\mu\rho}\mathfrak{g}_{\nu\sigma}-\frac{1}{2}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\rho\sigma}\right)\mathfrak{g}^{\lambda\tau}\partial_\lambda\mathfrak{g}^{\mu\nu}\partial_\tau\mathfrak{g}^{\rho\sigma} \\ &+\mathfrak{g}_{\mu\nu}\Big(\partial_\rho\mathfrak{g}^{\mu\sigma}\partial_\sigma\mathfrak{g}^{\nu\rho}-\partial_\rho\mathfrak{g}^{\mu\rho}\partial_\sigma\mathfrak{g}^{\nu\sigma}\Big)\Bigg]. \end{aligned} \tag{2.3}$$

Expanding around Minkowski space-time we pose $\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ which defines the metric perturbation variable $h^{\mu\nu}$. The action appears then as an infinite non-linear power series in h, where indices on h and on partial derivatives ∂ are lowered and raised with the Minkowski metric $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The Lagrangian density $\mathcal{L}_g$ can take various forms obtained from each others by integrations by parts. For our purpose the best form starts at

[2] We also denote $S_g = \int \mathrm{d}t\, L_g$ and $L_g = \int \mathrm{d}^3\mathbf{x}\,\mathcal{L}_g$. The Lagrangian L_g is defined modulo a total time derivative and the Lagrangian density $\mathcal{L}_g$ modulo a space-time derivative.

quadratic order by terms like $\sim h\Box h$, where $\Box = \eta^{\rho\sigma}\partial^2_{\rho\sigma}$ is the flat d'Alembertian operator. So the general structure of the Lagrangian we shall use is $\mathcal{L}_g \sim h\Box h + h\partial h\partial h + h\,h\partial h\partial h + \cdots$. See (3.1) for the explicit expressions of the quadratic and cubic terms.

The Einstein field equations derived from the harmonic gauge fixed action read

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}\,, \tag{2.4a}$$

$$\tau^{\mu\nu} \equiv |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Sigma^{\mu\nu}[h, \partial h, \partial^2 h]\,. \tag{2.4b}$$

The above quantity $\tau^{\mu\nu}$ denotes the pseudo stress-energy tensor of the matter and gravitational fields, with $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\delta S_m/\delta g_{\mu\nu}$ and with the gravitational source term $\Sigma^{\mu\nu}$, at least quadratic in h and its first and second derivatives, being given by

$$\Sigma^{\mu\nu} = \Lambda^{\mu\nu} - H^\mu H^\nu - H^\rho\partial_\rho h^{\mu\nu} - \frac{1}{2}\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\rho\sigma}H^\rho H^\sigma + 2\mathfrak{g}_{\rho\sigma}\mathfrak{g}^{\lambda(\mu}\partial_\lambda h^{\nu)\rho}H^\sigma\,, \tag{2.5}$$

where $\Lambda^{\mu\nu}$ takes the standard expression valid in harmonic coordinates while the "harmonicity" is defined by $H^\mu \equiv \partial_\nu h^{\mu\nu} = -\sqrt{-g}\,\Gamma^\mu$.[3] As we see, the gravitational source term contains all required harmonicities H^μ, which will not be assumed to be zero in the PN iteration of the field equations (2.4).

The Fokker action is obtained by inserting back into (2.1)–(2.2) an explicit PN iterated solution of the field equations (2.4) given as a functional of the particle's trajectories, *i.e.*, an explicit PN metric $g_{\mu\nu}(\mathbf{x}; \boldsymbol{y}_B(t), \boldsymbol{v}_B(t), \cdots)$. Here the ellipsis indicate extra variables coming from the fact that we solve Eqs. (2.4) including all harmonicity terms and without replacement of accelerations, so that the equations of motion are off-shell at this stage and the solution for the metric depends also on accelerations $\boldsymbol{a}_B(t)$, derivative of accelerations $\boldsymbol{b}_B(t)$, and so on. In particular, the metric in the matter action evaluated at the location of the particle A will be some $(g_{\mu\nu})_A = g_{\mu\nu}(\boldsymbol{y}_A(t); \boldsymbol{y}_B(t), \boldsymbol{v}_B(t), \cdots)$. Thus, the Fokker generalized PN action, depending not only on positions and velocities but also on accelerations and their derivatives, is given by

$$\begin{aligned} S_{\rm F}\left[\boldsymbol{y}_B(t), \boldsymbol{v}_B(t), \cdots\right] &= \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\,\mathcal{L}_g\left[\mathbf{x}; \boldsymbol{y}_B(t), \boldsymbol{v}_B(t), \cdots\right] \\ &\quad - \sum_A m_A c^2 \int \mathrm{d}t \sqrt{-g_{\mu\nu}\left(\boldsymbol{y}_A(t); \boldsymbol{y}_B(t), \boldsymbol{v}_B(t), \cdots\right) v_A^\mu v_A^\nu/c^2}\,, \end{aligned} \tag{2.6}$$

[3] The expression of $\Lambda^{\mu\nu}$ is given by Eq. (24) in [2]. Later we shall also need its generalization to d space dimensions as given by (175) in [2]. The harmonicity terms shown in (2.5) are the same in d dimensions.

where $\mathcal{L}_g$ is the Lagrangian density of the gravitational action (2.3). As is well known, it is always possible to eliminate from a generalized PN action a contribution that is quadratic in the accelerations by absorbing it into a "double-zero" term which does not contribute to the dynamics [21]. The argument can be extended to any term polynomial in the accelerations (and their derivatives). The PN equations of motion of the particles are obtained as the generalized Lagrange equations

$$\frac{\delta S_{\rm F}}{\delta \boldsymbol{y}_B} \equiv \frac{\partial L_{\rm F}}{\partial \boldsymbol{y}_B} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\rm F}}{\partial \boldsymbol{v}_B}\right) + \cdots = 0\,, \tag{2.7}$$

where $L_{\rm F}$ is the corresponding Lagrangian ($S_{\rm F} = \int \mathrm{d}t\, L_{\rm F}$). Once they have been constructed, the equations (2.7) can be order reduced by replacing iteratively all the higher-order accelerations by their expressions coming from the PN equations of motion themselves. The classical Fokker action should be equivalent, in the "tree-level" approximation, to the effective action used by the effective field theory [44, 64, 74, 75].

B. Fokker action in the PN approximation

In the Fokker action (2.6) the gravitational term integrates over the whole space a solution of the Einstein field equations obtained by PN iteration. The problem is that the PN solution is valid only in the near zone of the source — made here of a system of particles. Let us denote by $\overline{h}$ the PN expansion of the full-fledge metric perturbation h.[4] Outside the near zone of the source, $\overline{h}$ is not expected to agree with h and, in fact, will typically diverge at infinity.[5] On the other hand, the *multipole expansion* of the metric perturbation, that we denote by $\mathcal{M}(h)$, will agree with h in all the exterior region of the source, but will blow up when formally extended inside the near zone, and diverge when $r \to 0$. Indeed $\mathcal{M}(h)$ is a vacuum solution of the field equation differing from the true solution inside the matter source. The PN expansion $\overline{h}$ and the multipole expansion $\mathcal{M}(h)$ are matched together in their overlapping domain of validity, namely the exterior part of the near zone. Note that

[4] We are here dealing with an explicit solution of the Einstein field equations (2.4) for insertion into the Fokker action [see (2.6)]. Hence the metric perturbation depends on the particles, $h(\mathbf{x}; \boldsymbol{y}_A(t), \boldsymbol{v}_A(t), \cdots)$, as does its PN expansion $\overline{h}(\mathbf{x}; \boldsymbol{y}_A(t), \boldsymbol{v}_A(t), \cdots)$ and multipole expansion $\mathcal{M}(h)(\mathbf{x}; \boldsymbol{y}_A(t), \boldsymbol{v}_A(t), \cdots)$ considered below. For simplicity we shall not indicate the dependence on the particles.

[5] For instance, we know that $\overline{h}$ cannot be "asymptotically flat" starting at the 2PN or 3PN order, depending on the adopted coordinate system [76].

such overlapping region always exist for PN sources. The equation that realizes this match states that the near zone expansion ($r/c \to 0$) of the multipole expansion is identical, in a sense of formal series, to the multipole expansion ($a/r \to 0$, with a being the size of the source) of the PN expansion. It reads (see [2] for more details)

$$\overline{\mathcal{M}(h)} = \mathcal{M}\big(\overline{h}\big)\,. \tag{2.8}$$

The question we want to answer now is: How to transform the Fokker action (2.6) into an expression involving integrals over PN expansions that are obtained by formal PN iteration of the field equations in the near zone and can be computed in practice? Obviously, the problem concerns only the gravitational part of the action $S_g = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\,\mathcal{L}_g$. Note that the PN expansion of the Lagrangian density has the structure $\overline{\mathcal{L}}_g \sim \overline{h}\Box\overline{h} + \overline{h}\partial\overline{h}\partial\overline{h} + \cdots$. Similarly, we can define the multipole expansion of the integrand, which takes the form $\mathcal{M}(\mathcal{L}_g) \sim \mathcal{M}(h)\Box\mathcal{M}(h) + \cdots$. We are now in a position to state the following lemma.

Lemma 1: *The gravitational part of the Fokker Lagrangian can be written as a space integral over the looked-for* PN *Lagrangian density plus an extra contribution involving the multipole expansion,*

$$L_g = \operatorname*{FP}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g + \operatorname*{FP}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)\,. \tag{2.9}$$

A regulator $(r/r_0)^B$ and a finite part (FP) at $B = 0$, with r_0 being an arbitrary constant and B a complex number, cure the divergencies of the PN *expansion when $r \equiv |\boldsymbol{x}| \to +\infty$ in the first term while dealing with the singular behaviour of the multipole expansion when $r \to 0$ in the second term. The constant r_0 represents an IR scale in the first term and a UV scale in the second; it cancels out between the two terms.*

This lemma relies on the common general structure of the two sides of the matching equation (2.8), which implies a similar structure for the gravitational part of the Lagrangian density, namely (see *e.g.* [2])

$$\overline{\mathcal{M}(\mathcal{L}_g)} = \mathcal{M}(\overline{\mathcal{L}}_g) \sim \sum \hat{n}_L\, r^a (\ln r)^b F(t)\,, \tag{2.10}$$

where $\hat{n}_L = \mathrm{STF}(n_L)$ denotes an angular factor made of the symmetric-trace-free (STF) product of unit vectors $n^i = x^i/r$, with $L = i_1 \cdots i_\ell$ and $n_L = n_{i_1} \cdots n_{i_\ell}$. The powers of r

can take any positive or negative integer values $a \in \mathbb{Z}$, while the powers of the logarithm are positive integers $b \in \mathbb{N}$. The functions $F(t)$ denote very complicated multi-linear functionals of the multipole moments describing the source. The formal structure (2.10) can be either seen as a near-zone expansion when $r \to 0$ or as a far-zone expansion when $r \to +\infty$.

To prove (2.9), we consider the difference between L_g and the second term, namely

$$\Delta_g \equiv L_g - \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)\,. \tag{2.11}$$

Since L_g is perfectly convergent, it does not need any regularization; the regulator $(r/r_0)^B$ and the finite part at $B = 0$ can be inserted into it without altering the result. Hence

$$\Delta_g = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \Big[\mathcal{L}_g - \mathcal{M}(\mathcal{L}_g)\Big]\,. \tag{2.12}$$

Now we remark that the difference between $\mathcal{L}_g$ and its multipole expansion $\mathcal{M}(\mathcal{L}_g)$ is zero in the exterior region and is therefore of compact support, limited to the PN source, which is always smaller than the near zone size. Thus, we can replace it by the near-zone or PN expansion, so that

$$\Delta_g = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \Big[\overline{\mathcal{L}_g - \mathcal{M}(\mathcal{L}_g)}\Big]\,. \tag{2.13}$$

Finally the integral over a formal near-zone expansion of a multipolar expansion, *i.e.*, an object like $\overline{\mathcal{M}(\mathcal{L}_g)}$, multiplied by a regulator $(r/r_0)^B$, is always zero by analytic continuation in B. To see why it is so, we evaluate the integral by inserting the formal structure (2.10). After angular integration there remains a series of radial integrals of the type $\int_0^{+\infty} \mathrm{d}r\, r^{B+a+2} (\ln r)^b$ which are all separately zero by analytic continuation in B. Indeed, one may split the previous integral into near-zone $\int_0^{\mathcal{R}}$ and far-zone $\int_{\mathcal{R}}^{+\infty}$ contributions. The near-zone integral is computed for $\Re(B) > -a-3$ and analytically continued for any $B \in \mathbb{C}$, except for a multiple pole at $B = -a-3$. Likewise, the far-zone integral is computed for $\Re(B) < -a-3$ and analytically continued for any $B \in \mathbb{C}$, except $-a-3$. The two analytic continuations cancel each other and the result is exactly zero for any $B \in \mathbb{C}$, without poles (see [2] for more details). Finally, our lemma is proved,

$$\Delta_g = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g\,. \tag{2.14}$$

We now examine the fate of the second, multipolar term in (2.9) and show that it is actually negligible at the 4PN order. For this purpose, we shall prove that this term is non-zero only for "hereditary" terms depending on the whole past history of the source. Recall

indeed that the multipolar expansion $\mathcal{M}(h)$ is constructed from a post-Minkowskian (PM) algorithm starting from the most general solution of the linear Einstein field equations in the vacuum region outside the source (see [2] for a review, as well as Appendix A below). This linear solution is a functional of the multipole moments of the source, *i.e.*, the two series of mass-type and current-type moments $I_L(u)$ and $J_L(u)$ that describe the source ($L = i_1 \cdots i_\ell$, with ℓ being the multipole order), evaluated at the retarded time of harmonic coordinates $u = t - r/c$.[6] It is "instantaneous" in the sense that it depends on the state of the source, characterized by the moments I_L and J_L, only at time u. The PM iteration of this solution generates many terms that are likewise instantaneous and many hereditary terms that involve an integration over the past of the source, say $\int_{-\infty}^{u} \mathrm{d}v\, Q(1+\frac{u-v}{r})[I_L(v) \text{ or } J_L(v)]$, where Q is typically a Legendre function of the second kind [77–79]. One feature of the instantaneous terms is that, for them, the dependence on u can be factorized out through some function $G(u)$ which is a multi-linear product of the multipole moments $I_L(u)$ or $J_L(u)$ and their derivatives. By contrast, for hereditary terms, such a factorization is in general impossible.

This motivates our definition of instantaneous terms in the multipole expansion $\mathcal{M}(\mathcal{L}_g)$ (supposed to be generated by a PM algorithm) as being those with general structure of type

$$\mathcal{M}(\mathcal{L}_g)\big|_{\text{inst}} = \sum \frac{\hat{n}_L}{r^k} (\ln r)^q\, G(u)\,, \tag{2.15}$$

where $G(u)$ is any functional of the moments $I_L(u)$ or $J_L(u)$ and their time-derivatives (or anti time-derivatives), while k, q are positive integers with $k \geqslant 2$. By contrast the hereditary terms will have a more complicated structure. For instance, recalling that $\mathcal{M}(\mathcal{L}_g)$ is highly non-linear in $\mathcal{M}(h)$, the hereditary terms could consist of the interactions between instantaneous terms and tail terms producing the so-called "tails-of-tails". The corresponding

[6] The retarded cone in harmonic coordinates differs from a null coordinate cone by the famous logarithmic deviation, say $U = u - \frac{2GM}{c^3} \ln(\frac{c^2 r}{2GM}) + \mathcal{O}(\frac{1}{r})$. Such logarithmic deviation is taken into account in the formalism but in the form of a PM expansion, *i.e.*, it is formally expanded when $G \to 0$; it is then responsible for the appearance of powers of logarithms.

structure would be[7]

$$\mathcal{M}(\mathcal{L}_g)\big|_{\rm hered} = \sum \frac{\hat{n}_L}{r^k}(\ln r)^q\, H(u) \int_{-\infty}^{u} \mathrm{d}v\, Q\Big(1+\frac{u-v}{r}\Big) K(v)\,, \tag{2.16}$$

where $H(u)$ and $K(u)$ are multi-linear functionals of $I_L(u)$ and $J_L(u)$. Obviously, more complicated structures are possible. For hereditary terms in the previous sense, the dependence over u cannot be factorized out independently from r. Now, we have the following lemma.

Lemma 2: *The second term in the gravitational part of the Lagrangian* (2.9) *gives no contribution to the action for any instantaneous contribution of type* (2.15)*,*

$$\int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)\big|_{\rm inst} = 0\,. \tag{2.17}$$

Thus, only hereditary contributions of type (2.16) *or more complicated will contribute.*

The proof goes on in one line. Plugging (2.15) into the action, changing variable from t to u and using the factorization of the function $G(u)$, we get after angular integration a series of radial integrals of the type $\int_0^{+\infty} \mathrm{d}r\, r^{B+2-k}(\ln r)^q$, which are zero by analytic continuation in B as before.

We emphasize as a caveat that the object $\mathcal{M}(\mathcal{L}_g)$, made of instantaneous and hereditary pieces (2.15) and (2.16), should be carefully distinguished from $\overline{\mathcal{M}(\mathcal{L}_g)}$ whose general structure was given in (2.10). The multipole expansion $\mathcal{M}(\mathcal{L}_g)$ is defined all over the exterior zone and can be constructed by means of a PM algorithm. At any PM order and for a given set of multipole moments I_L and J_L, $\mathcal{M}(\mathcal{L}_g)$ is always made of a finite number of terms like (2.15) or (2.16). On the contrary, $\overline{\mathcal{M}(\mathcal{L}_g)}$ represents a formal *infinite* Taylor series when $r \to 0$ which, as we have seen from the matching equation (2.8), can also be interpreted as a formal series $\mathcal{M}(\overline{\mathcal{L}_g})$ when $r \to +\infty$. In such a formal sense, $\overline{\mathcal{M}(\mathcal{L}_g)}$ is in fact valid "everywhere".

Finally we are in a position to show that the multipolar contribution to the action — *i.e.*, the second term in (2.9) — is negligible at the 4PN order. Indeed, with the choice we have made to write the original action by starting at quadratic order with terms $\sim h\Box h$

[7] In our computation we consider only the conservative part of the dynamics and neglect the usual radiation reaction terms. Then, in the instantaneous terms (2.15), we should replace the retarded argument by the advanced one, while in hereditary terms of type (2.16) we should consider an appropriate symmetrization between retarded and advanced integrals.

(after suitable integration by parts), we see that the multipole expansion of the Lagrangian density, which is at least quadratic in $\mathcal{M}(h)$, takes the form

$$\mathcal{M}(\mathcal{L}_g) \sim \mathcal{M}(h)\Box\mathcal{M}(h) + \mathcal{M}(h)\partial\mathcal{M}(h)\partial\mathcal{M}(h) + \cdots . \tag{2.18}$$

Furthermore, $\mathcal{M}(h)$ is a *vacuum* solution of the field equations (2.4), physically valid only in the exterior of the source. Hence $\Box\mathcal{M}(h) = \mathcal{M}(\Sigma)$ with no matter source terms, and this quantity is therefore of the type

$$\Box\mathcal{M}(h) \sim \mathcal{M}(h)\partial^2\mathcal{M}(h) + \partial\mathcal{M}(h)\partial\mathcal{M}(h) + \cdots . \tag{2.19}$$

Combining (2.18) and (2.19) we see that $\mathcal{M}(\mathcal{L}_g)$ is at least *cubic* in $\mathcal{M}(h)$. In a PM expansion of $\mathcal{M}(h)$ [see (A5) in the Appendix], this term is at least of order $\mathcal{O}(G^3)$. Now, from (2.17), we know that $\mathcal{M}(\mathcal{L}_g)$ must necessarily be made of some multipole interaction involving hereditary terms, as displayed in (2.16), and these must be cubic. But we know that at dominant order such terms are the so-called "tails-of-tails", made of multipole interactions $M \times M \times I_L(u)$ or $M \times M \times J_L(u)$ (M is the ADM mass), which arise at least at the 5.5PN order [78, 79]. Therefore, in our calculation limited to 4PN, we are able to completely neglect the multipolar contribution in the Lagrangian (2.9), which becomes a pure functional of the PN expansion $\overline{h}$ of the metric perturbation up to the 4PN order,

$$L_g = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g \,. \tag{2.20}$$

Note that since the constant r_0 cancels out from the two terms of (2.9), the term (2.20) at 4PN order must *in fine* be independent of that constant. We shall explicitly verify the independence of our final Lagrangian over the IR cut-off scale r_0.

III. THE TAIL EFFECT AT 4PN ORDER

Let us recall that there is an imprint of tails in the local PN dynamics of the source at the 4PN order. The effect appears as a tail-induced modification of the dissipative radiation reaction force at the relative 1.5PN order beyond the leading 2.5PN contribution [65, 66]. Associated with it there exists a non dissipative piece that contributes to the conservative dynamics at the 4PN order. Here we shall show how to consistently include this piece into the Fokker action, starting from the result (2.20). To this end we first need the explicit

expressions for the parts of (2.20) that are quadratic and cubic in $\overline{h}$, namely

$$L_g^{(2)} = \frac{c^4}{32\pi G}\underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\left[\frac{1}{2}\overline{h}_{\mu\nu}\Box\overline{h}^{\mu\nu} - \frac{1}{4}\overline{h}\Box\overline{h}\right], \tag{3.1a}$$

$$\begin{aligned} L_g^{(3)} = \frac{c^4}{32\pi G}\underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\Bigg[&\overline{h}^{\rho\sigma}\Big(-\frac{1}{2}\partial_\rho\overline{h}_{\mu\nu}\partial_\sigma\overline{h}^{\mu\nu} + \frac{1}{4}\partial_\rho\overline{h}\partial_\sigma\overline{h}\Big) - \overline{h}_{\mu\nu}\Big(\partial_\rho\overline{h}^{\mu\sigma}\partial_\sigma\overline{h}^{\nu\rho} - \overline{H}^\mu\overline{H}^\nu\Big)\\ &+ \overline{h}_{\mu\nu}\Big(\partial_\rho\overline{h}^\mu_\sigma\partial^\rho\overline{h}^{\nu\sigma} - \frac{1}{2}\partial_\rho\overline{h}\partial^\rho\overline{h}^{\mu\nu}\Big)\Bigg]. \end{aligned} \tag{3.1b}$$

We shall insert in (3.1) the general expression for the PN expansion of the field in the near zone obtained by solving the matching equation (2.8) to any PN order [80, 81]. This solution incorporates all tails and related effects (both dissipative and conservative). It is built from a particular B-dependent solution of the wave equation, defined from the PN expansion of the pseudo stress-energy tensor (2.4b), $\overline{\tau}^{\mu\nu}$, by

$$\overline{h}^{\mu\nu}_{\mathrm{part}} \equiv \frac{16\pi G}{c^4}\underset{B=0}{\mathrm{FP}}\mathcal{I}^{-1}\Big[\Big(\frac{r}{s'_0}\Big)^B\overline{\tau}^{\mu\nu}\Big], \tag{3.2}$$

where the action of the operator $\mathcal{I}^{-1}$ of the "instantaneous" potentials (in the terminology of [66]) is given by

$$\mathcal{I}^{-1}\Big[\Big(\frac{r}{s'_0}\Big)^B\overline{\tau}^{\mu\nu}\Big] = \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2k}\Delta^{-k-1}\Big[\Big(\frac{r}{s'_0}\Big)^B\overline{\tau}^{\mu\nu}\Big], \tag{3.3}$$

in terms of the k-th iterated Poisson integral operator,

$$\Delta^{-k-1}\Big[\Big(\frac{r}{s'_0}\Big)^B\overline{\tau}^{\mu\nu}\Big] = -\frac{1}{4\pi}\int \mathrm{d}^3\mathbf{x}'\left(\frac{r'}{s'_0}\right)^B\frac{|\mathbf{x}-\mathbf{x}'|^{2k-1}}{(2k)!}\,\overline{\tau}^{\mu\nu}(\mathbf{x}',t)\,. \tag{3.4}$$

The general PN solution that matches an exterior solution with retarded boundary conditions at infinity is then the sum of the particular solution (3.2) and of an homogeneous multipolar solution regular inside the source, *i.e.*, of the type retarded minus advanced, and expanded in the near zone,[8]

$$\overline{h}^{\mu\nu} = \overline{h}^{\mu\nu}_{\mathrm{part}} - \frac{2G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\partial_L\left\{\frac{\mathcal{A}_L^{\mu\nu}(t-r/c) - \mathcal{A}_L^{\mu\nu}(t+r/c)}{r}\right\}. \tag{3.5}$$

Note that the particular solution (3.2) involves the scale s'_0. Similarly, as we shall see, the homogeneous solution in (3.5) will also depend on the scale s'_0 (through $s_0 = s'_0\,e^{-11/12}$ introduced below).

[8] Here and below, the presence of an overbar denoting the near-zone expansion $r \to 0$ is explicitly understood on the regular retarded-minus-advanced homogeneous solutions like the second term in (3.5).

The multipole moments $\mathcal{A}_L(t)$ in (3.5) are STF in $L = i_1 \cdots i_\ell$ and can be called radiation-reaction moments. They are composed of two parts,

$$\mathcal{A}^{\mu\nu}_L(t) = \mathcal{F}^{\mu\nu}_L(t) + \mathcal{R}^{\mu\nu}_L(t)\,. \tag{3.6}$$

The first one, $\mathcal{F}_L$, corresponds essentially to linear radiation reaction effects and yields the usual radiation damping terms at half integral 2.5PN and 3.5PN orders. These terms will not contribute to the conservative dynamics (they yield total time derivatives in the action) and we ignore them.

Important for our purpose is the second part, $\mathcal{R}_L$, which depends on boundary conditions imposed at infinity. It is a functional of the multipole expansion of the gravitational source term in the Einstein field equations, *i.e.*, $\mathcal{M}(\Sigma)$, and is given by Eq. (4.11) in [80]. The function $\mathcal{R}_L$ is responsible for the tail effects in the near zone metric and its contribution to (3.5) starts precisely at 4PN order. We evaluate it for quadrupolar tails, corresponding to the interaction between the total ADM mass M of the source and its STF quadrupole moment I_{ij}. Denoting the corresponding homogeneous solution by

$$\overline{\mathcal{H}}^{\mu\nu} = -\frac{2G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\partial_L\left\{\frac{\mathcal{R}^{\mu\nu}_L(t-r/c)-\mathcal{R}^{\mu\nu}_L(t+r/c)}{r}\right\}, \tag{3.7}$$

the relevant calculation at the quadrupole level was done in Eq. (3.64) of [66]:

$$\overline{\mathcal{H}}^{00} = -\frac{4G^2M}{c^5}\int_0^{+\infty}\mathrm{d}\tau\,\ln\left(\frac{c\tau}{2s_0}\right)\partial_{ij}\left\{\frac{I^{(2)}_{ij}(t-\tau-r/c)-I^{(2)}_{ij}(t-\tau+r/c)}{r}\right\}, \tag{3.8a}$$

$$\overline{\mathcal{H}}^{0i} = \frac{4G^2M}{c^6}\int_0^{+\infty}\mathrm{d}\tau\,\ln\left(\frac{c\tau}{2s_0}\right)\partial_{j}\left\{\frac{I^{(3)}_{ij}(t-\tau-r/c)-I^{(3)}_{ij}(t-\tau+r/c)}{r}\right\}, \tag{3.8b}$$

$$\overline{\mathcal{H}}^{ij} = -\frac{4G^2M}{c^7}\int_0^{+\infty}\mathrm{d}\tau\,\ln\left(\frac{c\tau}{2s_0}\right)\left\{\frac{I^{(4)}_{ij}(t-\tau-r/c)-I^{(4)}_{ij}(t-\tau+r/c)}{r}\right\}, \tag{3.8c}$$

with the shorthand $s_0 = s'_0\,e^{-11/12}$. For systems of particles the quadrupole moment reads $I_{ij} = \sum_A m_A y^{\langle i}_A y^{j\rangle}_A$, where $\langle\rangle$ denotes the STF projection. Time-derivatives are indicated as $I^{(n)}_{ij}$. Here, note that the constant s_0 in the logarithms *a priori* differs from r_0 [the IR cut-off in (2.20)] and we pose

$$s_0 = r_0\,e^{-\alpha}\,. \tag{3.9}$$

In this work we shall view the parameter α in (3.9) as an "ambiguity" parameter reflecting some incompleteness of the present formalism. We do not seem to be able to control this

ambiguity,[9] which we therefore leave as arbitrary. The parameter α is the equivalent of the parameter C in [67] and we shall later fix it, like in [67], by requiring that the conserved energy for circular orbits contains the correct self-force limit already known by numerical [69, 70] and analytical [71] calculations [see (4.32)]. We have checked that if we integrate by part the quadratic contributions in the PN Lagrangian (3.1a), so that we start with terms $\sim r^B \partial\overline{h}\partial\overline{h}$, the surface terms that are generated by the presence of the regulator factor r^B do not contribute to the dynamics modulo a mere redefinition of α.

At the leading 4PN order the expressions (3.8) reduce to

$$\overline{\mathcal{H}}^{00} = \frac{8G^2M}{15c^{10}} x^i x^j \int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2s_0}\right) I_{ij}^{(7)}(t-\tau) + \mathcal{O}\left(12\right) , \tag{3.10a}$$

$$\overline{\mathcal{H}}^{0i} = -\frac{8G^2M}{3c^{9}} x^j \int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2s_0}\right) I_{ij}^{(6)}(t-\tau) + \mathcal{O}\left(11\right) , \tag{3.10b}$$

$$\overline{\mathcal{H}}^{ij} = \frac{8G^2M}{c^{8}} \int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2s_0}\right) I_{ij}^{(5)}(t-\tau) + \mathcal{O}\left(10\right) , \tag{3.10c}$$

where we denote the PN remainder by $\mathcal{O}(n) \equiv \mathcal{O}(c^{-n})$. We insert the PN solution $\overline{h} = \overline{h}_{\rm part} + \overline{\mathcal{H}}$ into the action (2.20) and compute the contributions of the tail part $\overline{\mathcal{H}}$ (the instantaneous parts are discussed later). The quadratic terms in the action [see (3.1a)] will yield some $\sim \overline{\mathcal{H}}\Box\overline{h}_{\rm part}$ that are very simple to compute since at 4PN order we can use the leading expressions for $\overline{h}_{\rm part}$ [see *e.g.* (4.14) below]. Furthermore we find that some contributions $\sim \overline{\mathcal{H}}\,\partial\overline{h}_{\rm part}\,\partial\overline{h}_{\rm part}$ coming from the cubic part of the action must also be included at 4PN order.[10] Finally, inserting $\overline{\mathcal{H}}$ into the matter part S_m of the action makes obviously further contributions. We thus obtain the following 4PN tail effect in the total Fokker action as (skipping the PN remainder)

$$S_{\rm F}^{\rm tail} = \sum_A m_A c^2 \int \mathrm{d}t \left[-\frac{1}{8}\overline{\mathcal{H}}_A^{00} + \frac{1}{2c}\overline{\mathcal{H}}_A^{0i} v_A^i - \frac{1}{4c^2}\overline{\mathcal{H}}_A^{ij} v_A^i v_A^j\right] - \frac{1}{16\pi G}\int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, \overline{\mathcal{H}}^{ij}\partial_i U \partial_j U \,. \tag{3.11}$$

Most terms have a compact support and have been straightforwardly evaluated for particles with mass m_A and ordinary coordinate velocity v_A^i ($A = 1, 2$). However the last term in (3.11) is non compact and contains the Newtonian potential $U = \sum_A G m_A/|\mathbf{x} - \boldsymbol{y}_A|$.

[9] Notice that we do not control the "bulk" PN near-zone metric outside the particles; the present formalism is incomplete in this sense.

[10] From (3.1b), these cubic terms are easily identified as $\propto \overline{\mathcal{H}}^{ij}\,\partial_i\overline{h}^{00}_{\rm part}\,\partial_j\overline{h}^{00}_{\rm part} - \frac{1}{2}\overline{\mathcal{H}}^{ij}\,\partial_i\overline{h}_{\rm part}\,\partial_j\overline{h}_{\rm part}$.

Next we substitute (3.10) into (3.11) and obtain after some integrations by parts,

$$S_{\rm F}^{\rm tail} = -\frac{2G^2M}{5c^8}\int_{-\infty}^{+\infty} {\rm d}t\, I_{ij}(t) \int_0^{+\infty} {\rm d}\tau \ln\left(\frac{c\tau}{2s_0}\right) I_{ij}^{(7)}(t-\tau) + S_\eta\,. \tag{3.12}$$

We observe that the non-compact support term in (3.11) has nicely combined with the other terms to give a bilinear expression in the time derivatives of the quadrupole moment I_{ij}. The last term S_η denotes an irrelevant gauge term associated with a harmonic gauge transformation with vector η^i at the 4PN order. Such gauge term is due to a replacement of accelerations in the action that we did in order to arrive at the form (3.12). For completeness we give the "zero-on-shell" form of this gauge term as

$$S_\eta = -\sum_A m_A \int_{-\infty}^{+\infty} {\rm d}t \left[a_A^i - (\partial_i U)_A\right]\eta_A^i\,, \tag{3.13a}$$

$$\text{where}\quad \eta^i = -\frac{2G^2M}{c^8}x^j \int_0^{+\infty} {\rm d}\tau \ln\left(\frac{c\tau}{2s_0}\right) I_{ij}^{(5)}(t-\tau)\,. \tag{3.13b}$$

An important point to notice is that the result (3.12) can be rewritten in a manifestly time-symmetric way. Thus the procedure automatically selects some "conservative" part of the tail at 4PN order — the dissipative part giving no contribution to the action. Indeed we can alternatively write (ignoring from now on the gauge term)

$$S_{\rm F}^{\rm tail} = -\frac{G^2M}{5c^8}\int_{-\infty}^{+\infty} {\rm d}t\, I_{ij}(t) \int_0^{+\infty} {\rm d}\tau \ln\left(\frac{c\tau}{2s_0}\right)\left[I_{ij}^{(7)}(t-\tau) - I_{ij}^{(7)}(t+\tau)\right]\,, \tag{3.14}$$

which can also be transformed to a simpler form (after integrations by parts) with the help of the Hadamard *partie finie* (Pf) [82, 83], as

$$S_{\rm F}^{\rm tail} = \frac{G^2M}{5c^8}\operatorname*{Pf}_{2s_0/c}\iint \frac{{\rm d}t{\rm d}t'}{|t-t'|}\, I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')\,. \tag{3.15}$$

The dependence on the scale s_0 [see (3.9)] enters here *via* the arbitrary constant present in the definition of the Hadamard partie finie.[11] The result (3.15) agrees with the non-local action for the 4PN tail term which has been considered in [67] [see Eq. (4.4) there] and investigated in the effective field theory approach [74, 75]. Note however that while this contribution was added by hand to the 4PN local action in [67], we have shown here how to

[11] For any regular function $f(t)$ tending to zero sufficiently rapidly when $t \to \pm\infty$ we have

$$\operatorname*{Pf}_{\tau_0}\int_{-\infty}^{+\infty} {\rm d}t' \frac{f(t')}{|t-t'|} \equiv \int_0^{+\infty} {\rm d}\tau \ln\left(\frac{\tau}{\tau_0}\right)\left[f^{(1)}(t-\tau) - f^{(1)}(t+\tau)\right]\,.$$

derive it from scratch. Varying the action (3.15) with respect to the particle world-lines we obtain

$$\frac{\delta S_{\rm F}^{\rm tail}}{\delta y_A^i(t)} = -\frac{4G^2M}{5c^8} m_A y_A^j(t) \underset{2s_0/c}{\rm Pf} \int_{-\infty}^{+\infty} \frac{{\rm d}t'}{|t-t'|} I_{ij}^{(6)}(t') \,, \tag{3.16}$$

which coincides with the conservative part of the known 4PN tail contribution in the equations of motion [65, 66].

IV. PN ITERATION OF THE FOKKER ACTION

A. The method "$n+2$"

In the previous section we inserted the explicit PN solution $\overline{h} = \overline{h}_{\rm part} + \overline{\mathcal{H}}$ given by (3.5) into the Fokker action, and showed that the regular homogeneous solution $\overline{\mathcal{H}}$ produces the expected tails at 4PN order [see (3.15)]. We now deal with the terms generated by the particular solution in that decomposition. For simplicity, since the tails have now been determined, we shall just call that particular solution $\overline{h} = \overline{h}_{\rm part}$.

We first check that the variation of the Fokker action with PN gravitational term (2.20) yields back the PN expansion of the Einstein field equations. Indeed, because of the factor r^B we have to worry about the surface term that is generated when performing the variation with respect to $\overline{h}$. Schematically we have the structure $\mathcal{L}_g \sim r^B(\overline{h}\Box\overline{h} + \overline{h}\partial\overline{h}\partial\overline{h} + \cdots)$. When varying for instance the first term we get a contribution $\sim r^B\overline{h}\Box\delta\overline{h}$ on which we must shift the box operator to the left side modulo a surface term. However the surface term will contain the regulator r^B, so we see that it is actually rigourously zero by analytic continuation in B, since it is zero when starting from the case where $\Re(B)$ is a large negative number. Computing then the functional derivative of the Fokker action with respect to the PN expansion of the field, we still have some factors r^B but in some local (non integrated) expression, on which the FP prescription reduces to taking the value at $B = 0$. Thus we obtain the PN field equations as expected, say

$$\frac{\delta S_{\rm F}}{\delta\overline{h}} \sim c^4\Big[\Box\overline{h} - \overline{\Sigma} - c^{-4}\overline{\mathcal{T}}\Big] \,, \tag{4.1}$$

where $\overline{\Sigma}$ denotes the non-linear gravitational source term and $\mathcal{T} \sim |g|T$ symbolizes the matter tensor [see (2.4)]. In anticipation of the PN counting we address below, we have inserted into (4.1) the appropriate PN factor $c^4/16\pi G \sim c^4$.

We now discuss our practical method by which we control the PN expansion of the components of the metric perturbation $\overline{h}$ in order to obtain the Fokker action accurate to order nPN. As we shall see, thanks to the properties of the Fokker action [72], we essentially need to insert the metric perturbation at half the PN order which would have been naively expected.[12] To this end we decompose the PN metric perturbation according to

$$\overline{h}^{\mu\nu} \longrightarrow \begin{cases} \overline{h}^{00ii} \equiv \overline{h}^{00} + \overline{h}^{ii}, \\ \overline{h}^{0i}, \\ \overline{h}^{ij}. \end{cases} \tag{4.2}$$

Written in terms of these variables the (gauge fixed) gravitational action takes the form[13]

$$S_g = \frac{c^4}{64\pi G} \underset{B=0}{\mathrm{FP}} \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \left[\frac{1}{2}\overline{h}^{00ii}\Box\overline{h}^{00ii} - 2\overline{h}^{0i}\Box\overline{h}^{0i} + \overline{h}^{ij}\Box\overline{h}^{ij} - \overline{h}^{ii}\Box\overline{h}^{jj} + \mathcal{O}\big(\overline{h}^3\big)\right]. \tag{4.3}$$

Similarly the matter action reads at dominant order as

$$S_m = \sum_A m_A c^2 \int \mathrm{d}t \left[-1 + \frac{v_A^2}{2c^2} - \frac{1}{4}\overline{h}_A^{00ii} + \frac{v_A^i}{c}\,\overline{h}_A^{0i} - \frac{v_A^i v_A^j}{2c^2}\,\overline{h}_A^{ij} + \frac{v_A^2}{2c^2}\,\overline{h}_A^{ii} + \mathcal{O}\big(\overline{h}_A^2, c^{-2}\overline{h}_A\big)\right], \tag{4.4}$$

where the remainder term includes both higher-order terms in $\overline{h}$ as well as sub-dominant PN corrections. Varying independently with respect to these components of $\overline{h}$, we recover the fact that to lowest order $(\overline{h}^{00ii}, \overline{h}^{0i}, \overline{h}^{ij}) = \mathcal{O}(2,3,4)$, where we recall that $\mathcal{O}(n) = \mathcal{O}(c^{-n})$.

Consider first the usual PN iteration scheme, in which one solves the field equations up to order n, *i.e.*, up to order c^{-n} *included*, where n is an *even* integer. This means that $(\overline{h}^{00ii}, \overline{h}^{0i}, \overline{h}^{ij})$ are known up to order $\mathcal{O}(n+2, n+1, n)$ included, corresponding for n even to the usual *conservative* expansion — neglecting the radiation reaction dissipative terms.[14] We collectively denote by $\overline{h}_n[\boldsymbol{y}_A]$ the PN solution of the field equation up to that order, functional of the trajectories of the particles $\boldsymbol{y}_A(t)$ together with their velocities, accelerations and derivatives of accelerations, not indicated here. From (4.1), we see that the PN order of the functional derivative of the Fokker action evaluated for the approximate

[12] This point has been suggested to us by T. Damour (private communication).

[13] We present here the expression in 3 dimensions. Later we shall use dimensional regularization, so we shall need the easily generalized d-dimensional expression.

[14] For this discussion we can neglect conservative half-integral PN approximations, which arise to higher orders [78, 79].

solution $\overline{h}_n[\boldsymbol{y}_A]$ will be given by the commited error in that solution. Hence we have for n *even* (and ignoring the non-conservative odd PN orders)

$$\frac{\delta S_{\rm F}}{\delta \overline{h}^{00ii}}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big] = \mathcal{O}\big(n\big)\,, \tag{4.5a}$$

$$\frac{\delta S_{\rm F}}{\delta \overline{h}^{0i}}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big] = \mathcal{O}\big(n-1\big)\,, \tag{4.5b}$$

$$\frac{\delta S_{\rm F}}{\delta \overline{h}^{ij}}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big] = \mathcal{O}\big(n-2\big)\,. \tag{4.5c}$$

If now we write the complete solution as $\overline{h}[\boldsymbol{y}_B] = \overline{h}_n[\boldsymbol{y}_B] + \overline{r}_{n+2}$, introducing some uncontrolled PN remainder term

$$\overline{r}_{n+2} = (\overline{r}^{00ii}_{n+4}, \overline{r}^{0i}_{n+3}, \overline{r}^{ij}_{n+2}) = \mathcal{O}(n+4, n+3, n+2)\,, \tag{4.6}$$

the Fokker action expanded around the known approximate solution reads[15]

$$\begin{aligned} S_{\rm F}\big[\overline{h}[\boldsymbol{y}_B],\boldsymbol{y}_A\big] &= S_{\rm F}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big] \\ &\quad + \underset{B=0}{\rm FP}\int {\rm d}t \int {\rm d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B \left[\frac{\delta S_{\rm F}}{\delta \overline{h}^{00ii}}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big]\overline{r}^{00ii}_{n+4}\right. \\ &\qquad \left. + \frac{\delta S_{\rm F}}{\delta \overline{h}^{0i}}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big]\overline{r}^{0i}_{n+3} + \frac{\delta S_{\rm F}}{\delta \overline{h}^{ij}}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big]\overline{r}^{ij}_{n+2} + \cdots\right]. \end{aligned} \tag{4.7}$$

The ellipsis stand for the quadratic and higher-order terms in the remainders $\overline{r}_{n+2}$. Inserting both the orders of magnitude estimates (4.5) as well as the orders of the remainders (4.6) we readily obtain

$$S_{\rm F}\big[\overline{h}[\boldsymbol{y}_B],\boldsymbol{y}_A\big] = S_{\rm F}\big[\overline{h}_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big] + \mathcal{O}\left(2n\right)\,, \tag{4.8}$$

which means that the Fokker action has been determined at the $(n-1)$PN order. This is not yet the nPN accuracy we were aiming for.

However, we notice that in this scheme the term $\overline{h}^{ij}$ is responsible for the dominant error $\mathcal{O}(2n)$, together with a term of the same order, associated with the second variation $(\delta^2 S_{\rm F}/\delta\overline{h}^{ij}/\delta\overline{h}^{kl})\,\overline{r}^{ij}_{n+2}\overline{r}^{kl}_{n+2}$. Thus, if one pushes by one order the precision of the component $\overline{h}^{ij}$, denoting $\overline{h}'_n[\boldsymbol{y}_A]$ the corresponding solution which is now accurate up to order $\mathcal{O}(n+2, n+1, n+2)$ included, we see that

$$\frac{\delta S_{\rm F}}{\delta \overline{h}}\big[\overline{h}'_n[\boldsymbol{y}_B],\boldsymbol{y}_A\big] = \mathcal{O}\big(n, n-1, n\big)\,, \tag{4.9a}$$

[15] The complete justification of this expansion is actually not trivial because of the presence of the regulator $(r/r_0)^B$ coming from the PN gravitational term (2.20) and the integrations by parts that are necessary in order to arrive at (4.7). We deal with this point in Appendix A.

$$\text{and} \quad \overline{r}'_{n+2} = \mathcal{O}\big(n+4, n+3, n+4\big)\,. \tag{4.9b}$$

Here the remainders are such that $\overline{h}[\boldsymbol{y}_B] = \overline{h}'_n[\boldsymbol{y}_B] + \overline{r}'_{n+2}$. With the estimates (4.9) we now obtain our looked for nPN precision, namely

$$S_{\rm F}\big[\overline{h}[\boldsymbol{y}_B], \boldsymbol{y}_A\big] = S_{\rm F}\big[\overline{h}'_n[\boldsymbol{y}_B], \boldsymbol{y}_A\big] + \mathcal{O}\left(2n+2\right)\,. \tag{4.10}$$

Concerning the terms with higher order functional derivatives — the ellipsis in (4.7) — we can remark that the derivatives are at most a factor c^4 multiplied by a remainder term that is squared at least. In the scheme (4.9) the dominant source of error is now the term $\overline{h}^{0i}$.

Since we ignore non-conservative odd PN terms, solving for $(\overline{h}^{00ii}, \overline{h}^{0i}, \overline{h}^{ij})$ to order $\mathcal{O}(n+1, n+2, n+2)$ with n an *odd* integer gives

$$\frac{\delta S_{\rm F}}{\delta \overline{h}}\big[\overline{h}''_n[\boldsymbol{y}_B], \boldsymbol{y}_A\big] = \mathcal{O}\big(n-1, n, n-1\big)\,, \tag{4.11a}$$

$$\text{and} \quad \overline{r}''_{n+2} = \mathcal{O}\big(n+3, n+4, n+3\big)\,, \tag{4.11b}$$

hence the error is still $\mathcal{O}(2n+2)$. In conclusion, we find that in order to control the Fokker action to the nPN order, it is necessary and sufficient to insert the components of the metric perturbation

$$\overline{h} = (\overline{h}^{00ii}, \overline{h}^{0i}, \overline{h}^{ij}) \;\text{ up to order }\; \begin{cases} \mathcal{O}\big(n+2, n+1, n+2\big) & \text{when } n \text{ is even}\,, \\ \mathcal{O}\big(n+1, n+2, n+1\big) & \text{when } n \text{ is odd}\,. \end{cases} \tag{4.12}$$

Since in both cases all the components of $\overline{h}$ (for the conservative dynamics) are to be computed up to order $\mathcal{O}(n+2)$ we call the PN iteration up to that order the "method $n+2$".

B. Metric potentials in d dimensions

From the previous result, we see that at the 4PN order we need the components of the metric perturbation up to order $\mathcal{O}(6,5,6)$ included. To that order we shall parametrize the metric by means of usual PN potentials (see *e.g.* [2]). But since we use dimensional regularization for treating the local divergencies we provide the requested expression of the metric in d spatial dimensions. To this end the appropriate generalization of the variables (4.2) is

$$\overline{h}^{00ii} = 2\frac{(d-2)\overline{h}^{00} + \overline{h}^{ii}}{d-1}\,, \quad \overline{h}^{0i}\,, \quad \overline{h}^{ij}\,. \tag{4.13}$$

We have (with $\hat{W} = \hat{W}_{ii}$ and $\hat{Z} = \hat{Z}_{ij}$)

$$\overline{h}^{00ii} = -\frac{4}{c^2}V - \frac{4}{c^4}\left[\frac{d-1}{d-2}V^2 - 2\frac{d-3}{d-2}K\right] \tag{4.14a}$$
$$- \frac{8}{c^6}\left[2\hat{X} + V\hat{W} + \frac{1}{3}\left(\frac{d-1}{d-2}\right)^2 V^3 - 2\frac{d-3}{d-1}V_iV_i - 2\frac{(d-1)(d-3)}{(d-2)^2}KV\right] + \mathcal{O}\left(8\right) ,$$

$$\overline{h}^{0i} = -\frac{4}{c^3}V_i - \frac{4}{c^5}\left(2\hat{R}_i + \frac{d-1}{d-2}VV_i\right) + \mathcal{O}\left(7\right) , \tag{4.14b}$$

$$\overline{h}^{ij} = -\frac{4}{c^4}\left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}\right) - \frac{16}{c^6}\left(\hat{Z}_{ij} - \frac{1}{2}\delta_{ij}\hat{Z}\right) + \mathcal{O}\left(8\right) . \tag{4.14c}$$

Each of these potentials obeys a flat space-time wave equation (in d dimensions) sourced by lower order potentials in the same family, and by appropriate matter density components. The list of requested wave equations is

$$\Box V = -4\pi G\,\sigma\,, \tag{4.15a}$$

$$\Box K = -4\pi G\,\sigma\,V\,, \tag{4.15b}$$

$$\Box\hat{X} = -4\pi G\left[\frac{V\sigma_{ii}}{d-2} + \frac{2(d-3)}{d-1}\sigma_iV_i + \left(\frac{d-3}{d-2}\right)^2\sigma\left(\frac{V^2}{2} + K\right)\right] + \hat{W}_{ij}\,\partial^2_{ij}V$$
$$+ 2V_i\,\partial_t\partial_iV + \frac{d-1}{2(d-2)}V\partial_t^2V + \frac{d(d-1)}{4(d-2)^2}\left(\partial_tV\right)^2 - 2\partial_iV_j\,\partial_jV_i + \Box\delta\hat{X}\,, \tag{4.15c}$$

$$\Box V_i = -4\pi G\,\sigma_i\,, \tag{4.15d}$$

$$\Box\hat{R}_i = -\frac{4\pi G}{d-2}\left[\frac{5-d}{2}\,V\sigma_i - \frac{d-1}{2}\,V_i\,\sigma\right] - \frac{d-1}{d-2}\,\partial_kV\partial_iV_k - \frac{d(d-1)}{4(d-2)^2}\,\partial_tV\partial_iV\,, \tag{4.15e}$$

$$\Box\hat{W}_{ij} = -4\pi G\left(\sigma_{ij} - \delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right) - \frac{d-1}{2(d-2)}\partial_iV\partial_jV\,, \tag{4.15f}$$

$$\Box\hat{Z}_{ij} = -\frac{4\pi G}{d-2}\,V\left(\sigma_{ij} - \delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right) - \frac{d-1}{d-2}\,\partial_tV_{(i}\,\partial_{j)}V + \partial_iV_k\,\partial_jV_k + \partial_kV_i\,\partial_kV_j$$
$$- 2\partial_kV_{(i}\,\partial_{j)}V_k - \frac{\delta_{ij}}{d-2}\,\partial_kV_m\left(\partial_kV_m - \partial_mV_k\right) - \frac{d(d-1)}{8(d-2)^3}\,\delta_{ij}\left(\partial_tV\right)^2$$
$$+ \frac{(d-1)(d-3)}{2(d-2)^2}\,\partial_{(i}V\partial_{j)}K\,. \tag{4.15g}$$

The presence of additional terms proportional to the harmonicity H^μ in the gravitational source term (2.5) leads in principle to differences in these wave equations with respect to previous works which used harmonic coordinates [38]. At the order we are considering, the only such additional contribution enters the potential $\hat{X}$. It is denoted by $\delta\hat{X}$ above. The corresponding source reads

$$\Box\delta\hat{X} = \partial_iV\left[\partial_tV_i + \partial_j\left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}\right)\right]\,. \tag{4.16}$$

When the equations of motion are satisfied the above potentials are linked by the following differential identities coming from the harmonic gauge condition,

$$\partial_t\left\{\frac{d-1}{2(d-2)}V+\frac{1}{2c^2}\left[\hat{W}+\left(\frac{d-1}{d-2}\right)^2V^2-\frac{2(d-1)(d-3)}{(d-2)^2}K\right]\right\}$$
$$+\partial_i\left\{V_i+\frac{2}{c^2}\left[\hat{R}_i+\frac{d-1}{2(d-2)}VV_i\right]\right\}=\mathcal{O}\left(4\right)\,, \tag{4.17a}$$

$$\partial_t\left\{V_i+\frac{2}{c^2}\left[\hat{R}_i+\frac{d-1}{2(d-2)}VV_i\right]\right\}$$
$$+\partial_j\left\{\hat{W}_{ij}-\frac{1}{2}\,\hat{W}\delta_{ij}+\frac{4}{c^2}\left[\hat{Z}_{ij}-\frac{1}{2}\,\hat{Z}\delta_{ij}\right]\right\}=\mathcal{O}\left(4\right)\,. \tag{4.17b}$$

Note that we generally do not use these relations, which are true only "on-shell", at the level of the Fokker action. The only relation we are allowed to use for simplifications is

$$\frac{d-1}{2(d-2)}\partial_t V+\partial_i V_i=\mathcal{O}(2)\,, \tag{4.18}$$

since it will hold for the Newtonian potentials V and V_i regardless of the equations of motion. According to Eqs. (3.2) and (3.5), in order to recover the "particular" solution, one should integrate the latter wave equations by means of the operator of symmetric potentials $\mathcal{I}^{-1}$, and in principle, one should implement the calculation by means of a factor $(r/r'_0)^B$ to cure possible IR divergencies. But at this relatively low level $\mathcal{O}(6,5,6)$ we find that the IR regulator is in fact not necessary, and we can use the usual symmetric propagator $\Delta^{-1}+c^{-2}\partial_t^2\Delta^{-2}+\cdots$. The matter source terms are defined by

$$\sigma=2\frac{(d-2)T^{00}+T^{ii}}{(d-1)c^2}\,,\qquad \sigma_i=\frac{T^{0i}}{c}\,,\qquad \sigma_{ij}=T^{ij}\,, \tag{4.19}$$

from the components of the stress-energy tensor of the point particles,

$$T^{\mu\nu}=\sum_A\frac{m_A\,v_A^\mu v_A^\nu}{\sqrt{-(g_{\rho\sigma})_A\,v_A^\rho v_A^\sigma/c^2}}\frac{\delta^{(d)}(\mathbf{x}-\boldsymbol{y}_A)}{\sqrt{-g}}\,. \tag{4.20}$$

Finally the constant G is related to Newton's constant $G_{\rm N}$ in three dimensions by

$$G=G_{\rm N}\,\ell_0^{d-3}\,, \tag{4.21}$$

where ℓ_0 denotes the characteristic length scale associated with dimensional regularization.

C. Implementation of the calculation

Having determined in (4.14)–(4.15) the metric components for insertion into the Fokker action (2.1)–(2.2), we tackle the difficult (and very lengthy) calculation of all the spatial

integrals in the gravitational part S_g of the action.[16] To reach the 4PN precision we must include non-linear terms in the action up to the sixth non-linear level, say

$$\mathcal{L}_g \sim c^4\left[\overline{h}\Box\overline{h} + \overline{h}\partial\overline{h}\partial\overline{h} + \cdots + \overline{hhhh}\partial\overline{h}\partial\overline{h}\right] + \mathcal{O}\left(10\right) . \tag{4.22}$$

The matter part S_m of the action is much simpler and will not be discussed.

Following previous works on the 3PN equations of motion [34, 38] we shall proceed in several steps. The potentials (4.15) are first computed for any point in 3-dimensional space and then inserted into the gravitational part of the action. The computation of potentials extensively uses the famous function $g = \ln(r_1 + r_2 + r_{12})$, solution of the elementary Poisson equation $\Delta g = r_1^{-1} r_2^{-1}$ [9], which permits to deal with quadratic source terms of type $\sim \partial V \partial V$. One needs also to integrate a cubic source term $\sim \hat{W}\partial^2 V$ and for that we use more complicated elementary solutions given by Eqs. (6.3)–(6.5) in [24].

The integration is then implemented by means of the Hadamard regularization (HR)[17] to treat the UV divergencies associated with point particles. We thus compute with HR the spatial integral (with non-compact support) of the terms in (4.22), say some generic function $F(\mathbf{x})$ resulting from the PN iteration performed in 3 dimensions,

$$I = \operatorname*{Pf}_{s_1,s_2} \int \mathrm{d}^3\mathbf{x}\, F(\mathbf{x}) . \tag{4.23}$$

The function F is singular at the two points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, and the Hadamard partie finie Pf depends on two UV scales denoted s_A. We assume that the integral extends on some finite volume surrounding the singularities so that we do not include the IR regulator r^B at this stage (see below for discussion of the IR divergencies). The HR is simple and very convenient for practical calculations but is unfortunately plagued with ambiguities starting at the 3PN order. Therefore, in a second step, we shall correct for the possible ambiguities of HR by adding to the HR result the *difference* "DR−HR" between the corresponding result of the more powerful *dimensional* regularization (DR) [85, 86] and the one of HR. While at 3PN and 4PN orders the HR result contains logarithmic divergences yielding ambiguities, the DR result gives some simple poles, *i.e.*, $\propto 1/\varepsilon$ where $\varepsilon = d - 3$. The poles are followed by a finite part $\propto \varepsilon^0$ which is free of ambiguities, and all terms of order $\mathcal{O}(\varepsilon)$ are neglected.

[16] Extensive use is made of the algebraic software Mathematica together with the tensor package *xAct* [84].
[17] Or, more precisely, the so-called pure-Hadamard-Schwartz regularization [38]. See [35] for precise definitions of various concepts of Hadamard's partie finie.

We perform the similar PN iteration in d dimensions to obtain a generic non-compact d dimensional integral of some function $F^{(d)}(\mathbf{x})$, say

$$I^{(d)} = \int \mathrm{d}^d\mathbf{x}\, F^{(d)}(\mathbf{x})\,. \tag{4.24}$$

When $r_1 \to 0$ the function $F^{(d)}$ admits a singular expansion more complicated than in 3 dimensions, as it involves complex powers of r_1 of the type $p + q\varepsilon$ (instead of merely p),

$$F^{(d)}(\mathbf{x}) = \sum_{p,q} r_1^{p+q\varepsilon} \underset{1}{f}{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1) + o(r_1^N)\,, \tag{4.25}$$

where p and q are relative integers whose values are limited by some $p_0 \leqslant p \leqslant N$ and $q_0 \leqslant q \leqslant q_1$ (with $p_0, q_0, q_1 \in \mathbb{Z}$). The coefficients $f_1{}^{(\varepsilon)}_{p,q}$ depend on the direction of approach to the singularity $\mathbf{n}_1 = (\mathbf{x} - \boldsymbol{y}_1)/r_1$, and are linked to their counterparts $f_{1\,p}$ associated with the function F in 3 dimensions by

$$\sum_{q=q_0}^{q_1} \underset{1}{f}{}^{(0)}_{p,q}(\mathbf{n}_1) = \underset{1}{f}{}_p(\mathbf{n}_1)\,. \tag{4.26}$$

One can show that at the 4PN order the functions $F^{(d)}$ have no poles as $\varepsilon \to 0$, so the limit $\varepsilon = 0$ in (4.26) is well defined.

The point is that the difference DR−HR can be computed purely *locally*, *i.e.*, in the vicinity of the two particles, as it is entirely determined, in the limit $\varepsilon \to 0$, by the coefficients $f_1{}^{(\varepsilon)}_{p,q}$ of the local expansion of the function $F^{(d)}$. This is clear because the parts of the integrals outside the singularities cancel out in the difference when $\varepsilon \to 0$. Denoting such difference by $\mathcal{D}I = I^{(d)} - I$, for any of the non-compact support integrals composing the gravitational action, we have the basic formula

$$\mathcal{D}I = \frac{1}{\varepsilon} \sum_{q=q_0}^{q_1} \left[\frac{1}{q+1} + \varepsilon \ln s_1\right] \left\langle \underset{1}{f}{}^{(\varepsilon)}_{-3,q} \right\rangle_{2+\varepsilon} + 1 \leftrightarrow 2 + \mathcal{O}(\varepsilon)\,. \tag{4.27}$$

Here $1 \leftrightarrow 2$ is the particle label permutation, s_1 and s_2 are the HR scales in (4.23), the $\mathcal{O}(\varepsilon)$ remainder is neglected, and the spherical angular integrals read

$$\left\langle f \right\rangle_{d-1} = \int \mathrm{d}\Omega_{d-1}(\mathbf{n}_1)\, f(\mathbf{n}_1)\,, \tag{4.28}$$

with $\mathrm{d}\Omega_{d-1}$ being the usual differential surface element in $d-1$ dimensions. Notice the sum ranging over the integer q in (4.27) and the problematic case $q = -1$. An important test of the calculation (and more generally of the adequacy of DR to treat the classical problem of

point particles in GR), is that the spherical integral (4.28) is always zero in the case of the offending value $q = -1$.

The potentials (4.15) in d dimensions are in principle computed with d-dimensional generalizations of the elementary solutions used in HR, notably the function $g^{(d)}$ which generalizes the function $g = \ln(r_1+r_2+r_{12})$. This function is known in explicit closed form for any d (see the Appendix C in [38]). Here we need only its local expansion when $r_1 \to 0$. In practice, the local expansion of a potential is obtained by integrating term by term the local expansion of its source, and adding the appropriate homogeneous solution. We obtain in Appendix B the local expansion of the function $g^{(d)}$. We have checked that at the 4PN order we do not need to consider the d-dimensional generalizations of the elementary solutions (6.3)–(6.5) in [24]. We also found that the final 4PN results are unchanged if we add to the potentials some arbitrary homogeneous solutions at order ε, provided that the harmonic coordinate conditions (4.17) for the potentials remain satisfied when the potentials are "on-shell".

Once the HR calculation has been completed and the difference "DR−HR" added,[18] the next step consists in renormalizing the result by absorbing the poles $\propto 1/\varepsilon$ into appropriate shifts of the trajectories of the particles. There is a lot of freedom for such shifts. Here we adopt some non-minimal prescription in order to recover the earlier shifts at 3PN order [38], which yielded precisely the 3PN harmonic coordinate equations of motion in [34]. The latter 3PN equations of motion depend on two gauge constants r'_1 and r'_2 that we therefore introduce into the shifts in replacement of the characteristic DR length scale ℓ_0. For convenience we extend this prescription to 4PN order in the simplest way, so that ℓ_0 disappears from the Lagrangian and the logarithmic terms (in harmonic coordinates) are only of the form $\ln(r_{12}/r'_1)$ or $\ln(r_{12}/r'_2)$, where r_{12} is the separation between particles, and are symmetric under $1 \leftrightarrow 2$ exchange. Our 4PN shifts read then

$$\boldsymbol{\xi}_1 = \frac{11}{3}\frac{G_{\rm N}^2 m_1^2}{c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{\overline{q}^{1/2} r'_1}{\ell_0}\right) - \frac{327}{1540}\right]\boldsymbol{a}^{(d)}_{1,\rm N} + \frac{1}{c^8}\boldsymbol{\xi}_{1,\,4\rm PN}\,, \tag{4.29a}$$

$$\boldsymbol{\xi}_2 = 1 \leftrightarrow 2\,. \tag{4.29b}$$

At 3PN order we recognize the shift given by Eq. (1.13) in [38], where ℓ_0 is defined by (4.21), $\overline{q} = 4\pi e^{\gamma_{\rm E}}$ depends on Euler's constant $\gamma_{\rm E} \simeq 0.577$, and $\boldsymbol{a}^{(d)}_{1,\rm N}$ is the Newtonian acceleration of the particle 1 in d dimensions. The complete expression of the shift at 4PN order is given

[18] Notice that the scales s_A cancel out in the final DR result.

in Appendix C. After applying the shifts (4.29) the poles $\propto 1/\varepsilon$ cancel out and the result is UV finite.

We also found that our "brute" Lagrangian depends on the individual positions $\boldsymbol{y}_A$ of the particles. Such dependence is pure gauge and we removed it by including appropriate terms in the shift, so that the shifted Lagrangian depends only on the relative position $\boldsymbol{y}_{12}$ and is manifestly translation invariant. More generally we found that our initial Lagrangian is not manifestly Poincaré invariant, but that we can adjust the shift (4.29) so that it becomes Poincaré invariant *in a manifest way* (modulo a total time derivative). The (global) Lorentz-Poincaré invariance is a very satisfying property of our final 4PN dynamics.

Finally we discuss the very important problem of IR divergencies, which appear specifically at the 4PN order. As we see in the complete formula for the 4PN shift [Eq. (C3) in Appendix C], besides the UV logarithms $\ln(r_{12}/r'_1)$ and $\ln(r_{12}/r'_2)$, there are also some logarithms $\ln(r_{12}/r_0)$ at the 4PN order, where r_0 was introduced in the gravitational part of the action [see Eq. (2.20)] as an IR cut-off dealing with the divergences of three-dimensional volume integrals such as (4.23), caused by the PN expansion $\overline{h}$ diverging at infinity. The fact that the constant r_0 can be completely removed from the calculation by applying the shift (4.29) constitutes a very important test of the calculation. This is made possible by the presence of the 4PN non-local tails (3.14)–(3.15). To see that, we rewrite the logarithmic kernel in the tail integrals (containing the constant s_0) as

$$\ln\left(\frac{c\tau}{2s_0}\right) = \ln\left(\frac{c\tau}{2r_{12}}\right) + \ln\left(\frac{r_{12}}{r_0}\right) + \alpha\,, \tag{4.30}$$

where α links s_0 to r_0 and is defined by (3.9). We find that the second term of (4.30) combines with the IR divergences of the 3-dimensional volume integrals to exactly produce a term removable by a shift, hence the $\ln(r_{12}/r_0)$ contributions in (C3). With the first term in (4.30) we shall rewrite the tail integrals using the separation r_{12},[19] being careful that r_{12} is no longer a constant and will have to be varied and participate to the dynamics. Finally our end result will not only be UV finite but also IR finite.

The constant α which remains is the analogue of the constant C in [67]. It does not seem possible to determine its value within the present method. Like in [67] we shall compute it by comparison with self-force calculations, which have determined the 4PN term in the

[19] Specifically, our choice is to insert r_{12} into Eq. (3.15) of the tail term, *i.e.*, after integrations by parts.

conserved energy for circular orbits [69–71]. Let us check that α is a pure numerical constant, *i.e.*, does not depend on the masses. Since α is dimensionless and is necessarily a symmetric function of the two masses m_1 and m_2, it can only depend on the symmetric mass ratio $\nu = m_1 m_2/(m_1+m_2)^2$. Thus, we can write very generally (with a finite or infinite sum)

$$\alpha = \sum_n \alpha_n \, \nu^n \,. \tag{4.31}$$

In the Lagrangian α is in factor of $\sim (I_{ij}^{(3)})^2$. We derive the corresponding terms in the acceleration of the particles and look at the mass dependence of these terms. Imposing that the acceleration should be a polynomial in the two masses separately,[20] we find that the only admissible case is indeed a pure constant $\alpha = \alpha_0$. Finally we adjust α so that the conserved energy for circular orbits (that we shall compute in the sequel paper [73], see also Sec. V D) agrees with self-force calculations in the small mass ratio limit — see the coefficient of ν at 4PN order in Eq. (5.5) of [67]. Anticipating the result we find

$$\alpha = \frac{811}{672} \,. \tag{4.32}$$

V. LAGRANGIAN OF COMPACT BINARIES AT THE 4PN ORDER

A. Result in harmonic coordinates

The Lagrangian in harmonic coordinates at the 4PN order will be a generalized one, depending on the positions of particles $\boldsymbol{y}_A$ and velocities $\boldsymbol{v}_A = \mathrm{d}\boldsymbol{y}_A/\mathrm{d}t$, and also accelerations $\boldsymbol{a}_A = \mathrm{d}\boldsymbol{v}_A/\mathrm{d}t$, derivatives of accelerations $\boldsymbol{b}_A$, and so on. However, by adding suitable double-zero or multi-zero terms [21] we have removed all terms that are non-linear in accelerations and derivatives of accelerations. Furthermore, by adding suitable total time derivatives we have eliminated the dependence on derivatives of accelerations. Note that the process is iterative, since the latter time derivatives reintroduce some terms non-linear in accelerations, that need to be removed by further double-zeros. Thus the generalized 4PN harmonic-coordinate Lagrangian depends on $\boldsymbol{y}_A$ and $\boldsymbol{v}_A$, and is linear in accelerations $\boldsymbol{a}_A$.[21] As said in Sec. IV C, after a suitable shift the Lagrangian depends only on the relative separation

[20] This can be justified from a diagrammatic expansion of the N-body problem based on the post-Minkowskian approximation [87].

[21] An exception is the 4PN tail piece which will be left as a functional of $\boldsymbol{y}_A$, $\boldsymbol{v}_A$, $\boldsymbol{a}_A$ and $\boldsymbol{b}_A$.

$r_{12} = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$ between the particles in harmonic coordinates. We denote the corresponding unit direction by $\boldsymbol{n}_{12} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/r_{12}$. We systematically use parenthesis to denote ordinary scalar products, *e.g.* $(n_{12}v_1) = \boldsymbol{n}_{12} \cdot \boldsymbol{v}_1$ and $(a_1 v_2) = \boldsymbol{a}_1 \cdot \boldsymbol{v}_2$.

We write the full 4PN Lagrangian in the form

$$L = L_{\rm N} + \frac{1}{c^2} L_{\rm 1PN} + \frac{1}{c^4} L_{\rm 2PN} + \frac{1}{c^6} L_{\rm 3PN} + \frac{1}{c^8} L_{\rm 4PN} + \mathcal{O}\left(10\right) , \tag{5.1}$$

where the pieces up to 3PN order are known from previous works (see *e.g.* [2]) as

$$L_{\rm N} = \frac{G m_1 m_2}{2 r_{12}} + \frac{m_1 v_1^2}{2} + 1 \leftrightarrow 2 \,, \tag{5.2a}$$

$$\begin{aligned} L_{\rm 1PN} = {} & -\frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{m_1 v_1^4}{8} \\ & + \frac{G m_1 m_2}{r_{12}} \left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2} v_1^2 - \frac{7}{4}(v_1 v_2) \right) + 1 \leftrightarrow 2 \,, \end{aligned} \tag{5.2b}$$

$$\begin{aligned} L_{\rm 2PN} = {} & \frac{G^3 m_1^3 m_2}{2 r_{12}^3} + \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} \\ & + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left(\frac{7}{2}(n_{12}v_1)^2 - \frac{7}{2}(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 + \frac{1}{4} v_1^2 - \frac{7}{4}(v_1 v_2) + \frac{7}{4} v_2^2 \right) \\ & + \frac{G m_1 m_2}{r_{12}} \left(\frac{3}{16}(n_{12}v_1)^2 (n_{12}v_2)^2 - \frac{7}{8}(n_{12}v_2)^2 v_1^2 + \frac{7}{8} v_1^4 + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) \right. \\ & \left. - 2 v_1^2 (v_1 v_2) + \frac{1}{8}(v_1 v_2)^2 + \frac{15}{16} v_1^2 v_2^2 \right) + \frac{m_1 v_1^6}{16} \\ & + G m_1 m_2 \left(-\frac{7}{4}(a_1 v_2)(n_{12}v_2) - \frac{1}{8}(n_{12}a_1)(n_{12}v_2)^2 + \frac{7}{8}(n_{12}a_1) v_2^2 \right) + 1 \leftrightarrow 2 \,, \end{aligned} \tag{5.2c}$$

$$\begin{aligned} L_{\rm 3PN} = {} & \frac{G^2 m_1^2 m_2}{r_{12}^2} \left(\frac{13}{18}(n_{12}v_1)^4 + \frac{83}{18}(n_{12}v_1)^3 (n_{12}v_2) - \frac{35}{6}(n_{12}v_1)^2 (n_{12}v_2)^2 - \frac{245}{24}(n_{12}v_1)^2 v_1^2 \right. \\ & + \frac{179}{12}(n_{12}v_1)(n_{12}v_2) v_1^2 - \frac{235}{24}(n_{12}v_2)^2 v_1^2 + \frac{373}{48} v_1^4 + \frac{529}{24}(n_{12}v_1)^2 (v_1 v_2) \\ & - \frac{97}{6}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) - \frac{719}{24} v_1^2 (v_1 v_2) + \frac{463}{24}(v_1 v_2)^2 - \frac{7}{24}(n_{12}v_1)^2 v_2^2 \\ & \left. - \frac{1}{2}(n_{12}v_1)(n_{12}v_2) v_2^2 + \frac{1}{4}(n_{12}v_2)^2 v_2^2 + \frac{463}{48} v_1^2 v_2^2 - \frac{19}{2}(v_1 v_2) v_2^2 + \frac{45}{16} v_2^4 \right) \\ & + G m_1 m_2 \left(\frac{3}{8}(a_1 v_2)(n_{12}v_1)(n_{12}v_2)^2 + \frac{5}{12}(a_1 v_2)(n_{12}v_2)^3 + \frac{1}{8}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)^3 \right. \\ & + \frac{1}{16}(n_{12}a_1)(n_{12}v_2)^4 + \frac{11}{4}(a_1 v_1)(n_{12}v_2) v_1^2 - (a_1 v_2)(n_{12}v_2) v_1^2 \\ & - 2(a_1 v_1)(n_{12}v_2)(v_1 v_2) + \frac{1}{4}(a_1 v_2)(n_{12}v_2)(v_1 v_2) \\ & + \frac{3}{8}(n_{12}a_1)(n_{12}v_2)^2 (v_1 v_2) - \frac{5}{8}(n_{12}a_1)(n_{12}v_1)^2 v_2^2 + \frac{15}{8}(a_1 v_1)(n_{12}v_2) v_2^2 \\ & \left. - \frac{15}{8}(a_1 v_2)(n_{12}v_2) v_2^2 - \frac{1}{2}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2) v_2^2 \right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{5}{16}(n_{12}a_1)(n_{12}v_2)^2v_2^2\Big) + \frac{5m_1v_1^8}{128} \\
&+ \frac{G^2m_1^2m_2}{r_{12}}\Big(- \frac{235}{24}(a_2v_1)(n_{12}v_1) - \frac{29}{24}(n_{12}a_2)(n_{12}v_1)^2 - \frac{235}{24}(a_1v_2)(n_{12}v_2) \\
&- \frac{17}{6}(n_{12}a_1)(n_{12}v_2)^2 + \frac{185}{16}(n_{12}a_1)v_1^2 - \frac{235}{48}(n_{12}a_2)v_1^2 \\
&- \frac{185}{8}(n_{12}a_1)(v_1v_2) + \frac{20}{3}(n_{12}a_1)v_2^2\Big) \\
&+ \frac{Gm_1m_2}{r_{12}}\Big(- \frac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3 + \frac{1}{8}(n_{12}v_1)(n_{12}v_2)^3v_1^2 + \frac{5}{8}(n_{12}v_2)^4v_1^2 \\
&- \frac{11}{16}(n_{12}v_1)(n_{12}v_2)v_1^4 + \frac{1}{4}(n_{12}v_2)^2v_1^4 + \frac{11}{16}v_1^6 \\
&- \frac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1v_2) + (n_{12}v_1)(n_{12}v_2)v_1^2(v_1v_2) \\
&+ \frac{3}{8}(n_{12}v_2)^2v_1^2(v_1v_2) - \frac{13}{16}v_1^4(v_1v_2) + \frac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2 \\
&+ \frac{1}{16}(v_1v_2)^3 - \frac{5}{8}(n_{12}v_1)^2v_1^2v_2^2 - \frac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2v_2^2 + \frac{1}{16}v_1^4v_2^2 \\
&- \frac{1}{32}v_1^2(v_1v_2)v_2^2\Big) \\
&- \frac{3G^4m_1^4m_2}{8r_{12}^4} + \frac{G^4m_1^3m_2^2}{r_{12}^4}\left(-\frac{9707}{420} + \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right) \\
&+ \frac{G^3m_1^2m_2^2}{r_{12}^3}\Big(\frac{383}{24}(n_{12}v_1)^2 - \frac{889}{48}(n_{12}v_1)(n_{12}v_2) \\
&- \frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2 - \frac{305}{72}v_1^2 + \frac{41}{64}\pi^2(v_1v_{12}) + \frac{439}{144}(v_1v_2)\Big) \\
&+ \frac{G^3m_1^3m_2}{r_{12}^3}\Big(- \frac{8243}{210}(n_{12}v_1)^2 + \frac{15541}{420}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}(n_{12}v_2)^2 + \frac{15611}{1260}v_1^2 \\
&- \frac{17501}{1260}(v_1v_2) + \frac{5}{4}v_2^2 + 22(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right) \\
&- \frac{22}{3}(v_1v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\Big) + 1 \leftrightarrow 2\,. \qquad (5.2d)
\end{aligned}
$$

Next we present the 4PN term. As we have discussed this term is the sum of an instantaneous contribution and a non-local tail piece, say

$$L_{\rm 4PN} = L_{\rm 4PN}^{\rm inst} + L_{\rm 4PN}^{\rm tail}\,. \qquad (5.3)$$

The tail piece has been found in Eqs. (3.14)–(3.15), but here we have to replace the Hadamard partie-finie scale s_0 therein by the particle separation r_{12}. Specifically, we insert r_{12} into the form (3.15) of the action (after integrations by parts), so the Lagrangian reads

$$L_{\rm 4PN}^{\rm tail} \;=\; \frac{G^2M}{5}\,I_{ij}^{(3)}(t)\underset{2r_{12}/c}{\rm Pf}\int_{-\infty}^{+\infty}\frac{{\rm d}t'}{|t-t'|}I_{ij}^{(3)}(t')$$

$$= \frac{G^2 M}{5} I_{ij}^{(3)}(t) \int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2r_{12}}\right) \left[I_{ij}^{(4)}(t-\tau) - I_{ij}^{(4)}(t+\tau)\right] . \tag{5.4}$$

Again, note that when varying the Lagrangian we shall have to take into account the variation of the "constant" $r_{12} = |\boldsymbol{y}_1(t) - \boldsymbol{y}_2(t)|$ in (5.4). The Lagrangian is defined up to a total time derivative, and with the choice made in (5.4), the tail term is a functional of $\boldsymbol{y}_A$, $\boldsymbol{v}_A$, accelerations $\boldsymbol{a}_A$ and also derivatives of accelerations $\boldsymbol{b}_A = \mathrm{d}\boldsymbol{a}_A/\mathrm{d}t$. Spliting for convenience the very long instantaneous contribution according to powers of G as

$$L_{\rm 4PN}^{\rm inst} = L_{\rm 4PN}^{(0)} + G\, L_{\rm 4PN}^{(1)} + G^2\, L_{\rm 4PN}^{(2)} + G^3\, L_{\rm 4PN}^{(3)} + G^4\, L_{\rm 4PN}^{(4)} + G^5\, L_{\rm 4PN}^{(5)} , \tag{5.5}$$

we find

$$L_{\rm 4PN}^{(0)} = \frac{7}{256} m_1 v_1^{10} + 1 \leftrightarrow 2 , \tag{5.6a}$$

$$\begin{aligned}
L_{\rm 4PN}^{(1)} = {} & m_1 m_2 \Bigg[\frac{5}{128}(a_2 n_{12})(n_{12} v_1)^6 + (n_{12} v_1)^5 \Bigg\{ -\frac{13}{64}(a_2 v_2) + \frac{5}{64}(a_2 n_{12})(n_{12} v_2) \Bigg\} \\
& + \frac{33}{16}(a_1 v_1)(n_{12} v_2)(v_1 v_2)^2 + \frac{9}{16}(a_1 n_{12})(v_1 v_2)^3 + (a_2 n_{12})(n_{12} v_1)^4 \Bigg\{ \frac{11}{64}(v_1 v_2) \\
& - \frac{27}{128} v_1^2 \Bigg\} + (a_2 n_{12}) \Bigg\{ (v_1 v_2)^2 v_1^2 + \frac{49}{64}(v_1 v_2) v_1^4 - \frac{75}{128} v_1^6 \Bigg\} \\
& + (n_{12} v_1)^3 \Bigg\{ \frac{3}{32}(a_1 v_2)(n_{12} v_2)^2 - \frac{5}{32}(a_1 n_{12})(n_{12} v_2)^3 - \frac{2}{3}(a_2 v_2)(v_1 v_2) \\
& + \frac{77}{96}(a_2 v_2) v_1^2 + (a_2 n_{12}) \Big[\frac{1}{2}(n_{12} v_2)(v_1 v_2) - \frac{11}{32}(n_{12} v_2) v_1^2 \Big] - \frac{5}{32}(a_1 v_2) v_2^2 \\
& + \frac{15}{32}(a_1 n_{12})(n_{12} v_2) v_2^2 \Bigg\} + (n_{12} v_1)^2 \Bigg\{ \frac{9}{32}(a_1 v_1)(n_{12} v_2)^3 \\
& + \frac{9}{32}(a_1 n_{12})(n_{12} v_2)^2 (v_1 v_2) + (a_2 n_{12}) \Big[-\frac{1}{2}(v_1 v_2)^2 - \frac{13}{32}(v_1 v_2) v_1^2 + \frac{53}{128} v_1^4 \Big] \\
& - \frac{15}{32}(a_1 n_{12})(v_1 v_2) v_2^2 + (n_{12} v_2) \Big[-\frac{11}{16}(a_1 v_2)(v_1 v_2) - \frac{23}{32}(a_1 v_1) v_2^2 \Big] \Bigg\} \\
& + v_1^2 \Bigg\{ -\frac{19}{32}(a_1 v_1)(n_{12} v_2)^3 - \frac{9}{32}(a_1 n_{12})(n_{12} v_2)^2 (v_1 v_2) + \frac{23}{32}(a_1 n_{12})(v_1 v_2) v_2^2 \\
& + (n_{12} v_2) \Big[\frac{33}{16}(a_1 v_2)(v_1 v_2) + \frac{93}{32}(a_1 v_1) v_2^2 \Big] \Bigg\} \\
& + (n_{12} v_1) \Bigg\{ -\frac{9}{16}(a_1 v_1)(n_{12} v_2)^2 (v_1 v_2) + \frac{27}{16}(a_1 v_2)(v_1 v_2)^2 \\
& - \frac{11}{16}(a_1 n_{12})(n_{12} v_2)(v_1 v_2)^2 - \frac{123}{64}(a_2 v_2) v_1^4 + (a_2 n_{12}) \Big[-(n_{12} v_2)(v_1 v_2) v_1^2 \\
& + \frac{31}{64}(n_{12} v_2) v_1^4 \Big] + \frac{23}{16}(a_1 v_1)(v_1 v_2) v_2^2 + v_1^2 \Big[-\frac{9}{32}(a_1 v_2)(n_{12} v_2)^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{9}{32}(a_1n_{12})(n_{12}v_2)^3+\frac{7}{2}(a_2v_2)(v_1v_2)+\frac{23}{32}(a_1v_2)v_2^2-\frac{23}{32}(a_1n_{12})(n_{12}v_2)v_2^2\Big]\Big\}\\
&+(a_2v_1)\Big\{\frac{13}{64}(n_{12}v_1)^5+\frac{11}{64}(n_{12}v_1)^4(n_{12}v_2)+2(n_{12}v_2)(v_1v_2)v_1^2\\
&+(n_{12}v_1)^2\Big[-(n_{12}v_2)(v_1v_2)-\frac{13}{32}(n_{12}v_2)v_1^2\Big]+\frac{49}{64}(n_{12}v_2)v_1^4\\
&+(n_{12}v_1)^3\Big[\frac{1}{4}(n_{12}v_2)^2-\frac{1}{16}(v_1v_2)-\frac{77}{96}v_1^2-\frac{1}{3}v_2^2\Big]+(n_{12}v_1)\Big[\frac{3}{2}(v_1v_2)^2+\frac{123}{64}v_1^4\\
&+v_1^2\big(-\frac{1}{2}(n_{12}v_2)^2-\frac{3}{16}(v_1v_2)+\frac{7}{4}v_2^2\big)\Big]\Big\}+\frac{1}{r_{12}}\Big\{-\frac{17}{32}(v_1v_2)^4+\frac{75}{128}v_1^8\\
&+v_1^6\Big[-\frac{11}{32}(n_{12}v_2)^2-\frac{5}{4}(v_1v_2)+\frac{5}{8}v_2^2\Big]+(n_{12}v_1)^5\Big[\frac{35}{128}(n_{12}v_2)^3-\frac{75}{128}(n_{12}v_2)v_2^2\Big]\\
&+(n_{12}v_1)^4\Big[-\frac{35}{256}(n_{12}v_2)^4-\frac{75}{128}(n_{12}v_2)^2(v_1v_2)+\frac{75}{128}(v_1v_2)v_2^2\Big]\\
&+v_1^2\Big[\frac{37}{16}(n_{12}v_2)^2(v_1v_2)^2+\frac{51}{32}(v_1v_2)^3-\frac{7}{4}(v_1v_2)^2v_2^2\Big]\\
&+(n_{12}v_1)^3\Big[\frac{15}{16}(n_{12}v_2)^3(v_1v_2)+\frac{39}{32}(n_{12}v_2)(v_1v_2)^2+v_1^2\big(-\frac{55}{64}(n_{12}v_2)^3\\
&+\frac{99}{64}(n_{12}v_2)v_2^2\big)\Big]+(n_{12}v_1)^2\Big[-\frac{57}{32}(n_{12}v_2)^2(v_1v_2)^2-\frac{29}{32}(v_1v_2)^3\\
&+v_1^2\big(\frac{45}{64}(n_{12}v_2)^4+(n_{12}v_2)^2(\frac{99}{64}(v_1v_2)-\frac{3}{4}v_2^2)-\frac{99}{64}(v_1v_2)v_2^2\big)\Big]\\
&+(n_{12}v_1)\Big[\frac{11}{8}(n_{12}v_2)(v_1v_2)^3+v_1^4\big(\frac{93}{128}(n_{12}v_2)^3-\frac{185}{128}(n_{12}v_2)v_2^2\big)\\
&+v_1^2\big(-\frac{21}{8}(n_{12}v_2)^3(v_1v_2)+(n_{12}v_2)(-\frac{73}{32}(v_1v_2)^2+\frac{19}{8}(v_1v_2)v_2^2)\big)\Big]\\
&+v_1^4\Big[-\frac{15}{128}(n_{12}v_2)^4+\frac{3}{4}(v_1v_2)^2+(n_{12}v_2)^2\big(\frac{3}{128}(v_1v_2)+\frac{23}{64}v_2^2\big)-\frac{7}{128}(v_1v_2)v_2^2\\
&+\frac{3}{256}v_2^4\Big]\Big\}\Big]+1\leftrightarrow 2\,,
\end{aligned}
\tag{5.6b}
$$

$$
\begin{aligned}
L^{(2)}_{\rm 4PN}=m_1^2m_2\Bigg[&\frac{1}{r_{12}}\Big\{\Big[-\frac{4247}{960}(a_1n_{12})-\frac{2}{3}(a_2n_{12})\Big](n_{12}v_1)^4+(n_{12}v_1)^3\Big[-\frac{29}{12}(a_2v_1)\\
&-\frac{4501}{480}(a_1v_2)+\frac{51}{8}(a_2v_2)+\frac{519}{80}(a_1n_{12})(n_{12}v_2)-\frac{25}{6}(a_2n_{12})(n_{12}v_2)\Big]\\
&-\frac{367}{10}(a_2v_2)(n_{12}v_2)(v_1v_2)+\Big[-\frac{13129}{480}(a_1v_2)(n_{12}v_2)+\frac{437}{30}(a_2v_2)(n_{12}v_2)\Big]v_1^2\\
&+(a_1v_2)\Big[\frac{8653}{480}(n_{12}v_2)^3+(n_{12}v_2)\big(\frac{8291}{240}(v_1v_2)-\frac{6669}{160}v_2^2\big)\Big]+(a_2v_1)\Big[\frac{42}{5}(n_{12}v_2)^3\\
&-\frac{107}{12}(n_{12}v_2)v_1^2+(n_{12}v_2)\big(\frac{112}{15}(v_1v_2)-\frac{367}{20}v_2^2\big)\Big]+(a_2n_{12})\Big[\frac{126}{5}(n_{12}v_2)^2(v_1v_2)\\
&+\frac{56}{15}(v_1v_2)^2+\frac{19}{12}v_1^4-\frac{367}{20}(v_1v_2)v_2^2+v_1^2\big(-\frac{47}{5}(n_{12}v_2)^2-\frac{107}{12}(v_1v_2)+\frac{437}{60}v_2^2\big)\Big]\\
&+(n_{12}v_1)^2\Big[\frac{1}{2}(a_2v_1)(n_{12}v_2)+\frac{10463}{480}(a_1v_2)(n_{12}v_2)-\frac{77}{15}(a_2v_2)(n_{12}v_2)
\end{aligned}
$$

$$
\begin{aligned}
&+ (a_2 n_{12})\big(\frac{28}{5}(n_{12}v_2)^2 + \frac{1}{2}(v_1v_2) - \frac{89}{12}v_1^2 - \frac{77}{30}v_2^2\big) + (a_1 n_{12})\big(-\frac{9661}{480}(n_{12}v_2)^2 \\
&- \frac{94}{15}(v_1v_2) + \frac{3017}{240}v_1^2 + \frac{2177}{240}v_2^2\big)\Big] + (a_1v_1)\Big[\frac{2507}{160}(n_{12}v_1)^3 - \frac{16183}{480}(n_{12}v_1)^2(n_{12}v_2) \\
&- \frac{2543}{160}(n_{12}v_2)^3 + \frac{16589}{480}(n_{12}v_2)v_1^2 + (n_{12}v_1)\big(\frac{6261}{160}(n_{12}v_2)^2 + \frac{5213}{80}(v_1v_2) \\
&- \frac{16429}{480}v_1^2 - \frac{5113}{160}v_2^2\big) + (n_{12}v_2)\big(-\frac{13129}{240}(v_1v_2) + \frac{18191}{480}v_2^2\big)\Big] \\
&+ (n_{12}v_1)\Big[\big(\frac{7063}{160}(a_1v_2) - \frac{583}{24}(a_2v_2)\big)v_1^2 + (a_1n_{12})\big(\frac{2233}{240}(n_{12}v_2)^3 - \frac{489}{40}(n_{12}v_2)v_1^2 \\
&+ (n_{12}v_2)(\frac{37}{60}(v_1v_2) - \frac{56}{15}v_2^2)\big) + (a_2v_2)\big(\frac{127}{10}(n_{12}v_2)^2 + \frac{329}{30}(v_1v_2) - \frac{83}{5}v_2^2\big) \\
&+ (a_2v_1)\big(-\frac{77}{15}(n_{12}v_2)^2 + \frac{1}{4}(v_1v_2) + \frac{299}{12}v_1^2 + \frac{329}{60}v_2^2\big) + (a_1v_2)\big(-\frac{13807}{480}(n_{12}v_2)^2 \\
&- \frac{4469}{240}(v_1v_2) + \frac{9511}{480}v_2^2\big) + (a_2n_{12})\big(-\frac{184}{15}(n_{12}v_2)^3 + \frac{55}{3}(n_{12}v_2)v_1^2 \\
&+ (n_{12}v_2)(-\frac{154}{15}(v_1v_2) + \frac{127}{10}v_2^2)\big)\Big] + (a_1n_{12})\Big[-\frac{1811}{960}(n_{12}v_2)^4 + \frac{1341}{80}(v_1v_2)^2 \\
&+ \frac{9143}{960}v_1^4 + (n_{12}v_2)^2\big(\frac{31}{10}(v_1v_2) - \frac{3}{5}v_2^2\big) - \frac{4277}{240}(v_1v_2)v_2^2 + v_1^2\big(\frac{109}{30}(n_{12}v_2)^2 \\
&- \frac{1213}{240}(v_1v_2) + \frac{1363}{480}v_2^2\big) + \frac{1603}{320}v_2^4\Big]\Big\} + \frac{1}{r_{12}^2}\Big\{\frac{11}{40}(n_{12}v_1)^6 + \frac{109}{40}(n_{12}v_1)^5(n_{12}v_2) \\
&+ \frac{110}{3}(n_{12}v_2)^4(v_1v_2) - \frac{527}{120}(v_1v_2)^3 + \frac{33}{16}v_1^6 + (n_{12}v_1)^4\Big[10(n_{12}v_2)^2 + \frac{105}{16}(v_1v_2) \\
&- \frac{727}{48}v_1^2 - \frac{65}{6}v_2^2\Big] + v_1^4\Big[\frac{37}{3}(n_{12}v_2)^2 + \frac{175}{16}(v_1v_2) - \frac{287}{48}v_2^2\Big] - \frac{649}{60}(v_1v_2)^2v_2^2 \\
&+ (n_{12}v_1)^3\Big[-\frac{237}{10}(n_{12}v_2)^3 + \frac{541}{12}(n_{12}v_2)v_1^2 + (n_{12}v_2)\big(-\frac{91}{6}(v_1v_2) + \frac{1691}{60}v_2^2\big)\Big] \\
&+ (n_{12}v_1)\Big[-\frac{92}{5}(n_{12}v_2)^5 - \frac{207}{8}(n_{12}v_2)v_1^4 + v_1^2\big(\frac{794}{15}(n_{12}v_2)^3 + (n_{12}v_2)(\frac{617}{6}(v_1v_2) \\
&- \frac{2513}{40}v_2^2)\big) + (n_{12}v_2)^3\big(-\frac{1052}{15}(v_1v_2) + \frac{113}{3}v_2^2\big) + (n_{12}v_2)\big(-\frac{1109}{60}(v_1v_2)^2 \\
&+ \frac{1144}{15}(v_1v_2)v_2^2 - \frac{171}{8}v_2^4\big)\Big] + v_1^2\Big[-\frac{78}{5}(n_{12}v_2)^4 - \frac{293}{24}(v_1v_2)^2 + \frac{2959}{240}(v_1v_2)v_2^2 \\
&+ (n_{12}v_2)^2\big(-\frac{623}{15}(v_1v_2) + \frac{1819}{60}v_2^2\big) - \frac{1169}{240}v_2^4\Big] + (n_{12}v_2)^2\Big[\frac{189}{5}(v_1v_2)^2 \\
&- \frac{273}{4}(v_1v_2)v_2^2 + \frac{3}{16}v_2^4\Big] + \frac{75}{8}(v_1v_2)v_2^4 + (n_{12}v_1)^2\Big[\frac{148}{5}(n_{12}v_2)^4 - \frac{3}{2}(v_1v_2)^2 \\
&+ \frac{231}{16}v_1^4 + (n_{12}v_2)^2\big(\frac{2063}{60}(v_1v_2) - \frac{1253}{30}v_2^2\big) - \frac{3503}{240}(v_1v_2)v_2^2 + v_1^2\big(-\frac{883}{12}(n_{12}v_2)^2 \\
&- \frac{1009}{16}(v_1v_2) + \frac{1693}{48}v_2^2\big) + \frac{989}{120}v_2^4\Big] + \frac{115}{32}v_2^6\Big\}\Big] + 1 \leftrightarrow 2\,, \qquad (5.6c)
\end{aligned}
$$

$$
L^{(3)}_{\rm 4PN} = \frac{m_1^3 m_2}{r_{12}^2}\Bigg[\Big\{\frac{2582267}{16800}(a_1n_{12}) - \frac{89763}{1400}(a_2n_{12})\Big\}(n_{12}v_1)^2 + \Big\{\frac{1111}{25200}
$$

$$+\frac{110}{3}\ln\Big[\frac{r_{12}}{r_1'}\Big]\Big\}(a_2v_1)(n_{12}v_2)+\frac{487591}{25200}(a_1v_2)(n_{12}v_2)-6(a_2v_2)(n_{12}v_2)$$
$$+(a_1v_1)\Big\{-\frac{163037}{1200}(n_{12}v_1)+\Big[\frac{15929}{1400}-\frac{110}{3}\ln\big(\frac{r_{12}}{r_1'}\big)\Big](n_{12}v_2)\Big\}$$
$$+(n_{12}v_1)\Big\{\frac{435011}{5040}(a_2v_1)+\Big[\frac{31309}{560}+44\ln\big(\frac{r_{12}}{r_1'}\big)\Big](a_1v_2)+\Big[-\frac{212641}{6300}$$
$$-44\ln\big(\frac{r_{12}}{r_1'}\big)\Big](a_2v_2)-\frac{268169}{2100}(a_1n_{12})(n_{12}v_2)+\Big[\frac{5421}{700}$$
$$+22\ln\big(\frac{r_{12}}{r_1'}\big)\Big](a_2n_{12})(n_{12}v_2)\Big\}+(a_1n_{12})\Big\{\frac{27203}{1200}(n_{12}v_2)^2+\Big[\frac{888179}{6300}$$
$$-44\ln\big(\frac{r_{12}}{r_1'}\big)\Big](v_1v_2)+\Big[-\frac{1391897}{12600}+44\ln\big(\frac{r_{12}}{r_1'}\big)\Big]v_1^2-\frac{89129}{2016}v_2^2\Big\}$$
$$+(a_2n_{12})\Big\{\frac{51}{2}(n_{12}v_2)^2+\Big[\frac{1111}{25200}+\frac{110}{3}\ln\big(\frac{r_{12}}{r_1'}\big)\Big](v_1v_2)+\Big[\frac{128867}{8400}$$
$$-\frac{110}{3}\ln\big(\frac{r_{12}}{r_1'}\big)\Big]v_1^2-3v_2^2\Big\}\Big]+\frac{m_1^3m_2}{r_{12}^3}\Big[-\frac{906349}{3360}(n_{12}v_1)^4+\frac{399851}{672}(n_{12}v_1)^3(n_{12}v_2)$$
$$+\frac{85}{2}(n_{12}v_2)^4+\Big\{-\frac{34003}{525}+\frac{110}{3}\ln\Big[\frac{r_{12}}{r_1'}\Big]\Big\}(v_1v_2)^2+\Big\{-\frac{1195969}{16800}$$
$$+\frac{55}{3}\ln\Big[\frac{r_{12}}{r_1'}\Big]\Big\}v_1^4+(n_{12}v_2)^2\Big\{\Big[-\frac{28403}{1680}+99\ln\big(\frac{r_{12}}{r_1'}\big)\Big](v_1v_2)-\frac{131}{4}v_2^2\Big\}$$
$$+\Big\{-\frac{193229}{25200}-\frac{44}{3}\ln\Big[\frac{r_{12}}{r_1'}\Big]\Big\}(v_1v_2)v_2^2+v_1^2\Big\{\Big[\frac{735527}{8400}-99\ln\big(\frac{r_{12}}{r_1'}\big)\Big](n_{12}v_2)^2$$
$$+\Big[\frac{7879619}{50400}-55\ln\big(\frac{r_{12}}{r_1'}\big)\Big](v_1v_2)+\Big[-\frac{540983}{25200}+\frac{44}{3}\ln\big(\frac{r_{12}}{r_1'}\big)\Big]v_2^2\Big\}$$
$$+(n_{12}v_1)^2\Big\{\Big[-\frac{46577}{140}-55\ln\big(\frac{r_{12}}{r_1'}\big)\Big](n_{12}v_2)^2-\frac{1160909}{2400}(v_1v_2)+\Big[\frac{1732751}{4200}$$
$$-55\ln\big(\frac{r_{12}}{r_1'}\big)\Big]v_1^2+\Big[\frac{77801}{840}+66\ln\big(\frac{r_{12}}{r_1'}\big)\Big]v_2^2\Big\}+(n_{12}v_1)\Big\{\Big[-\frac{9559}{280}$$
$$+55\ln\big(\frac{r_{12}}{r_1'}\big)\Big](n_{12}v_2)^3+\Big[-\frac{2617007}{5600}+165\ln\big(\frac{r_{12}}{r_1'}\big)\Big](n_{12}v_2)v_1^2+(n_{12}v_2)\Big[\big(\frac{65767}{150}$$
$$-88\ln(\frac{r_{12}}{r_1'})\big)(v_1v_2)+\big(-\frac{129667}{4200}-88\ln(\frac{r_{12}}{r_1'})\big)v_2^2\Big]\Big\}+\frac{139}{16}v_2^4\Big]$$
$$+\frac{m_1^2m_2^2}{r_{12}^2}\Big[\big((\frac{17811527}{33600}-\frac{8769}{512}\pi^2)(a_1n_{12})+(-\frac{12448339}{33600}+\frac{1017}{64}\pi^2)(a_2n_{12})\big)(n_{12}v_1)^2$$
$$+(a_1v_1)\big((-\frac{3168457}{10080}-\frac{2095}{256}\pi^2)(n_{12}v_1)+(\frac{11535007}{50400}+\frac{177}{64}\pi^2)(n_{12}v_2)\big)$$
$$+(n_{12}v_1)\big((\frac{12111653}{50400}+\frac{133}{8}\pi^2)(a_2v_1)+(\frac{12496303}{50400}+\frac{1023}{64}\pi^2)(a_1v_2)+(-\frac{4383363}{5600}$$
$$+\frac{2157}{64}\pi^2)(a_1n_{12})(n_{12}v_2)\big)+\big(\frac{3213347}{100800}+\frac{55}{32}\pi^2\big)(a_2n_{12})v_1^2+(a_1n_{12})\big((\frac{1263331}{16800}$$

$$
\begin{aligned}
&+\frac{1107}{64}\pi^2)(v_1v_2)-\frac{11}{4608}(29656+1989\pi^2)v_1^2\big)\Big]+\frac{m_1^2m_2^2}{r_{12}^3}\Big[\big(-\frac{465431}{480}\\
&+\frac{27075}{1024}\pi^2\big)(n_{12}v_1)^4+\big(\frac{10701209}{3360}-\frac{53445}{512}\pi^2\big)(n_{12}v_1)^3(n_{12}v_2)+\big(-\frac{8248733}{50400}\\
&-\frac{8379}{512}\pi^2\big)(v_1v_2)^2+(n_{12}v_1)^2\big((-\frac{14873539}{6720}+\frac{79815}{1024}\pi^2)(n_{12}v_2)^2+(-\frac{27374071}{16800}\\
&-\frac{9033}{512}\pi^2)(v_1v_2)+(\frac{2079017}{2100}-\frac{2037}{512}\pi^2)v_1^2\big)+(n_{12}v_1)\big((\frac{1040673}{700}\\
&+\frac{4587}{256}\pi^2)(n_{12}v_2)(v_1v_2)+(-\frac{5303279}{3360}+\frac{7953}{512}\pi^2)(n_{12}v_2)v_1^2\big)+\big(-\frac{1177829}{10080}\\
&-\frac{4057}{1024}\pi^2\big)v_1^4+v_1^2\big((\frac{12260653}{16800}-\frac{6057}{512}\pi^2)(n_{12}v_2)^2+(\frac{17958959}{50400}+\frac{11049}{512}\pi^2)(v_1v_2)\\
&+(-\frac{7672087}{100800}-\frac{1283}{1024}\pi^2)v_2^2\big)\Big]+1\leftrightarrow 2\,,
\end{aligned}
\tag{5.6d}
$$

$$
\begin{aligned}
L^{(4)}_{\rm 4PN}=&\frac{1}{r_{12}^3}\bigg\{m_1^4m_2\Big[\frac{1691807}{25200}(a_1n_{12})-\frac{149}{6}(a_2n_{12})\Big]+m_1^3m_2^2\Big[\big(-\frac{2470667}{16800}\\
&+\frac{1099}{96}\pi^2\big)(a_1n_{12})+\big(\frac{9246557}{50400}-\frac{555}{64}\pi^2\big)(a_2n_{12})\Big]\bigg\}+\frac{1}{r_{12}^4}\bigg\{m_1^4m_2\Big[\big(\frac{2146}{75}\\
&-\frac{880}{3}\ln(\frac{r_{12}}{r'_1})\big)(n_{12}v_1)^2+\big(\frac{3461}{50}+\frac{880}{3}\ln(\frac{r_{12}}{r'_1})\big)(n_{12}v_1)(n_{12}v_2)-\frac{1165}{12}(n_{12}v_2)^2\\
&+\big(-\frac{11479}{300}-\frac{220}{3}\ln(\frac{r_{12}}{r'_1})\big)(v_1v_2)+\big(\frac{317}{25}+\frac{220}{3}\ln(\frac{r_{12}}{r'_1})\big)v_1^2+\frac{1237}{48}v_2^2\Big]\\
&+m_1^3m_2^2\Big[\big(\frac{9102109}{16800}-\frac{3737}{96}\pi^2-\frac{286}{3}\ln(\frac{r_{12}}{r'_1})\big)(n_{12}v_1)^2+\big(-\frac{1409257}{1680}+\frac{179}{4}\pi^2\\
&+44\ln(\frac{r_{12}}{r'_1})+64\ln(\frac{r_{12}}{r'_2})\big)(n_{12}v_1)(n_{12}v_2)+\big(\frac{5553521}{16800}-\frac{559}{96}\pi^2+\frac{110}{3}\ln(\frac{r_{12}}{r'_1})\\
&-64\ln(\frac{r_{12}}{r'_2})\big)(n_{12}v_2)^2+\big(\frac{1637809}{6300}-\frac{2627}{192}\pi^2-\frac{154}{3}\ln(\frac{r_{12}}{r'_1})-16\ln(\frac{r_{12}}{r'_2})\big)(v_1v_2)\\
&+\big(-\frac{1887121}{12600}+\frac{527}{48}\pi^2+\frac{121}{3}\ln(\frac{r_{12}}{r'_1})\big)v_1^2+\big(-\frac{44389}{450}+\frac{173}{64}\pi^2+\frac{22}{3}\ln(\frac{r_{12}}{r'_1})\\
&+16\ln(\frac{r_{12}}{r'_2})\big)v_2^2\Big]\bigg\}+1\leftrightarrow 2\,,
\end{aligned}
\tag{5.6e}
$$

$$
\begin{aligned}
L^{(5)}_{\rm 4PN}=&\frac{3}{8}\frac{m_1^5m_2}{r_{12}^5}+\frac{m_1^3m_2^3}{r_{12}^5}\big(\frac{587963}{5600}-\frac{71}{32}\pi^2-\frac{110}{3}\ln(\frac{r_{12}}{r'_1})\big)+\frac{m_1^4m_2^2}{r_{12}^5}\big(\frac{1690841}{25200}+\frac{105}{32}\pi^2\\
&-\frac{242}{3}\ln(\frac{r_{12}}{r'_1})-16\ln(\frac{r_{12}}{r'_2})\big)+1\leftrightarrow 2\,.
\end{aligned}
\tag{5.6f}
$$

These expressions depend linearly on accelerations $\boldsymbol{a}_A$ and do not contain derivatives of accelerations. The only remaining constants are the two UV scales r'_1 and r'_2 which are gauge constants and will disappear from physical invariant results. The correct value of α given by (4.32) has been inserted.

We have checked that the full 4PN Lagrangian is invariant under global Lorentz-Poincaré

transformations. Indeed, the tail part of the Lagrangian is separately Lorentz invariant. We have transformed the variables $\boldsymbol{y}_A$, $\boldsymbol{v}_A$ and $\boldsymbol{a}_A$ in Eqs. (5.2) and (5.6) according to a Lorentz boost (with constant boost velocity), and verified that the Lagrangian is merely changed at linear order by a total time derivative irrelevant for the dynamics.

Finally, we have verified that our 4PN Lagrangian, when restricted to terms up to quadratic order in Newton's constant G, *i.e.*, for $L^{(0)}_{\rm 4PN}$, $L^{(1)}_{\rm 4PN}$ and $L^{(2)}_{\rm 4PN}$, is equivalent to the Lagrangian obtained using the effective field theory by Foffa & Sturani [64].

B. Removal of accelerations from the Lagrangian

We shall now perform a shift of the particle's dynamical variables (or "contact" transformation) to a new Lagrangian whose instantaneous part will be ordinary, in the sense that it depends only on positions and velocities. Furthermore, the shift will be such that the logarithms $\ln(r_{12}/r'_1)$ and $\ln(r_{12}/r'_2)$ are cancelled. This directly shows that the scales r'_A are pure gauge constants. Here we simply report the resulting ordinary Lagrangian, which is comparatively much simpler than the harmonic one (the shift is too lengthy to be presented here). Our choice for this ordinary Lagrangian is that it is the closest possible one from the ADM Lagrangian (see the discussion in Sec. V C). We have

$$\tilde{L} = \tilde{L}_{\rm N} + \frac{1}{c^2}\tilde{L}_{\rm 1PN} + \frac{1}{c^4}\tilde{L}_{\rm 2PN} + \frac{1}{c^6}\tilde{L}_{\rm 3PN} + \frac{1}{c^8}\tilde{L}_{\rm 4PN} + \mathcal{O}\left(10\right) , \tag{5.7}$$

where $\tilde{L}_{\rm N}$ and $\tilde{L}_{\rm 1PN}$ are actually unchanged since the shift starts only at the 2PN order, and

$$\tilde{L}_{\rm N} = \frac{Gm_1m_2}{2r_{12}} + \frac{m_1v_1^2}{2} + 1 \leftrightarrow 2\,, \tag{5.8a}$$

$$\tilde{L}_{\rm 1PN} = -\frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2)\right) + 1 \leftrightarrow 2\,, \tag{5.8b}$$

$$\begin{aligned}\tilde{L}_{\rm 2PN} = {}& \frac{1}{16}m_1v_1^6 + \frac{Gm_1m_2}{r_{12}}\big(\frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 + \frac{1}{8}(v_1v_2)^2 + (n_{12}v_1)(\frac{3}{4}(n_{12}v_2)(v_1v_2)\\ & - \frac{1}{4}(n_{12}v_2)v_1^2) + \frac{7}{8}v_1^4 + v_1^2(-\frac{5}{8}(n_{12}v_2)^2 - \frac{7}{4}(v_1v_2) + \frac{11}{16}v_2^2)\big)\\ & + \frac{G^2m_1^2m_2}{r_{12}^2}\big(\frac{15}{8}(n_{12}v_1)^2 - \frac{15}{4}(v_1v_2) + \frac{11}{8}v_1^2 + 2v_2^2\big) + \frac{1}{4}\frac{G^3m_1^3m_2}{r_{12}^3} + \frac{5}{8}\frac{G^3m_1^2m_2^2}{r_{12}^3}\\ & + 1 \leftrightarrow 2,\end{aligned} \tag{5.8c}$$

$$
\begin{aligned}
\tilde{L}_{\rm 3PN} &= \frac{5}{128}m_1v_1^8 + \frac{Gm_1m_2}{r_{12}}\biggl\{-\frac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3 + \frac{1}{16}(v_1v_2)^3 + \frac{11}{16}v_1^6 \\
&+ (n_{12}v_1)\Bigl[\frac{5}{16}(n_{12}v_2)(v_1v_2)^2 - \frac{3}{16}(n_{12}v_2)v_1^4 + v_1^2\bigl(\frac{9}{16}(n_{12}v_2)^3 + (n_{12}v_2)(\frac{3}{4}(v_1v_2) \\
&- \frac{9}{32}v_2^2)\bigr)\Bigr] + (n_{12}v_1)^2\Bigl[-\frac{15}{32}(n_{12}v_2)^2(v_1v_2) + v_1^2\bigl(\frac{3}{16}(n_{12}v_2)^2 - \frac{5}{16}v_2^2\bigr)\Bigr] \\
&+ v_1^4\Bigl[-\frac{5}{16}(n_{12}v_2)^2 - \frac{21}{16}(v_1v_2) + \frac{7}{8}v_2^2\Bigr] + v_1^2\Bigl[-\frac{1}{16}(n_{12}v_2)^2(v_1v_2) + \frac{1}{8}(v_1v_2)^2 \\
&- \frac{15}{32}(v_1v_2)v_2^2\Bigr]\biggr\} + \frac{G^2m_1^2m_2}{r_{12}^2}\biggl\{-\frac{5}{12}(n_{12}v_1)^4 - \frac{13}{8}(n_{12}v_1)^3(n_{12}v_2) + \frac{341}{48}(v_1v_2)^2 \\
&+ \frac{21}{16}v_1^4 + (n_{12}v_1)\Bigl[\frac{1}{4}(n_{12}v_2)v_1^2 + (n_{12}v_2)\bigl(\frac{1}{3}(v_1v_2) - v_2^2\bigr)\Bigr] - \frac{71}{8}(v_1v_2)v_2^2 \\
&+ (n_{12}v_1)^2\Bigl[-\frac{23}{24}(n_{12}v_2)^2 - \frac{1}{2}(v_1v_2) + \frac{13}{16}v_1^2 + \frac{29}{24}v_2^2\Bigr] + v_1^2\Bigl[\frac{5}{6}(n_{12}v_2)^2 - \frac{97}{16}(v_1v_2) \\
&+ \frac{43}{12}v_2^2\Bigr] + \frac{47}{16}v_2^4\biggr\} + \frac{G^3m_1^2m_2^2}{r_{12}^3}\biggl\{\frac{1}{64}\Bigl[292 + 3\pi^2\Bigr](n_{12}v_1)^2 + \Bigl[-11 \\
&- \frac{3}{64}\pi^2\Bigr](n_{12}v_1)(n_{12}v_2) + \frac{1}{64}\Bigl[472 + \pi^2\Bigr](v_1v_2) + \Bigl[-\frac{265}{48} - \frac{1}{64}\pi^2\Bigr]v_1^2\biggr\} \\
&+ \frac{G^3m_1^3m_2}{r_{12}^3}\biggl\{-5(n_{12}v_1)^2 - \frac{1}{8}(n_{12}v_1)(n_{12}v_2) - \frac{27}{8}(v_1v_2) + \frac{173}{48}v_1^2 + \frac{13}{8}v_2^2\biggr\} \\
&- \frac{1}{8}\frac{G^4m_1^4m_2}{r_{12}^4} + \frac{1}{96}\biggl\{-908 + 63\pi^2\biggr\}\frac{G^4m_1^3m_2^2}{r_{12}^4} + 1 \leftrightarrow 2\,. \qquad \text{(5.8d)}
\end{aligned}
$$

Next the 4PN term is of the form

$$
\tilde{L}_{\rm 4PN} = \tilde{L}^{\rm inst}_{\rm 4PN} + L^{\rm tail}_{\rm 4PN}\,, \qquad \text{(5.9a)}
$$

$$
\tilde{L}^{\rm inst}_{\rm 4PN} = \tilde{L}^{(0)}_{\rm 4PN} + G\,\tilde{L}^{(1)}_{\rm 4PN} + G^2\,\tilde{L}^{(2)}_{\rm 4PN} + G^3\,\tilde{L}^{(3)}_{\rm 4PN} + G^4\,\tilde{L}^{(4)}_{\rm 4PN} + G^5\,\tilde{L}^{(5)}_{\rm 4PN}\,, \qquad \text{(5.9b)}
$$

where the tail piece $L^{\rm tail}_{\rm 4PN}$ is exactly the same as in Eqs. (5.4) and where

$$
\tilde{L}^{(0)}_{\rm 4PN} = \frac{7}{256}m_1v_1^{10} + 1 \leftrightarrow 2\,, \qquad \text{(5.10a)}
$$

$$
\begin{aligned}
\tilde{L}^{(1)}_{\rm 4PN} &= \frac{m_1m_2}{r_{12}}\biggl\{-\frac{25}{64}(n_{12}v_1)^3(n_{12}v_2)(v_1v_2)^2 + \frac{3}{64}(n_{12}v_1)^2(v_1v_2)^3 + \frac{75}{128}v_1^8 \\
&+ v_1^6\Bigl[-\frac{5}{32}(n_{12}v_1)(n_{12}v_2) - \frac{15}{64}(n_{12}v_2)^2 - \frac{35}{32}(v_1v_2) + \frac{45}{64}v_2^2\Bigr] \\
&+ (n_{12}v_1)^5\Bigl[\frac{35}{256}(n_{12}v_2)^3 - \frac{55}{256}(n_{12}v_2)v_2^2\Bigr] + (n_{12}v_1)^4\Bigl[\frac{85}{256}(n_{12}v_2)^2(v_1v_2) \\
&+ \frac{23}{256}(v_1v_2)v_2^2\Bigr] + v_1^2\Bigl[-\frac{1}{8}(n_{12}v_2)^2(v_1v_2)^2 + \frac{9}{64}(v_1v_2)^3 + \frac{1}{32}(v_1v_2)^2v_2^2 \\
&+ (n_{12}v_1)^3\bigl(-\frac{85}{128}(n_{12}v_2)^3 + \frac{115}{128}(n_{12}v_2)v_2^2\bigr) + (n_{12}v_1)^2\bigl(\frac{5}{32}(n_{12}v_2)^4 \\
&+ (n_{12}v_2)^2(-\frac{135}{128}(v_1v_2) - \frac{21}{64}v_2^2) - \frac{19}{128}(v_1v_2)v_2^2\bigr) + (n_{12}v_1)\bigl(\frac{1}{2}(n_{12}v_2)^3(v_1v_2)
\end{aligned}
$$

$$+ (n_{12}v_2)\big(\frac{53}{64}(v_1v_2)^2 - \frac{1}{16}(v_1v_2)v_2^2\big)\Big] + v_1^4\Big[-\frac{7}{32}(n_{12}v_2)^4 + \frac{3}{32}(v_1v_2)^2$$
$$+ (n_{12}v_1)\big(\frac{183}{256}(n_{12}v_2)^3 + (n_{12}v_2)\big(\frac{9}{16}(v_1v_2) - \frac{167}{256}v_2^2\big)\big) + (n_{12}v_1)^2\big(\frac{9}{64}(n_{12}v_2)^2$$
$$- \frac{15}{64}v_2^2\big) + (n_{12}v_2)^2\big(-\frac{23}{256}(v_1v_2) + \frac{3}{16}v_2^2\big) - \frac{185}{256}(v_1v_2)v_2^2 + \frac{31}{128}v_2^4\Big]\Big\} + 1 \leftrightarrow 2\,, \tag{5.10b}$$

$$\tilde{L}^{(2)}_{\rm 4PN} = \frac{m_1^2m_2}{r_{12}^2}\Big\{-\frac{369}{160}(n_{12}v_1)^6 + \frac{549}{128}(n_{12}v_1)^5(n_{12}v_2) - \frac{21}{16}(n_{12}v_2)^2(v_1v_2)^2 - \frac{53}{96}(v_1v_2)^3$$
$$+ \frac{143}{64}v_1^6 + (n_{12}v_1)^4\Big[\frac{2017}{1280}(n_{12}v_2)^2 - \frac{1547}{256}(v_1v_2) + \frac{243}{64}v_1^2 - \frac{4433}{1920}v_2^2\Big]$$
$$+ \frac{335}{32}(v_1v_2)^2v_2^2 + v_1^4\Big[\frac{1869}{1280}(n_{12}v_2)^2 - \frac{1947}{256}(v_1v_2) + \frac{5173}{1280}v_2^2\Big]$$
$$+ (n_{12}v_1)^3\Big[-\frac{11}{8}(n_{12}v_2)^3 - \frac{81}{16}(n_{12}v_2)v_1^2 + (n_{12}v_2)\big(\frac{4531}{320}(v_1v_2) + \frac{205}{96}v_2^2\big)\Big]$$
$$+ (n_{12}v_1)\Big[\frac{7}{2}(n_{12}v_2)^3(v_1v_2) + \frac{295}{128}(n_{12}v_2)v_1^4 + v_1^2\big(\frac{841}{192}(n_{12}v_2)^3$$
$$+ (n_{12}v_2)(-\frac{771}{160}(v_1v_2) - \frac{125}{32}v_2^2)\big) + (n_{12}v_2)\big(\frac{37}{192}(v_1v_2)^2 + \frac{15}{4}(v_1v_2)v_2^2 - \frac{3}{2}v_2^4\big)\Big]$$
$$+ (n_{12}v_1)^2\Big[\frac{7}{4}(n_{12}v_2)^4 - \frac{5629}{1280}(v_1v_2)^2 - \frac{53}{16}v_1^4 + (n_{12}v_2)^2\big(-\frac{4013}{384}(v_1v_2) - \frac{45}{16}v_2^2\big)$$
$$+ \frac{527}{384}(v_1v_2)v_2^2 + v_1^2\big(-\frac{859}{160}(n_{12}v_2)^2 + \frac{875}{128}(v_1v_2) + \frac{2773}{1280}v_2^2\big) + \frac{11}{64}v_2^4\Big]$$
$$- \frac{381}{32}(v_1v_2)v_2^4 + v_1^2\Big[-\frac{7}{4}(n_{12}v_2)^4 + \frac{10087}{1280}(v_1v_2)^2 - \frac{5395}{384}(v_1v_2)v_2^2$$
$$+ (n_{12}v_2)^2\big(\frac{629}{384}(v_1v_2) + \frac{17}{16}v_2^2\big) + \frac{379}{64}v_2^4\Big] + \frac{59}{16}v_2^6\Big\} + 1 \leftrightarrow 2\,, \tag{5.10c}$$

$$\tilde{L}^{(3)}_{\rm 4PN} = \frac{m_1^3m_2}{r_{12}^3}\Big[-\frac{5015}{384}(n_{12}v_1)^4 + \frac{46493}{1920}(n_{12}v_1)^3(n_{12}v_2) + \frac{7359}{400}(v_1v_2)^2 + \frac{4799}{1152}v_1^4$$
$$+ (n_{12}v_1)\big(-\frac{6827}{640}(n_{12}v_2)v_1^2 + (n_{12}v_2)(\frac{23857}{2400}(v_1v_2) - \frac{31}{16}v_2^2)\big)$$
$$+ (n_{12}v_1)^2\big(\frac{3521}{960}(n_{12}v_2)^2 - \frac{6841}{384}(v_1v_2) + \frac{11923}{960}v_1^2 - \frac{2027}{1600}v_2^2\big) - \frac{357}{16}(v_1v_2)v_2^2$$
$$+ v_1^2\big(-\frac{13433}{4800}(n_{12}v_2)^2 - \frac{468569}{28800}(v_1v_2) + \frac{54061}{4800}v_2^2\big) + \frac{203}{32}v_2^4\Big]$$
$$+ \frac{m_1^2m_2^2}{r_{12}^3}\Big[\frac{3}{40960}\big(182752 - 625\pi^2\big)(n_{12}v_1)^4 + \big(-\frac{72}{5} - \frac{35655}{16384}\pi^2\big)(n_{12}v_1)^3(n_{12}v_2)$$
$$+ \big(\frac{2051549}{57600} - \frac{10631}{8192}\pi^2\big)(v_1v_2)^2 + (n_{12}v_1)^2\big((\frac{16523}{960} + \frac{36405}{16384}\pi^2)(n_{12}v_2)^2$$
$$+ (\frac{578461}{6400} - \frac{56955}{16384}\pi^2)(v_1v_2) + (-\frac{64447}{1600} + \frac{1107}{1024}\pi^2)v_1^2\big)$$
$$+ (n_{12}v_1)\big(\frac{7}{51200}(-668104 + 21975\pi^2)(n_{12}v_2)(v_1v_2)$$

$$
\begin{aligned}
&+\Big(\frac{1295533}{19200}-\frac{43869}{16384}\pi^2\Big)(n_{12}v_2)v_1^2\Big)+\Big(\frac{65463}{6400}-\frac{2877}{8192}\pi^2\Big)v_1^4\\
&+v_1^2\Big(\frac{1}{38400}(-1487258+79425\pi^2)(n_{12}v_2)^2+\Big(-\frac{836017}{14400}+\frac{40739}{16384}\pi^2\Big)(v_1v_2)\\
&+\Big(\frac{787817}{57600}-\frac{13723}{16384}\pi^2\Big)v_2^2\Big)\Big]+1\leftrightarrow 2\,, \qquad (5.10d)\\
\tilde{L}^{(4)}_{\rm 4PN}&=\frac{m_1^4m_2}{r_{12}^4}\Big[\frac{19341}{1600}(n_{12}v_1)^2-\frac{15}{8}(v_1v_2)-\frac{16411}{4800}v_1^2+\frac{31}{32}v_2^2\Big]\\
&+\frac{m_1^3m_2^2}{r_{12}^4}\Big[\Big(-\frac{3461303}{403200}-\frac{15857}{16384}\pi^2\Big)(n_{12}v_1)^2\\
&+\Big(\frac{46994113}{403200}-\frac{79385}{24576}\pi^2\Big)(n_{12}v_1)(n_{12}v_2)\\
&+\Big(-\frac{5615591}{134400}+\frac{35603}{24576}\pi^2\Big)(n_{12}v_2)^2+\Big(\frac{2827397}{57600}-\frac{171041}{24576}\pi^2\Big)(v_1v_2)\\
&+\Big(-\frac{1830673}{57600}+\frac{193801}{49152}\pi^2\Big)v_1^2+\Big(-\frac{1158323}{57600}+\frac{21069}{8192}\pi^2\Big)v_2^2\Big]+1\leftrightarrow 2\,, \qquad (5.10e)\\
\tilde{L}^{(5)}_{\rm 4PN}&=\frac{1}{16}\frac{m_1^5m_2}{r_{12}^5}+\Big(\frac{3421459}{50400}-\frac{6237}{1024}\pi^2\Big)\frac{m_1^4m_2^2}{r_{12}^5}+\Big(\frac{4121669}{50400}-\frac{44825}{6144}\pi^2\Big)\frac{m_1^3m_2^3}{r_{12}^5}\\
&+1\leftrightarrow 2\,. \qquad (5.10f)
\end{aligned}
$$

C. Comparison with the Hamiltonian formalism

In principle, by properly adjusting the contact transformation or shift from harmonic coordinates, the ordinary Lagrangian obtained in the previous section, Eqs. (5.7)–(5.10), should correspond to ADM like coordinates, and by an ordinary Legendre transformation we should obtain the (instantaneous part of the) ADM Hamiltonian. Concerning the tails we also need to find a shift (which will be non-local [68]) that removes the accelerations and derivatives of accelerations from the tail part of the Lagrangian (5.4), or, rather, from the corresponding action. Once the tail part of the Lagrangian becomes ordinary, we can obtain the corresponding tail part in the Hamiltonian.

The tail part of the action is

$$
S^{\rm tail}_{\rm F}=\frac{G^2M}{5c^8}\operatorname*{Pf}_{2r_{12}/c}\iint\frac{{\rm d}t{\rm d}t'}{|t-t'|}\,I^{(3)}_{ij}(t)\,I^{(3)}_{ij}(t')\,, \qquad (5.11)
$$

where the Hadamard scale s_0 in Eq. (3.15) has been replaced by $r_{12}=r_{12}(t)$; the time derivatives of the Newtonian quadrupole moment $I_{ij}=\sum_A m_A\,y_A^{\langle i}y_A^{j\rangle}$ are evaluated without

replacement of accelerations, *i.e.*,

$$I_{ij}^{(3)} = \sum_A 2m_A\Big(3v_A^{\langle i}a_A^{j\rangle} + y_A^{\langle i}b_A^{j\rangle}\Big)\,. \tag{5.12}$$

Here we look for a shift that transforms the action into the same expression but with the derivatives of the quadrupole evaluated using the Newtonian equations of motion, *i.e.*,

$$\hat{I}_{ij}^{(3)} = \frac{2Gm_1m_2}{r_{12}^2}\left(-4n_{12}^{\langle i}v_{12}^{j\rangle} + 3(n_{12}v_{12})n_{12}^{\langle i}n_{12}^{j\rangle}\right)\,. \tag{5.13}$$

Note that here $\hat{I}_{ij}^{(3)}$ is not the third time derivative of the quadrupole moment unless the equations of motion are satisfied. The requested shift is easy to find and we get, after removal of some double-zero terms which do not contribute to the dynamics,

$$S_{\rm F}^{\rm tail} = \hat{S}_{\rm F}^{\rm tail} + \sum_A m_A\int_{-\infty}^{+\infty}{\rm d}t\left[a_A^i - (\partial_i U)_A\right]\xi_A^i\,, \tag{5.14}$$

where $\hat{S}_{\rm F}^{\rm tail}$ is given by the same expression as (5.11) but with the derivatives of the quadrupole moment computed on-shell, Eq. (5.13), while the second term takes the form of a shift explicitly given by[22]

$$\xi_A^i = \frac{4G^2M}{5c^8}\left[2v_A^j\underset{2r_{12}/c}{\rm Pf}\int\frac{{\rm d}t'}{|t-t'|}\,\hat{I}_{ij}^{(3)}(t') - y_A^j\underset{2r_{12}/c}{\rm Pf}\int\frac{{\rm d}t'}{|t-t'|}\,\hat{I}_{ij}^{(4)}(t') + 2\frac{(n_{12}v_{12})}{r_{12}}y_A^j\hat{I}_{ij}^{(3)}\right]\,. \tag{5.15}$$

Once the total action $\hat{S}_{\rm F} = S_{\rm F}^{\rm inst} + \hat{S}_{\rm F}^{\rm tail}$ is ordinary, the (Fokker) Hamiltonian is defined by the usual Legendre transformation as

$$\hat{S}_{\rm F} = \int_{-\infty}^{+\infty}{\rm d}t\left[\sum_A p_A^iv_A^i - H\right]\,. \tag{5.16}$$

The Hamiltonian is a functional of positions $\boldsymbol{y}_A$ and momenta $\boldsymbol{p}_A$, and reads then $H = H^{\rm inst} + \hat{H}^{\rm tail}$, where the tail part is just the opposite of the tail part of the Lagrangian, as also found in Eq.(4.5) of [67],

$$\hat{H}^{\rm tail} = -\frac{G^2M}{5c^8}\,\hat{I}_{ij}^{(3)}(t)\underset{2r_{12}/c}{\rm Pf}\int_{-\infty}^{+\infty}\frac{{\rm d}t'}{|t-t'|}\hat{I}_{ij}^{(3)}(t')\,. \tag{5.17}$$

To prove this we notice that the tail term is a small 4PN quantity, and that its contribution in the velocity expressed as a function of the momentum cancels out in the Legendre

[22] Note that in the shift vector itself, it does not matter whether we replace the accelerations by the equations of motion or not.

transformation at leading order. In the right-hand side of (5.17), the velocities present in the quadrupole moment (5.13) are to be replaced with this approximation by $\boldsymbol{v}_A \to \boldsymbol{p}_A/m_A$.

An important point is that since the action is non-local in time the Hamiltonian is only defined in an "integrated" sense by Eq. (5.16) but not in a local sense [88, 89]. Thus, the Hamiltonian equations of motion will be valid in a sense of *functional derivatives*, and the value of the Hamiltonian "on-shell" does not yield in general a strictly conserved energy. Indeed, we shall find in the companion paper [73] that in order to obtain an energy E that consistently includes the non-local tails at the 4PN order and is strictly conserved, *i.e.*, $\mathrm{d}E/\mathrm{d}t = 0$ at any time, we must take into account an extra contribution with respect to the Hamiltonian computed on-shell. The latter extra contribution is however zero for circular orbits. We shall show in Sec. V D how to compute, in that case, the energy from the Hamiltonian.

We have compared our 4PN dynamics with the 4PN Hamiltonian published in Refs. [61–63, 67], but unfortunately we have not been able to match our results with these works. Moreover, we fundamentally disagree with Ref. [67] regarding the contribution of tails to the energy for circular orbits (see the details in Sec. V D), but taking into account that disagreement is not sufficient to explain the full discrepancy.

We did two comparisons. One at the level of the equations of motion, looking for a shift of the trajectories such that the equations of motion derived from the 4PN harmonic Lagrangian in Sec. V A are transformed into the equations of motion derived from the 4PN Hamiltonian published in Eqs. (A3)–(A4) of [67]. Our second comparison was directly at the level of the Lagrangian, constructing from the harmonic Lagrangian the ordinary Lagrangian (see the result in Sec. V B), then shifting the tail part according to Eq. (5.14), and constructing the 4PN Hamiltonian following (5.16).

However these comparisons failed. The best we could do was to match all the terms with powers G^0, G^1, G^2 (the terms G^0, G^1 and G^2 in our Lagrangian also match with those of Ref. [64]), G^3 and G^5, as well as many terms with powers G^4 in the acceleration, but there are residual terms with powers G^4 that are impossible to reconcile. When looking for the ADM Lagrangian, the closest one we could find is given by (5.8)–(5.10) in Sec. V B, but its Legendre transform disagrees with the published ADM Hamiltonian by G^4 and G^5 terms.

Finally the contact transformation which minimizes the number of irreconcilable terms in both formalisms gives the difference between our harmonic-transformed acceleration a_1^i

and their ADM acceleration $(a_1^i)_{\rm DJS}$ as

$$a_1^i - (a_1^i)_{\rm DJS} = \frac{2}{15}\frac{G^4 m\, m_1 m_2^2}{c^8 r_{12}^5}\left[-\frac{472}{3}v_{12}^i(n_{12}v_{12}) + n_{12}^i\Big(-\frac{1429}{7}(n_{12}v_{12})^2 + \frac{1027}{7}v_{12}^2\Big)\right], \quad (5.18)$$

where we denote $m = m_1 + m_2$ and $v_{12}^i = v_1^i - v_2^i$. Such a difference of accelerations cannot be eliminated by a further contact transformation. It corresponds to the following difference between Hamiltonians,

$$H - (H)_{\rm DJS} = \frac{G^4 m}{315\, c^8\, r_{12}^4}\bigg[1429\big(m_2^2(n_{12}p_1)^2 - 2\, m_1 m_2 (n_{12}p_1)(n_{12}p_2) + m_1^2(n_{12}p_2)^2\big) + 826\big(m_2^2 p_1^2 - 2\, m_1 m_2 (p_1 p_2) + m_1^2 p_2^2\big) + 902\frac{G m\, m_1^2 m_2^2}{r_{12}}\bigg]. \quad (5.19)$$

Our Hamiltonian H is defined by the sum of the tail part (5.17) and of the Legendre transformation of the ordinary Lagrangian given by (5.8)–(5.10). In conclusion, from Eqs. (5.18)–(5.19) we face a true discrepancy. Note, however, that this discrepancy concerns only a few terms; for many terms our Hamiltonian agrees with the Hamiltonian of [67].

Furthermore, we observe the paradoxical fact that the difference of accelerations (5.18) does not yield a zero contribution to the energy in the case of circular orbits. Similarly, the difference of Hamiltonians (5.19) does not vanish for circular orbits. This is inconsistent with the fact that the two groups agree on the conserved energy in that case. Recall that we have adjusted our ambiguity parameter α to the value $\alpha = \frac{811}{672}$ so that the 4PN energy for circular orbits (computed directly from the 4PN equations of motion in harmonic coordinates [73]) agrees with self-force calculations [see (4.32) and the preceding discussion]. On the other hand, the ambiguity parameter in Ref. [67], which is denoted by C, has been adjusted (to the value $C = -\frac{1681}{1536}$) using the same self-force results. This contradiction leads us to investigate the validity of the derivation of the conserved energy for circular orbits using the Hamiltonian formalism as presented in Ref. [67]. We address this point in the next section.

D. Energy for circular orbits computed with the Hamiltonian

As discussed in the previous section we can consider the non-local but ordinary Hamiltonian $H[\boldsymbol{y}_A, \boldsymbol{p}_A] = H^{\rm inst}(\boldsymbol{y}_A, \boldsymbol{p}_A) + \hat{H}^{\rm tail}[\boldsymbol{y}_A, \boldsymbol{p}_A]$, where the tail term given by (5.17) functionally depends on the canonical positions $\boldsymbol{y}_A$ and momenta $\boldsymbol{p}_A$. In the frame of the center of mass the Hamiltonian is a functional of $\boldsymbol{y} = r\boldsymbol{n} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$ and $\boldsymbol{p} \equiv \boldsymbol{p}_1 = -\boldsymbol{p}_2$. Next,

introducing polar coordinates (r,φ) in the binary's orbital plane and their conjugate momenta $(p_r = \boldsymbol{n}\cdot\boldsymbol{p}, p_\varphi)$, we make the substitution $\boldsymbol{p}^2 = p_r^2 + p_\varphi^2/r^2$ to obtain the reduced Hamiltonian $H_{\rm red}$ which is a (non-local) functional of the canonical variables r, p_r and p_φ.[23] For circular orbits we have $r = r_0$ (a constant) and $p_r^0 = 0$. The angular momentum p_φ^0 is then obtained as a function of the radius r_0 by solving the radial equation

$$\frac{\delta H_{\rm red}}{\delta r}\left[r_0, p_r^0 = 0, p_\varphi^0\right] = 0\,, \tag{5.20}$$

while the orbital frequency Ω of the circular motion is given by

$$\frac{\delta H_{\rm red}}{\delta p_\varphi}\left[r_0, p_r^0 = 0, p_\varphi^0\right] = \Omega\,. \tag{5.21}$$

The circular energy is then $E = H_{\rm red}[r_0, 0, p_\varphi^0(r_0)]$, the function $p_\varphi^0(r_0)$ representing here the solution of Eq. (5.20). Finally, by inverting Eq. (5.21), we can express the radius r_0 as a function of the frequency Ω, or rather, of the PN parameter $x = (Gm\Omega/c^3)^{2/3}$. This leads to the invariant circular energy $E(x)$.

The only tricky calculation is that of the contribution of the tail part of the Hamiltonian. Because of the non-locality, the differentiation occurring in Eqs. (5.20)–(5.21) should be performed in the sense of functional derivatives. As such, the functional variation of the tail term with respect to $r(t)$ (where t is the coordinate time with respect to which the binary's dynamics is measured) yields

$$\frac{\delta \hat{H}^{\rm tail}}{\delta r(t)} = -\frac{2G^2M}{5c^8}\left[\frac{\partial \hat{I}_{ij}^{(3)}(t)}{\partial r(t)}\underset{2r(t)/c}{\rm Pf}\int_{-\infty}^{+\infty}\frac{{\rm d}t'}{|t-t'|}\hat{I}_{ij}^{(3)}(t') - \frac{1}{r(t)}\left(\hat{I}_{ij}^{(3)}(t)\right)^2\right]. \tag{5.22}$$

The second term in the square brackets comes from the variation of the Hadamard partie finie scale $r(t) \equiv r_{12}(t)$ present in Eq. (5.17). Similarly, the functional variation with respect to $p_\varphi(t)$ reads

$$\frac{\delta \hat{H}^{\rm tail}}{\delta p_\varphi(t)} = -\frac{2G^2M}{5c^8}\frac{\partial \hat{I}_{ij}^{(3)}(t)}{\partial p_\varphi(t)}\underset{2r(t)/c}{\rm Pf}\int_{-\infty}^{+\infty}\frac{{\rm d}t'}{|t-t'|}\hat{I}_{ij}^{(3)}(t')\,. \tag{5.23}$$

We substitute into the radial equation (5.20) all the instantaneous contributions — this poses no problem since the partial derivatives are ordinary — and add to that the tail

[23] Because of the non-local tail term, the Hamiltonian depends also on φ, so that p_φ is not strictly conserved. However, we can neglect this dependence on φ and the variation of p_φ in the present calculation, since in particular p_φ is constant in the case of circular orbits.

piece (5.22). Solving iteratively for p_φ^0 as a function of r_0, we find the standard Newtonian result

$$p_\varphi^0(r_0) = m\nu\sqrt{Gmr_0} + \mathcal{O}(2)\,, \tag{5.24}$$

which we can insert back into the tail integral entering (5.22), because the tail term is a small 4PN quantity. At this stage, and *only at this stage*, we are allowed to reduce the tail integral and compute it in the case of circular orbits, *i.e.*, for $r = r_0$, $p_r^0 = 0$, $p_\varphi^0(r_0)$ being given by (5.24), and for $\dot\varphi = \omega_0 + \mathcal{O}(2)$, with $\omega_0 = \sqrt{Gm/r_0^3}$. A straightforward calculation [90] leads to the formula (modulo higher-order PN radiation-reaction corrections)

$$\left(\underset{2r(t)/c}{\mathrm{Pf}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t'}{|t-t'|}\hat I_{ij}^{(3)}(t')\right)\Big|_{[r_0,p_r^0=0,p_\varphi^0(r_0)]} = -2\left(\hat I_{ij}^{(3)}(t)\right)\Big|_{[r_0,p_r^0=0,p_\varphi^0(r_0)]}\left[\ln\left(\frac{4\omega_0 r_0}{c}\right)+\gamma_{\mathrm{E}}\right], \tag{5.25}$$

with γ_{E} denoting the Euler constant. This result is used to determine the contribution of the tail term in the link between p_φ^0 and r_0 at the 4PN order. Denoting such contribution by $\Delta p_\varphi^0(r_0)$ we explicitly find

$$\Delta p_\varphi^0(r_0) = \frac{G^2M}{5c^8\omega_0}\left\{\left(r\frac{\partial\big(\hat I_{ij}^{(3)}\big)^2}{\partial r}\right)\Big|_{[r_0,p_r^0=0,p_\varphi^0(r_0)]}\left[\ln\left(\frac{4\omega_0 r_0}{c}\right)+\gamma_{\mathrm{E}}\right] + \left(\big(\hat I_{ij}^{(3)}\big)^2\right)\Big|_{[r_0,p_r^0=0,p_\varphi^0(r_0)]}\right\}. \tag{5.26}$$

This is easily reduced by employing the Newtonian expression of the quadrupole moment, valid for a general orbit (*i.e.*, for any r, p_r and p_φ), hence

$$\big(\hat I_{ij}^{(3)}\big)^2 = \frac{G^2m^2}{r^4}\left(\frac{8}{3}p_r^2 + 32\frac{p_\varphi^2}{r^2}\right) + \mathcal{O}(2)\,. \tag{5.27}$$

Treating $(r,\varphi,p_r,p_\varphi)$ as independent variables, we differentiate (5.27) partially with respect to r and take $r = r_0$, $p_r = 0$, $p_\varphi = p_\varphi^0(r_0)$ afterwards. We get [using also $M = m + \mathcal{O}(2)$]

$$\Delta p_\varphi^0(r_0) = \frac{G^{9/2}m^{11/2}\nu^2}{5c^8r_0^{7/2}}\left(-192\left[\ln\left(\frac{4\omega_0 r_0}{c}\right)+\gamma_{\mathrm{E}}\right]+32\right). \tag{5.28}$$

Next, we consider the orbital frequency $\Omega \equiv \mathrm{d}\varphi/\mathrm{d}t$ given by Eq. (5.21). The tail contribution therein has been displayed in (5.23). Using Eq. (5.24) to Newtonian order we simply find $\Omega(r_0) = \omega_0 + \mathcal{O}(2)$. The tail term is consistently evaluated thanks to (5.25) as before. The tail induced modification of the frequency, say $\Delta\Omega(r_0)$, is then the sum of the direct effect of the tail term in (5.23) and of a contribution due to the tail modification of the angular momentum (5.28),

$$\Delta\Omega(r_0) = \frac{\Delta p_\varphi^0(r_0)}{m\nu r_0^2} + \frac{128}{5}\frac{G^{9/2}m^{9/2}\nu}{5c^8r_0^{11/2}}\left[\ln\left(\frac{4\omega_0 r_0}{c}\right)+\gamma_{\mathrm{E}}\right]$$

$$= \frac{G^{9/2}m^{9/2}\nu}{5c^8 r_0^{11/2}}\left(-64\left[\ln\left(\frac{4\omega_0 r_0}{c}\right)+\gamma_{\rm E}\right]+32\right). \tag{5.29}$$

With the two results (5.28)–(5.29) in hands the contribution of tails in the invariant energy for circular orbits expressed as a function of $x=(Gm\Omega/c^3)^{2/3}$ is readily found to be

$$\Delta E(x) = -\frac{224}{15}mc^2\nu^2 x^5\left[\ln(16x)+2\gamma_{\rm E}-\frac{4}{7}\right]. \tag{5.30}$$

This result fully agrees with our alternative derivation based on the direct construction of the conserved circular energy from the equations of motion in harmonic coordinates (see the companion paper [73]). Thus, by following the above Hamiltonian procedure, *i.e.*, by carefully taking into account the non-local character of the tail term during the variation of the Hamiltonian [see notably (5.22)], we have shown that our Hamiltonian H defined by the Legendre transformation of the ordinary Lagrangian (5.8)–(5.10) plus the tail part (5.17) leads to the correct conserved invariant energy for circular orbits. This calculation confirms our value $\alpha=\frac{811}{672}$ for the ambiguity parameter.

However, we find that, applying the same Hamiltonian procedure to the Hamiltonian $(H)_{\rm DJS}$, we do not recover the part of the invariant energy for circular orbits that is known from self-force calculations, unless the ambiguity parameter C is adjusted to a different value, which would then in turn change several coefficients in the Hamiltonian for general orbits [67]. We obtain that the value for which that Hamiltonian gives the correct circular energy is

$$C^{\rm new} = C+\frac{3}{7} = -\frac{7159}{10752}. \tag{5.31}$$

One possible explanation for the discrepancy could reside in the treatment of the non-local part of the Hamiltonian when reducing to circular orbits. Recall from Eq. (5.22) that one must evaluate the tail integral for circular orbits *after* the differentiation with respect to r. We think that the treatment of Ref. [67] effectively amounts to doing the reverse, *i.e.*, to computing first the tail integral for circular orbits by means of (5.25), and only then performing the differentiation with respect to r. Indeed, we have been told by G. Schäfer (private communication) that Ref. [67] uses a local version of the Hamiltonian computed with Eq. (5.25), and then differentiates it with respect to the independent canonical variables r, p_r and p_φ, using $\omega=p_\varphi/(m\nu r^2)$ for the circular orbit frequency, therefore arriving at

$$\left(\Delta p_\varphi^0\right)_{\rm DJS} = \frac{G^2M}{5c^8\omega_0}\left(r\frac{\partial}{\partial r}\left(\left(\hat{I}_{ij}^{(3)}\right)^2\left[\ln\left(\frac{4p_\varphi}{m\nu rc}\right)+\gamma_{\rm E}\right]\right)\right)\Bigg|_{[r_0,p_r^0=0,p_\varphi^0(r_0)]}, \tag{5.32}$$

instead of our prescription (5.26). If one now applies the derivative with respect to r, one finds that the tail induced contribution to the angular momentum as a function of r_0 in [67] differs from ours by the amount

$$\Delta p^0_\varphi - \left(\Delta p^0_\varphi\right)_{\rm DJS} = \frac{2}{5}\frac{G^2 M}{c^8\omega_0}\left(\left(\hat{I}^{(3)}_{ij}\right)^2\right)\Big|_{[r_0,p^0_r=0,p^0_\varphi(r_0)]} = \frac{64}{5}\frac{G^{9/2}m^{11/2}\nu^2}{c^8 r_0^{7/2}}\,. \tag{5.33}$$

Furthermore we find that the tail contribution to the orbital frequency $\Omega(r_0)$ as a function of the radius agrees with us, so that, in the end, the tail contribution in the invariant circular energy differs from ours by

$$\Delta E - (\Delta E)_{\rm DJS} = \omega_0\left[\Delta p^0_\varphi - \left(\Delta p^0_\varphi\right)_{\rm DJS}\right] = \frac{64}{5}mc^2\nu^2 x^5\,. \tag{5.34}$$

Finally the prescription (5.32), with which we disagree, leads to an incorrect invariant energy $E(x)$ for circular orbits when starting from our Hamiltonian or from the one in [67] but with C given by (5.31). On the other hand, if one applies this prescription, different from ours, for the circular orbit reduction of the Hamiltonian in [67], without modifying the constant C, one ends up with the correct $E(x)$.

Similarly, we disagree with the computation of the effective-one-body (EOB) potentials at the 4PN order in Ref. [68]. Indeed, a local ansatz has been made for the EOB Hamiltonian, since it has been obtained by evaluating the tail term on shell for an explicit solution of the motion (see Eqs. (4.10)–(4.11) in [68]), which means effectively using Eq. (5.25) in the case of circular orbits, and results in a local Hamiltonian. Note that the comparison to the self-force results of the work [67, 68] have been recently complemented by deriving and confirming with another method the EOB potential $D(u)$ [91]. However, the problem in the treatment of the non-locality described above might affect this comparison as well[24].

Still, if we now make the comparison with the Hamiltonian [67] but with the new value of the ambiguity parameter (5.31), we cannot reduce the difference to zero. Indeed, the right-hand sides of Eqs. (5.18)–(5.19) do not correspond to a mere rescaling of the ambiguity parameter. We get instead

$$a'^i_1 - (a^i_1)^{\rm new}_{\rm DJS} = \frac{2}{15}\frac{G^4 m\, m_1 m_2^2}{c^8 r_{12}^5}\left[\frac{680}{3}v^i_{12}(n_{12}v_{12}) + n^i_{12}\Big(-595(n_{12}v_{12})^2 + 85v^2_{12}\Big)\right], \tag{5.35a}$$

[24] After this work was submitted for publication, the authors of [67, 68] described in more details their method for reducing the dynamics to a local-in-time Hamiltonian in [92].

$$H' - (H)^{\rm new}_{\rm DJS} = \frac{1}{315}\frac{G^4 m}{c^8\, r_{12}^4}\bigg[4165\big(m_2^2(n_{12}p_1)^2 - 2\,m_1m_2(n_{12}p_1)(n_{12}p_2) + m_1^2(n_{12}p_2)^2\big) \\ - \,1190\big(m_2^2\,p_1^2 - 2\,m_1m_2(p_1p_2) + m_1^2\,p_2^2\big) + 1190\frac{Gm\,m_1^2m_2^2}{r_{12}}\bigg]. \quad \text{(5.35b)}$$

With respect to (5.18)–(5.19) we have performed for convenience an additional shift, hence we denote our new acceleration and Hamiltonian with a prime. When using the value (5.31) of the ambiguity parameter, the differences (5.35) can now be seen not to contribute to the conserved invariant energy for circular orbits, which resolves our paradox. Unfortunately, we have no explanation for the remaining discrepancy in Eqs. (5.35).

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Appendix A: Complement about the method "$n+2$"

We compute the Fokker action $S_{\rm F}$ for a full-fledge solution h of the field equations reducing to the PN expansion $\overline{h}$ in the near zone and to the multipole expansion $\mathcal{M}(h)$ in the far zone. These two expansions obey the matching equation $\overline{\mathcal{M}(h)} = \mathcal{M}\big(\overline{h}\big)$. We suppose that this solution is of the type

$$h = h_n + \delta h\,, \quad \text{(A1)}$$

where h_n is some known *approximate* solution and δh is a remainder (or error) term defined everywhere. We denote the approximate solution h_n with the label n because we assume that in the near zone the PN expansion of this solution agrees with the known solution considered in Sec. IV A, *i.e.*, $\overline{h}_n$. However we extend here that solution to the far zone as well, where it agrees with the multipole expansion $\mathcal{M}(h_n)$. Similarly the error in that solution is defined both in the near zone, *i.e.*, $\overline{\delta h}$, and in the far zone, $\mathcal{M}(\delta h)$.

Since we are considering here the true solution h (and not merely its PN expansion $\overline{h}$) there is no regulator r^B in the first place, and we can freely integrate by parts the action

and write the usual expansion

$$S_{\rm F}[h] = S_{\rm F}[h_n] + \int {\rm d}t \int {\rm d}^3\mathbf{x}\, \frac{\delta S_{\rm F}}{\delta h}[h_n]\, \delta h + \mathcal{O}\left(\delta h^2\right) . \tag{A2}$$

At this stage we apply the lemma (2.9) to the second term in (A2). We introduce the regulator r^B and transform it into an expression that integrates over the formal PN expansion, plus a contribution that integrates over the multipole expansion (with, say, $r_0 = 1$):

$$\int {\rm d}^3\mathbf{x}\, \frac{\delta S_{\rm F}}{\delta h}\, \delta h = \underset{B=0}{\rm FP} \int {\rm d}^3\mathbf{x}\, r^B\, \overline{\frac{\delta S_{\rm F}}{\delta h}}\, \overline{\delta h} + \underset{B=0}{\rm FP} \int {\rm d}^3\mathbf{x}\, r^B \mathcal{M}\left(\frac{\delta S_{\rm F}}{\delta h}\right) \mathcal{M}\left(\delta h\right) . \tag{A3}$$

The first term of (A3) corresponds exactly to the PN remainder that is investigated in Sec. IV A and yields to our method $n+2$ [see Eq. (4.7)]. Here we worry about the second, multipolar contribution in (A3) that was not considered in the arguments of Sec. IV A. We shall argue that its contribution is completely negligible when re-expanded in the near zone as compared with the 4PN order.

For the method $n+2$ we proved in Sec. IV A that if the PN solution $\overline{h_n}$ is known to order $\mathcal{O}(n+2, n+1, n+2)$ when n is even and $\mathcal{O}(n+1, n+2, n+1)$ when n is odd, then the contribution of the first, PN term in (A3) is very small, and the action is finally controlled up to nPN order [see Eq. (4.12)]. We evaluate the contribution due to the second, multipolar term in (A3), namely

$$T_{\rm F} = \underset{B=0}{\rm FP} \int {\rm d}t \int {\rm d}^3\mathbf{x}\, r^B \mathcal{M}\left(\frac{\delta S_{\rm F}}{\delta h}\right) \mathcal{M}\left(\delta h\right) . \tag{A4}$$

We must impose that the error made in the multipole expansion in the far zone, *i.e.*, $\mathcal{M}(\delta h)$, becomes equal when re-expanded into the near zone to the error assumed in the PN expansion, *i.e.*, $\overline{\delta h}$. Note that $\overline{\mathcal{M}(\delta h)}$ is not equal to $\mathcal{M}(\overline{\delta h})$ as the matching equation is only correct for the true, complete solution h [see Eq. (2.8)]. Since the multipole expansion is constructed from a post-Minkowskian (PM) expansion (see [2])

$$\mathcal{M}(h) = \sum_{m=1}^{+\infty} G^m h_{(m)} , \tag{A5}$$

we shall assume that the error $\mathcal{M}(\delta h)$ in the far zone corresponds to some high PM order m_0, *i.e.*, is of order $\mathcal{O}(G^{m_0})$. Thus we have

$$\mathcal{M}(h_n) = \sum_{m=1}^{m_0-1} G^m h_{(m)} , \tag{A6a}$$

$$\mathcal{M}(\delta h) = \sum_{m=m_0}^{+\infty} G^m h_{(m)} \,. \tag{A6b}$$

To determine what m_0 is we recall that the leading PN order of the near zone re-expansion of the PM coefficients is $\overline{h}_{(m)} = \mathcal{O}(2m, 2m+1, 2m)$.[25] Imposing then that this PN-expanded PM error is equal to the previous one assumed for $\overline{\delta h}$, we find the minimal PM order of the error in the far zone to be $2m_0 = n+2$ when n is even and $2m_0 = n+3$ when n is odd, thus (with [ ] being the integer part)

$$m_0 = \left[\tfrac{n+1}{2}\right] + 1 \,. \tag{A7}$$

To obtain the magnitude of the term (A4) we first notice that any term in the integrand which is instantaneous in the sense of having the structure (2.15) will yield zero contribution thanks to our lemma (2.17). Thus it remains only the hereditary contributions which have the more complicated structure given by (2.16). Next, we remark that since $\mathcal{M}(\delta h)$ is a small error PM term of order $\mathcal{O}(G^{m_0})$, the variation of the Fokker action $\mathcal{M}(\delta S_{\rm F}/\delta h)$ evaluated for the approximate solution h_n must necessarily also be a small PM term, because the Fokker action is stationary for the exact solution. More precisely we find [because of the extra factor $\sim c^4/G$ in front of (4.1)] that it must be of order $\mathcal{O}(G^{m_0-1})$, hence

$$\mathcal{M}\left(\frac{\delta S_{\rm F}}{\delta h}\right) = \sum_{m=m_0-1}^{+\infty} G^m k_{(m)} \,, \tag{A8}$$

with some PM coefficients $k_{(m)}$. Thus we conclude that only hereditary terms that are at least of order $\mathcal{O}(G^{2m_0-1})$ can contribute to the PM expansion of (A4). For our 4PN computation $n = 4$ thus $m_0 = 3$ from (A7), thus such hereditary terms must be $\mathcal{O}(G^5)$.

We had argued at the end of Sec. II B that cubic $\sim \mathcal{O}(G^3)$ hereditary terms will correspond for the leading multipole interactions to "tail-of-tails" and are dominantly of order 5.5PN when re-expanded in the near zone. Here the hereditary terms $\sim \mathcal{O}(G^5)$ should correspond minimally to say "tail-of-tail-of-tail-of-tails" and give an even smaller contribution in the near zone, presumably starting at the order 8.5PN. In conclusion we can neglect the term (A4) and our use of Eq. (4.7) for the method "$n+2$" is justified.

[25] Such statement can be proved by induction over the PM order m.

Appendix B: Local expansion of the function $g^{(d)}$ in d dimensions

In this Appendix we work out the local expansion near the singularities, say when $r_1 \to 0$, of the function $g^{(d)}$, defined in d dimensions by

$$g^{(d)} = \Delta^{-1}\left(r_1^{2-d} r_2^{2-d}\right) , \tag{B1}$$

where Δ^{-1} is the usual inverse Laplace operator in d dimensions. Such solution plays a crucial role when integrating the non compact support source terms in the elementary potentials (4.15) for d dimensions. The explicit form of this function is known and has been displayed in the Appendix C of [38]. Here we shall complete the latter work by providing the explicit expansion of $g^{(d)}$ when $r_1 \to 0$. This expansion is all what we need when computing the difference between DR and HR — since that difference can precisely be obtained solely from the local expansions $r_1 \to 0$ or $r_2 \to 0$ near the singularities [see notably Eq. (4.27)].

1. Derivation based on distribution theory

Following Ref. [38] we first obtain a local solution in an expanded form near the particle 1, denoted as $g^{(d)}_{\rm loc1}$, by expanding near $r_1 = 0$ the source of the Poisson equation for $g^{(d)}$ in (B1) and integrating that source term by term. For this purpose, we insert the well-known expansion when $r_1 \to 0$,

$$r_2^{2-d} = r_{12}^{2-d} \sum_{\ell=0}^{+\infty} \left(\frac{r_1}{r_{12}}\right)^{\ell} P_\ell^{(d)}(c_1) , \tag{B2}$$

where we have posed $c_1 = -\boldsymbol{n}_1 \cdot \boldsymbol{n}_{12} = \cos\theta_1$, following exactly the notation of Appendix C of [38], and denoted $P_\ell^{(d)}(c_1) = C_\ell^{(d/2-1)}(c_1)$ the Gegenbauer polynomial representing the appropriate generalization of the ℓth-degree Legendre polynomial in d dimensions

$$P_\ell^{(d)}(c_1) = \frac{(-2)^\ell \Gamma\left(\frac{d}{2}+\ell-1\right)}{\ell!\,\Gamma\left(\frac{d}{2}-1\right)} \hat{n}_1^L \hat{n}_{12}^L . \tag{B3}$$

After replacing Eq. (B2) into the right-hand side of (B1), we integrate term by term, using the fact that $P_\ell^{(d)}(c_1) \propto \hat{n}_1^L$, by means of the elementary formula

$$\Delta^{-1}\left(\hat{n}_1^L r_1^\alpha\right) = \frac{\hat{n}_1^L r_1^{\alpha+2}}{(\alpha-\ell+2)(\alpha+\ell+d)} . \tag{B4}$$

In this way, we arrive at the formal *local* expansion when $r_1 \to 0$,

$$g^{(d)}_{\rm loc1} = \frac{r_{12}^{2-d} r_1^{4-d}}{2(4-d)} \sum_{\ell=0}^{+\infty} \frac{1}{\ell+1} \left(\frac{r_1}{r_{12}}\right)^{\ell} P_\ell^{(d)}(c_1) . \tag{B5}$$

The trick now is to rewrite (B5) as an expression formally valid "everywhere", *i.e.*, not only in the vicinity of the singular point 1, namely the following integral extending along the segment of line joining the source point $\boldsymbol{y}_1$ to the field point $\boldsymbol{x}$,

$$g^{(d)}_{\text{loc1}} = \frac{r_1^{4-d}}{2(4-d)} \int_0^1 \mathrm{d}\lambda \big|\boldsymbol{y}_{12} + \lambda\, \boldsymbol{r}_1\big|^{2-d}. \tag{B6}$$

See the Appendix C in Ref. [38] for more details about this procedure (we recall that here $\boldsymbol{y}_{12} = \boldsymbol{y}_1 - \boldsymbol{y}_2$ and $\boldsymbol{r}_1 = \boldsymbol{x} - \boldsymbol{y}_1$).

Let us next add to $g^{(d)}_{\text{loc1}}$ given in the form of the line integral (B6) the appropriate homogeneous solution in such a way that the requested equation (B1) be satisfied *in the sense of distributions*. Computing $\Delta g^{(d)}_{\text{loc1}}$ in the sense of distributions we readily obtain [38]

$$\Delta g^{(d)}_{\text{loc1}} = r_1^{2-d} r_2^{2-d} + \frac{r_{12}^{4-d}}{2(4-d)} \int_0^1 \frac{\mathrm{d}\lambda}{\lambda^2} \Delta \big|\boldsymbol{r}_1 + \tfrac{1}{\lambda}\, \boldsymbol{y}_{12}\big|^{2-d}, \tag{B7}$$

showing that the true solution, valid in the sense of distributions, actually reads

$$g^{(d)} = g^{(d)}_{\text{loc1}} + g^{(d)}_{\text{hom1}}, \tag{B8}$$

where $g^{(d)}_{\text{hom1}}$ is obtained from the second term in (B7). Changing λ into $1/\lambda$ we can arrange this term as a semi infinite line integral extending from $\boldsymbol{x}$ up to infinity in the direction $\boldsymbol{n}_{12}$,

$$g^{(d)}_{\text{hom1}} = -\frac{r_{12}^{4-d}}{2(4-d)} \int_1^{+\infty} \mathrm{d}\lambda \, \big|\boldsymbol{r}_1 + \lambda\, \boldsymbol{y}_{12}\big|^{2-d}. \tag{B9}$$

This is an homogeneous solution in the sense that $\Delta g^{(d)}_{\text{hom1}} = 0$ in the sense of functions. One can prove that the sum $g^{(d)} = g^{(d)}_{\text{loc1}} + g^{(d)}_{\text{hom1}}$ is indeed symmetric in the exchange of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ although the two separate pieces are not.

Here we shall only need the expansion when $r_1 \to 0$. An easy calculation, inserting the expansion (B2) into (B9) and performing the integral over λ using analytic continuation in d (which is the essence of dimensional regularization) readily yields (with $\varepsilon = d - 3$)

$$g^{(d)}_{\text{hom1}} = -\frac{r_{12}^{-2\varepsilon}}{2(1-\varepsilon)} \sum_{\ell=0}^{+\infty} \frac{1}{\ell+\varepsilon} \left(\frac{r_1}{r_{12}}\right)^{\ell} P^{(d)}_{\ell}(c_1), \tag{B10}$$

where $\varepsilon = d - 3$. Finally the complete expansion of $g^{(d)}$ when $r_1 \to 0$ is obtained by adding the corresponding piece given by (B5), as

$$g^{(d)} = \frac{r_{12}^{-2\varepsilon}}{2(1-\varepsilon)} \sum_{\ell=0}^{+\infty} \left[\frac{1}{\ell+1} \left(\frac{r_1}{r_{12}}\right)^{1-\varepsilon} - \frac{1}{\ell+\varepsilon}\right] \left(\frac{r_1}{r_{12}}\right)^{\ell} P^{(d)}_{\ell}(c_1). \tag{B11}$$

2. Derivation based on asymptotic matching

It is instructive to present an alternative proof of Eq. (B11) based on the same asymptotic matching techniques as in the demonstration of Lemma 1 exposed in Sec. II B. We start with the definition of $g^{(d)}$ in the form of the d-dimensional Poisson integral, which we choose to be centred on the particle 1,

$$g^{(d)} = -\frac{\tilde{k}_d}{4\pi}\int \frac{\mathrm{d}^d\boldsymbol{r}'_1}{|\boldsymbol{r}_1-\boldsymbol{r}'_1|^{d-2}}\, r_1'^{2-d} r_2'^{2-d}, \tag{B12}$$

where $\tilde{k}_d$ stands for the constant factor $\Gamma(d/2-1)/\pi^{d/2-1}$ and $\boldsymbol{r}_1 = \mathbf{x}-\boldsymbol{y}_1$. In this definition, we decompose the source into two terms. The first one is taken to be the Taylor expansion of $S = r_1^{2-d} r_2^{2-d}$ near $r_1 = 0$, denoted as $\mathcal{T}_1(S)$ henceforth. The second term is thus the difference $\delta S(\mathbf{x},t) = S - \mathcal{T}_1(S)$. The key point consists in noticing that $\delta S(\mathbf{x},t)$ vanishes in some open ball of radius R_1 centered at the "origin" $\boldsymbol{r}_1 = 0$. This means that one can restrict the integration domain of the Poisson operator acting on δS to the set of points verifying $r'_1 > R_1$. Therefore, for $r_1 < R_1$, the Poisson kernel $|\boldsymbol{r}_1-\boldsymbol{r}'_1|^{2-d}$ may be replaced by its multipole expansion

$$\mathcal{M}\left(|\boldsymbol{r}_1-\boldsymbol{r}'_1|^{2-d}\right) = \sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!} r_1^L \partial_L r_1'^{2-d}, \tag{B13}$$

with the short notation $r_1^L = r_1^{i_1} r_1^{i_2}\cdots r_1^{i_\ell}$. After this operation, the integration domain may be extended again to the whole space, since the source is still zero for $r'_1 < R_1$. It is implicitly understood here that all Taylor and multipole expansions are actually performed at some finite but arbitrary high orders. The formal use of infinite series in the present discussion just allows us to elude technicalities related to the control of remainders. However, we have checked that truncations at finite orders do not change the backbone of our argument. In particular, we are formally allowed to commute the sum and integral symbols.

At this stage, we have shown that

$$\begin{aligned} g^{(d)} &= \Delta^{-1}\mathcal{T}_1\left(r_1^{2-d} r_2^{2-d}\right) \\ &\quad - \frac{\tilde{k}_d}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!} r_1^L \int \mathrm{d}^d\boldsymbol{r}'_1\, \partial_L\left(\frac{1}{r_1'^{d-2}}\right)\left[r_1'^{2-d} r_2'^{2-d} - \mathcal{T}_1\left(r_1'^{2-d} r_2'^{2-d}\right)\right]. \end{aligned} \tag{B14}$$

Because $\Delta^{-1}\mathcal{T}_1(r_1^{2-d} r_2^{2-d})$ is precisely what we have defined to be $g^{(d)}_{\mathrm{loc}1}$ in Eq. (B5), the expression in the second line is identified with the homogeneous solution $g^{(d)}_{\mathrm{hom}1}$. Now, the second

term within the square brackets is made of pieces of the form $\sim (\hat{n}_1'^L/r_1'^{d-2+\ell})(r_1'^{2-d})(r_1'^k\hat{n}_1'^K)$ for ℓ, k integers. Its radial integration leads to integrals $\int_0^{+\infty} \mathrm{d}r_1'\, r_1^{-1-\varepsilon}$. The latter are just zero by analytic continuation on the parameter $\varepsilon = d-3$, as explained in the proof a Lemma 1 (with ε playing the role of $-B$ there). Hence the homogeneous solution reads

$$g^{(d)}_{\rm hom1} = -\frac{\tilde{k}_d}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}r_1^L\int \mathrm{d}^d\boldsymbol{r}_1'\partial_L\left(\frac{1}{r_1'^{d-2}}\right)r_1'^{2-d}r_2'^{2-d}\,. \tag{B15}$$

Next we complete the proof by evaluating explicitly the integral entering the above formula with the help of the relation $\hat{\partial}_L r_1'^\alpha = (-2)^\ell\Gamma(\ell-\alpha/2)/\Gamma(-\alpha/2)r_1'^{\alpha-\ell}\hat{n}_1'^L$. We first obtain[26]

$$g^{(d)}_{\rm hom1} = -\frac{\tilde{k}_d}{4\pi}\sum_{\ell=0}^{+\infty}\frac{\Gamma(\ell+d/2-1)\Gamma(d-2)}{\Gamma(d/2-1)\Gamma(\ell+d-2)}\frac{r_1^L}{\ell!}\frac{\partial}{\partial\hat{y}_1^L}\int \mathrm{d}^d\boldsymbol{r}_1' r_1'^{4-2d}r_2'^{2-d}\,, \tag{B16}$$

and, in the last step, we compute $\int \mathrm{d}^d\boldsymbol{r}_1' r_1'^{4-d}r_2'^{2-d}$ by means of the Riesz formula, given *e.g.* by Eq. (B.19) of Ref. [38]. Using the relation (B3) we find that the ensuing expression for $g^{(d)}_{\rm hom1}$ is in full agreement with Eq. (B10).

Notice finally that the function $g^{(d)}$ in d dimensions contains a pole in the dimension coming from the monopole part of the expansion (B10) or (B11), namely $g^{(d)} = -\frac{1}{2}\varepsilon^{-1} + \mathcal{O}(\varepsilon^0)$. However, since in practical computations $g^{(d)}$ will always be differentiated, this pole is always cancelled out. Furthermore it was proved in Ref. [38] that the finite part of $g^{(d)}$ when $\varepsilon \to 0$ recovers the 3-dimensional result $\ln(r_1+r_2+r_{12})$ [9] up to some irrelevant additive constant, namely

$$g^{(d)} = -\frac{1}{2\varepsilon} - \frac{1}{2} + \ln\left(\frac{r_1+r_2+r_{12}}{2}\right) + \mathcal{O}(\varepsilon)\,. \tag{B17}$$

But here we only need the local expansion provided by (B11) up to order ε included.

Appendix C: The complete 4PN shift

In this Appendix we show the complete shift at 4PN order that removes, in particular, all the poles $\propto 1/\varepsilon$ and all the IR constants r_0.[27] Furthermore, this shift cancels the dependence

[26] Recall that the spatial multi-derivative $\partial_L(r^{2-d})$ is trace-free. The traces are actually made of derivatives of d-dimensional Dirac functions but one can check that, when inserted into the integral of (B15), they vanish by analytic continuation on ε.

[27] Another shift has been used in Secs. V B and V C to remove the accelerations in the harmonic Lagrangian and compute the Hamiltonian. This shift, however, is too long to be presented.

on the individual positions $\boldsymbol{y}_A$ of the particles and is such that the shifted equations of motion are manifestly Poincaré invariant (including spatial translations and boosts). It reads

$$\boldsymbol{\xi}_1 = \frac{11}{3}\frac{G^2\,m_1^2}{c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{\overline{q}^{1/2}r'_1}{\ell_0}\right) - \frac{327}{1540}\right]\boldsymbol{a}^{(d)}_{1,\,\mathrm{N}} + \frac{1}{c^8}\boldsymbol{\xi}_{1,\,4\mathrm{PN}}\,, \tag{C1}$$

where $G = G_{\mathrm{N}}$ in this Appendix, $\boldsymbol{a}^{(d)}_{1,\,\mathrm{N}}$ represents the Newtonian acceleration of 1 in d dimensions and we recall that $\overline{q} = 4\pi e^{\gamma_{\mathrm{E}}}$. For convenience we divide the 4PN piece of the shift in several pieces,

$$\xi^i_{1,\,4\mathrm{PN}} = \frac{1}{\varepsilon}\xi^{i\,(-1)}_{1,\,4\mathrm{PN}} + \xi^{(0,n_{12})}_{1,\,4\mathrm{PN}}n^i_{12} + \xi^{(0,v_1)}_{1,\,4\mathrm{PN}}v^i_1 + \xi^{(0,v_{12})}_{1,\,4\mathrm{PN}}v^i_{12}\,, \tag{C2}$$

with $v^i_{12} = v^i_1 - v^i_2$ and

$$\begin{aligned}
\xi^{i\,(-1)}_{1,\,4\mathrm{PN}} &= \frac{G^3m_1^2m_2v^i_{12}}{r_{12}^2}\big(11(n_{12}v_{12}) + \frac{11}{3}(n_{12}v_1)\big) + n^i_{12}\big(\frac{G^4}{r_{12}^3}\big(\frac{55}{3}m_1^3m_2 + \frac{22}{3}m_1^2m_2^2 + 4m_1m_2^3\big)\\
&+ \frac{G^3m_1^2m_2}{r_{12}^2}\big(\frac{11}{2}(n_{12}v_{12})^2 - 11(n_{12}v_{12})(n_{12}v_1) + \frac{11}{2}(n_{12}v_1)^2 - \frac{22}{3}v_{12}^2\big)\big)\,, \qquad \text{(C3a)}\\
\xi^{(0,n_{12})}_{1,\,4\mathrm{PN}} &= G^3m_1m_2^2\bigg\{\frac{1}{r_{12}^2}\Big[\big(-\frac{2539}{560} + \frac{72}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})^2\\
&+ \big(\frac{18759}{280} + \frac{96}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})(n_{12}v_1) + \big(-\frac{6253}{140} - \frac{64}{5}\ln(\frac{r_{12}}{r_0})\big)(v_{12}v_1)\\
&+ \big(\frac{5783}{840} - \frac{32}{5}\ln(\frac{r_{12}}{r_0})\big)v_{12}^2\Big] + \frac{1}{r_{12}^3}\Big[\big(\frac{519}{35} + 48\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})^2(n_{12}y_1)\\
&+ \big(-\frac{289}{35} - \frac{144}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})(v_{12}y_1) + \big(-\frac{171}{35} - \frac{48}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}y_1)v_{12}^2\Big]\bigg\}\\
&+ G^3m_1^2m_2\bigg\{\frac{1}{r_{12}^2}\Big[\big(-\frac{2761}{168} - \frac{33}{2}\ln(\frac{\bar{q}^{1/2}r'_1}{\ell_0}) - \frac{11}{2}\ln(\frac{r_{12}}{r'_1})\big)(n_{12}v_{12})^2\\
&+ \big(\frac{41299}{840} + 33\ln(\frac{\bar{q}^{1/2}r'_1}{\ell_0}) + \frac{96}{5}\ln(\frac{r_{12}}{r_0}) + 11\ln(\frac{r_{12}}{r'_1})\big)(n_{12}v_{12})(n_{12}v_1)\\
&+ \big(\frac{7489}{840} - \frac{33}{2}\ln(\frac{\bar{q}^{1/2}r'_1}{\ell_0}) - \frac{11}{2}\ln(\frac{r_{12}}{r'_1})\big)(n_{12}v_1)^2 + \big(-\frac{6253}{140} - \frac{64}{5}\ln(\frac{r_{12}}{r_0})\big)(v_{12}v_1)\\
&+ \big(\frac{3103}{168} + 22\ln(\frac{\bar{q}^{1/2}r'_1}{\ell_0}) + \frac{16}{5}\ln(\frac{r_{12}}{r_0})\big)v_{12}^2\Big] + \frac{1}{r_{12}^3}\Big[\big(\frac{519}{35} + 48\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})^2(n_{12}y_1)\\
&+ \big(-\frac{289}{35} - \frac{144}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})(v_{12}y_1) + \big(-\frac{171}{35} - \frac{48}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}y_1)v_{12}^2\Big]\bigg\}\\
&+ G^4m_1^2m_2^2\Big[\frac{1}{r_{12}^3}\big(-\frac{811}{210} - \frac{88}{3}\ln(\frac{\bar{q}^{1/2}r'_1}{\ell_0}) - \frac{16}{5}\ln(\frac{r_{12}}{r_0})\big)\\
&+ \frac{1}{r_{12}^4}\big(-\frac{114}{35} - \frac{32}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}y_1)\Big]\\
&+ G^4m_1^3m_2\Big[\frac{1}{r_{12}^3}\big(-\frac{811}{105} - \frac{220}{3}\ln(\frac{\bar{q}^{1/2}r'_1}{\ell_0}) - \frac{32}{5}\ln(\frac{r_{12}}{r_0})\big)
\end{aligned}$$

$$+\frac{1}{r_{12}^4}\big(-\frac{57}{35}-\frac{16}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}y_1)\Big]+G^4m_1m_2^3\Big[\frac{1}{r_{12}^3}\big(\frac{811}{210}-16\ln(\frac{\bar{q}^{1/2}r_1'}{\ell_0})$$
$$+\frac{16}{5}\ln(\frac{r_{12}}{r_0})\big)+\frac{1}{r_{12}^4}\big(-\frac{57}{35}-\frac{16}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}y_1)\Big], \tag{C3b}$$

$$\xi^{(0,v_1)}_{1,\,4\mathrm{PN}}=G^3\Big[\frac{m_1^2m_2}{r_{12}^2}\big(\frac{839}{2520}+\frac{32}{15}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})+\frac{m_1m_2^2}{r_{12}^2}\big(\frac{839}{2520}+\frac{32}{15}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})\Big], \tag{C3c}$$

$$\xi^{(0,v_{12})}_{1,\,4\mathrm{PN}}=G^3m_1m_2^2\Big\{\frac{1}{r_{12}^2}\Big[\big(\frac{2041}{336}-8\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})+\big(-\frac{6253}{140}-\frac{64}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_1)\Big]$$
$$+\frac{1}{r_{12}^3}\Big[\big(-\frac{289}{35}-\frac{144}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})(n_{12}y_1)+\big(\frac{228}{35}+\frac{64}{5}\ln(\frac{r_{12}}{r_0})\big)(v_{12}y_1)\Big]\Big\}$$
$$+G^3m_1^2m_2\Big\{\frac{1}{r_{12}^2}\Big[\big(\frac{23113}{1680}-33\ln(\frac{\bar{q}^{1/2}r_1'}{\ell_0})+\frac{11}{3}\ln(\frac{r_{12}}{r_1'})\big)(n_{12}v_{12})$$
$$+\big(-\frac{1398}{35}-11\ln(\frac{\bar{q}^{1/2}r_1'}{\ell_0})-\frac{64}{5}\ln(\frac{r_{12}}{r_0})-\frac{11}{3}\ln(\frac{r_{12}}{r_1'})\big)(n_{12}v_1)\Big]$$
$$+\frac{1}{r_{12}^3}\Big[\big(-\frac{289}{35}-\frac{144}{5}\ln(\frac{r_{12}}{r_0})\big)(n_{12}v_{12})(n_{12}y_1)+\big(\frac{228}{35}+\frac{64}{5}\ln(\frac{r_{12}}{r_0})\big)(v_{12}y_1)\Big]\Big\}. \tag{C3d}$$

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