Continuity of scalar fields with logarithmic correlations

S. G. Rajeev and Evan Ranken

Continuity of Scalar Fields With Logarithmic Correlations

S. G. Rajeev* and Evan Ranken†
Department of Physics and Astronomy
University of Rochester
Rochester NY 14627
(Dated: July 15, 2015)

We apply select ideas from the modern theory of stochastic processes in order to study the continuity/roughness of scalar quantum fields. A scalar field with logarithmic correlations (such as a massless field in 1+1 spacetime dimensions) has the mildest of singularities, making it a logical starting point. Instead of the usual inner product of the field with a smooth function, we introduce a moving average on an interval which allows us to obtain explicit results and has a simple physical interpretation. Using the mathematical work of Dudley, we prove that the averaged random process is in fact continuous, and give a precise modulus of continuity bounding the short-distance variation.

PACS numbers: 02.50.-r, 11.10.Cd , 11.10.Kk, 84.37.+q

I. INTRODUCTION

In traditional geometry, the distance between two points is the length of the shortest curve that joins them. This fits well with classical physics, as this shortest path is the one followed by a free particle. But in quantum physics, the shortest one is only the most likely of many paths that the particle can take. Moreover, no particle can follow a path connecting two space-like separated points. Taking these facts into account, we should hesitate to associate distance with the length of one particular curve. Instead, we can average over all paths connecting two points, yielding the Green's function of a quantum field (also called the two point function, correlation function or propagator.) A metric does emerge out of the correlation, but turns out to be non-Euclidean [1].

The idea of defining a metric from the correlation of a random process is a staple of modern stochastic analysis [2–5]. This can be illustrated with Brownian motion. The Brownian paths are continuous, but not differentiable with respect to the usual time parameter. A particle executing Brownian motion is knocked around by other particles in the medium. As the time between collisions tends to zero, the velocity at any instant is no longer a physical quantity. Furthermore, even the speed cannot be bounded; as $x \to y$ the probability of $\frac{|B(x)-B(y)|}{|x-y|}$ being bounded is zero (To make comparison with a quantum field easier, we call the time parameter of the Brownian process x rather than t. Since the diffusion constant has dimension $(\text{length})^2/\text{time}$, dimensional analysis suggests that

$$\frac{|B(x) - B(y)|}{\sqrt{|x - y|}} \tag{1.1}$$

would be a better quantity to measure the speed of a Brownian particle. But it turns out that even this is unbounded with probability 1 (more commonly stated as "almost surely" or a.s.) as $x \to y$. The proper way to quantify the time that has elapsed between two measurements is not $|x-y|$ or even $\sqrt{|x - y|}$. We seek a metric ω with respect to which the sample paths are locally Lipschitz continuous, meaning $\frac{|B(x)-B(y)|}{\omega(x,y)}$ is almost surely bounded as $x \to y$. The correct such "modulus of continuity," attributed to Lévy, is

$$\omega(x,y) \propto \sqrt{|x - y| \log \frac{1}{|x - y|}} \tag{1.2}$$

for small $|x - y|$. This quantifies the roughness of Brownian paths (One can bound the variations precisely with a proportionality constant $\sqrt{2}$, but we will generally ignore multiplicative constants in discussing continuity/roughness here).

We look at the spatial metric in the simplest relativistic theory, a massless scalar quantum field in $1 + 1$ dimensions. Such logarithmically correlated fields have generated interest in purely mathematical contexts, and have potential applications in areas ranging from finance to cosmology (see [6]). It is enough to understand the continuity of sample fields in the ground state; those in any state of finite energy will exhibit identical behavior over small distances (see Appendix B for a discussion of the ground state wavefunction).

A complication is that scalar quantum fields are random distributions rather than functions: $\phi(x)$ at some point in space is not a meaningful quantity. But we will show that a mild smoothing procedure (averaging over an interval) is enough to get around this difficulty, yielding a continuous but not differentiable function. This average can be viewed as a model for the potential measured by a device: such a measurement will always take place over some finite width. A peculiar property of the logarithmically correlated field is that the probability law of the average is independent of the size of the interval (size of the measuring device). That is, the field does not appear

---
* s.g.rajeev@rochester.edu; Also at the Department of Mathematics
† evan@pas.rochester.edu

any rougher if we average over smaller intervals.

We then obtain a result analogous to that of Levy: a metric in space with respect to which the scalar field is a.s. Lipschitz (we will use this term exclusively in the sense of local continuity). Our result is a particular case of the much deeper mathematical theory of regularity of random processes [2–5] . The idea of using a moving average (instead of an inner product with smooth test functions) seems to be new, and yields simple explicit results.

We then apply this moving average technique to other random fields of physical interest, noting that a new procedure is sometimes needed if the field considered has more severe divergences. For supplemental context, Appendix A discusses the intimate relationship of this work to the resistance metric on a lattice, connecting to an earlier paper [1], while Appendix B describes connections to a functional analytic approach to regularity of random processes.

Although we work with Gaussian fields in this paper, the short distance behavior is the same for asymptotically free interacting fields (up to sub-leading logarithmic corrections). The regularity of renormalizable but not asymptotically-free theories (such as QED or the Higgs model) can be quite different. The strength of interactions grow as distances shrink, possibly leading to a singularity (Landau pole). In the case of QED, we know that this is not physically significant, due to unification with weak interactions into a non-abelian gauge theory.

But the question of what happens to the self-interaction of a scalar quantum field (Higgs boson) at short distances is still open. In the absence of evidence at the LHC for supersymmetry or compositeness of the scalar, we have to consider the possibility that the Higgs model is truly the fundamental theory. The short distance behavior is dominated by interactions, necessitating new mathematical methods beyond perturbative renormalization. The extensive mathematical literature [2–5] on continuity of non-Gaussian processes ought to contain useful tools for physics. In order to apply this work to a full interacting theory, we must first know what happens in the simpler case of a free theory. This is part of the physical motivation for this paper.

II. CONTINUITY

A. Continuity of Random Processes

A random process $r(x)$ assigns a random variable to each value of x in some space X. The quantity

$$d(x, y) = \sqrt{\langle [r(x) - r(y)]^2 \rangle} \qquad (2.1)$$

satisfies the triangle inequality and so defines a metric on X (provided we identify any originally distinct points x, y for which $d(x, y) = 0$).

This metric need not be Euclidean or even Riemannian. A standard example is Brownian motion, where $d(x, y) = \sqrt{|x - y|}$, which is neither. We work out another simple case in Appendix A: when X is a finite graph, and d is the square root of the resistance metric [7–9].

One commonly successful approach to the study of continuity is to leave behind the intuitive structure associated with the space X, and begin instead by looking at structures related to the process of interest (such as the metric d above). At first, one might expect that the sample paths for a random process $r(x)$ will be necessarily continuous with respect to d. Although true for Brownian motion, almost sure continuity w.r.t. d does not hold in general. For Gaussian processes, a sufficient condition for continuity is the convergence of the Dudley integral [2–5]

$$J(\delta) = \int_0^\delta \sqrt{\log N(D, \epsilon)} d\epsilon, \quad \delta < D \qquad (2.2)$$

where $N(D, \epsilon)$ be the minimum number of balls of radius ϵ it takes to cover a ball of radius D in (X, d) (We suppress the D dependence of J for simplicity of notation). The possible divergence comes from the lower limit of the integral $\epsilon \to 0$.

Can we go beyond continuity? To speak of differentiable functions, a metric is not enough: we would need a differentiable structure on X which we do not have intrinsically. The closest analogue to differentiable functions on a metric space (X, d) are Lipschitz functions, for which $\frac{|f(x) - f(y)|}{d(x, y)}$ is bounded. For comparison, differentiable functions on the real line are Lipschitz, but not all Lipschitz functions are differentiable. Of course, all Lipschitz functions are continuous.

Even in cases where $J(\delta)$ converges, indicating that the sample paths are continuous, they may still not be Lipschitz with respect to the metric d above. Again, the Dudley integral comes to the rescue: using it we can define a more refined metric

$$\omega(x, y) = J(d(x, y)) \qquad (2.3)$$

Since $\sqrt{\log N(D, \epsilon)}$ is a decreasing function of ϵ, $J(\delta)$ is a convex function. Thus $J(d(x, y))$ satisfies the triangle inequality as well.

The sample paths of a Gaussian process for which $J(\delta)$ converges are [2] a.s. Lipschitz in this refined metric ω. Thus ω, rather than d, is the metric ("modulus of continuity") we must associate to a Gaussian random process.

What would one do if the Dudley integral does not converge? There is a more general theory [3] which gives necessary and sufficient conditions for continuity: a "majorising measure" must exist on X . We will not use this theory in this paper, but hope to return to it, as it can deal with more general cases than Gaussian processes (e.g., interacting quantum fields).

B. Quantum Fields

In this paper, we consider a quantum scalar field ϕ in the continuum limit. In the trivial case where ϕ is massless with 1 spatial dimension and no time dimension, the correlations of Brownian motion are reproduced and ϕ remains a continuous function. However, in any fully relativistic field theory, ϕ lives on a space of distributions, not functions. To get a sensible random variable, we must then take the inner product w.r.t. some test function h with zero average.

$$\phi[h] = \int \phi(x)h(x)dx, \quad \int h(x)dx = 0. \qquad (2.4)$$

We study the case where ϕ is a distribution with the weakest possible singularities; one might say we want a field that is "close" to being a function. The obvious candidate is the case of logarithmic correlations (For a recent review, see [6])

$$\langle \phi[h]\phi[h']\rangle = -\int \log|x-y|\; h(x)h'(y)dxdy. \qquad (2.5)$$

The corresponding Gaussian measure can be thought of as the (square of the) ground state wavefunction of a massless scalar field in $1+1$ dimensions. (More precisely, the continuum limit of the resistance metric of a row on an infinite square lattice, discussed in Appendix A).

The condition $\int h(x)dx = 0$ ensures that the covariance is unchanged if $\log|x-y|$ is replaced by $\log\lambda|x-y|$, meaning ϕ is scale invariant. Since ϕ has the physical meaning of a potential, observables such as $\phi[h]$ must be unchanged under a shift $\phi(x) \mapsto \phi(x) + a$, which equivalently suggests the requirement $\int h(x)dx = 0$.

C. Moving Average of a Quantum Field

Quantum fields which are only mildly singular can act on test functions which are not smooth or even continuous. It is not necessary to consider the whole space of test functions as in [4]; in this paper our test functions will be piecewise constant with compact support and zero mean.

We define a *moving average* of ϕ:

$$\bar{\phi}_s(u) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ \phi\left(s\left[u-w\right]\right) - \phi\left(s\left[0-w\right]\right) \right\} dw, \quad s > 0 \qquad (2.6)$$

This is the inner product of ϕ with a discontinuous test function h that has support on two intervals of width s based at su and at 0; the sign is chosen so that $\int h(x)dx = 0$. The probability law of $\bar{\phi}_s$ is not translation invariant: the second term ensures the boundary condition

$$\bar{\phi}_s(0) = 0. \qquad (2.7)$$

We will mostly work with the quantity

$$\bar{\phi}_s(u) - \bar{\phi}_s(v) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ \phi\left(s\left[u-w\right]\right) - \phi\left(s\left[v-w\right]\right) \right\} dw \qquad (2.8)$$

which has a translation invariant law. It is convenient to rescale the co-ordinate of the midpoint by the width (as we have already done), so that the variable u is dimensionless. Then the quantity

$$\rho(u,v) \equiv \sqrt{\langle\left[\bar{\phi}_s(u) - \bar{\phi}_s(v)\right]^2\rangle}, \qquad (2.9)$$

which is just a special case of (2.1), is finite and defines a metric. Moreover, it is independent of s in the logarithmically correlated case. This means the process $\bar{\phi}_s(u) - \bar{\phi}_s(v)$ has a probability law that is independent of s: a consequence of scale invariance, which is specific to logarithmic correlations. As an interesting aside, we note that $s\bar{\phi}_s(u)$ produces a solution to the wave equation in u and s.

The moving average does not depart from the essence of the standard idea of averaging over a test function. It is simply that a piecewise constant test function is especially convenient for a mildly singular quantum field as opposed to a smoother function. For more singular fields (e.g. scalar field in four dimensions) we would have to revert to more regular test functions.

III. LOGARITHMICALLY CORRELATED SCALAR FIELD IN 1 DIMENSION

In the logarithmically correlated case, we obtain explicit formula

$$\rho(u,v) = \rho(|u-v|) \qquad (3.1)$$

$$\rho(r) = \sqrt{L(r+1) + L(r-1) - 2L(r)} \qquad (3.2)$$

Where

$$L(r) = \frac{1}{2}r^2 \log r^2. \qquad (3.3)$$

Being a convex function of $r = |u-v|$, this $\rho(u,v)$ will satisfy the triangle inequality (not true of ρ^2, as seen in figure 1). Thus, ρ defines a translationally invariant metric.

Simple calculations (see IV) show that the Dudley integral J converges, so that $\bar{\phi}_s$ is a.s. continuous in ρ. Moreover we can construct a refinement

$$J\left(\rho(u,v)\right) \equiv \omega(u,v) \approx |u-v| \log\frac{1}{|u-v|} \qquad (3.4)$$
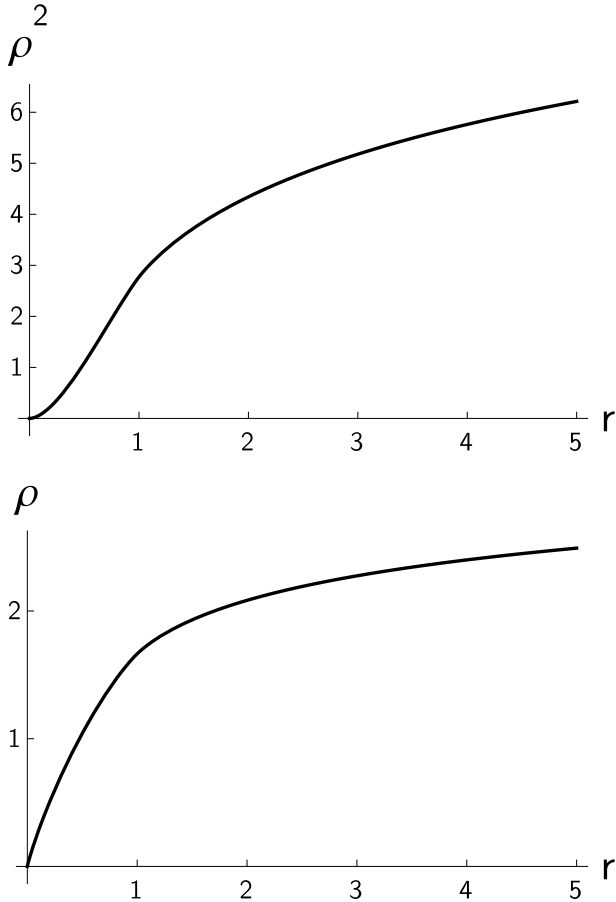
FIG. 1. The behavior of $\rho^2(r)$ and $\rho(r)$, setting $s = 1$.

FIG. 2. We show an approximate ϕ and two averages $\bar{\phi}_{\Lambda,s}$ where $\Lambda = 4000$, $L = 5$ and we average over width $s = 0.05$ (blue/dark) and $s = 0.2$ (orange/light). We see that despite the factor of 4 difference in averaging windows, the two appear interchangeable, demonstrating the scale invariance even for this approximate representation. The two appear to follow the same law, and the continuity is seen to be greatly improved.

with respect to which $\bar{\phi}_s$ is a.s. Lipschitz. This is a "modulus of continuity" for the moving average of a quantum field, analogous to that of Lévy for Brownian motion. (Note that there is no square root, however.)

We can obtain a crude picture of the moving average process by generating noise which has the same power spectrum as a log-correlated field, but with some high frequency cutoff. This is given by the Fourier series

$$\phi_\Lambda(x) = \sum_{m=1}^{\Lambda} \frac{1}{\sqrt{m}} \left[X_m \cos\left(\frac{\pi m x}{L}\right) + Y_m \sin\left(\frac{\pi m x}{L}\right) \right]$$

(3.5)

where X_m, Y_m are independent standard Gaussian variables. For large Λ (Ultra Violet cutoff) and L (the Infra Red cutoff, $-L < x < L$) this creates an intuitive "approximation" to the divergent field. Such a technique is often used to visualize white noise. While one must be careful claiming to "approximate" a distribution with a truncated series, ϕ_Λ gives us some sensible object on which to numerically test the properties of our moving average. This is carried out in figure 2. We see that $\bar{\phi}_s$ has the desired properties without requiring the full distribution.

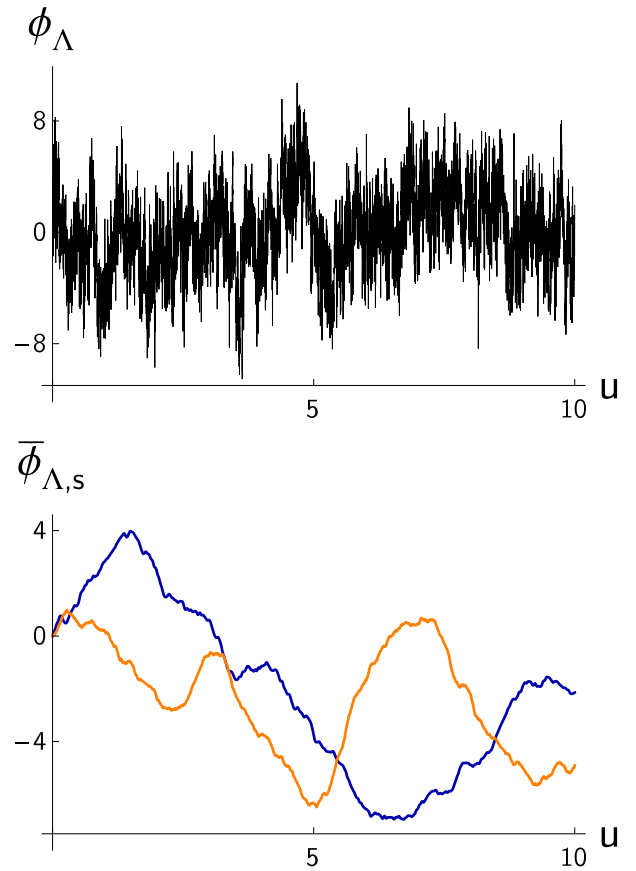While we focus on the log-correlated field for its math-

ematical simplicity, it is worth noting that such objects are not necessarily confined to the realm of mathematical fantasy. A free scalar field in one dimension can in principle be a good approximation for a real physical system, with one possible example being the electromagnetic field of certain optical fibers. If the refractive index of the fiber is chosen appropriately, only a finite number of transmission modes will be allowed. We can think of the wave equation as analogous to the Schrödinger equation, with the variable refractive index providing an effective potential. This potential can be chosen to allow only a finite number of bound states. Single-mode fibers have only one such state, leading to a system with one effective spatial dimension.

Even in the absence of light in the fiber (ground state of the electromagnetic field), there will be quantum fluctuations in the potential. In the absence of severe nonlinearities, these fluctuations can be modeled as two noninteracting scalar fields, one for each polarization mode.

If the wire is transparent over a sufficient frequency (maintaining its single-mode property and minimal dispersion for propagating waves), then the potential difference between two points will be a Gaussian random variable whose variance is approximately logarithmic with distance. The measurement of the potential would require a probe of finite size, so the averaging process employed in this paper provides a convincing model for the potential as seen by a measuring apparatus at a given instant. The considerations of this paper can be viewed as a model of the spatial regularity of the electromagnetic potential in such an optical fiber. This model could also, in principle, describe the ground state fluctuations of a quantum system confined to a very narrow region in 2 spatial dimensions, sometimes called a quantum wire.

Perhaps an experimental test of the sample field behavior in figure 2 is indeed possible. However, the details of realizing such a system and carrying out such measurements is highly non-trivial and not suited to the themes of this paper; we include this discussion mainly as a reminder that lower dimensional systems are often not so unphysical as they seem.

IV. EXPLICIT CALCULATIONS AND FURTHER EXAMPLES

A. Variance of $\bar{\phi}_s$ for the log-correlated field

The calculations that justify the above assertions are straightforward, but worth outlining as they help illuminate the properties discussed above. Because of the divergences, we can't use the standard approach directly to the quantum field, but only to its moving average. Begin with the observation that

$$F(a, b, c, d) \equiv -\int_a^b dx \int_c^d dy \log|x - y| \quad (4.1)$$

$$= \frac{3}{2}(a - b)(c - d)$$
$$+ \frac{1}{2}\left[L(c - a) - L(d - a) - L(c - b) + L(d - b)\right] \quad (4.2)$$

Where L is defined in (3.3). Note this quantity is not quite scale invariant: there is an "anomaly" proportional to $\log \lambda$.

$$L(\lambda r) = \lambda^2 L(r) + r^2 \log \lambda \quad (4.3)$$

$$F(\lambda a, \lambda b, \lambda c, \lambda d) = \lambda^2 F(a, b, c, d) - (b - a)(d - c)\log \lambda \quad (4.4)$$

Then

$$\langle \left[\bar{\phi}_s(u) - \bar{\phi}_s(v)\right]^2 \rangle = \quad (4.5)$$

$$= \frac{F(a, b, a, b)}{(b - a)^2} + \frac{F(c, d, c, d)}{(d - c)^2} - 2\frac{F(a, b, c, d)}{(b - a)(d - c)} \quad (4.6)$$

Where

$$\frac{F(a, b, a, b)}{(b - a)^2} = \frac{3}{2} - \frac{1}{2}\log(b - a)^2 \quad (4.7)$$

which only depends on the width of the interval $[a, b]$. We can then consider two intervals of equal width s, centered at su and sv, yielding

$$\langle \left[\bar{\phi}_s(u) - \bar{\phi}_s(v)\right]^2 \rangle = 3 - 2\log(s)$$
$$- \frac{2}{s^2} F\left(su - \frac{s}{2}, su + \frac{s}{2}, sv - \frac{s}{2}, sv + \frac{s}{2}\right) \quad (4.8)$$

From the scale transformation property above of F we can see that this quantity is independent of s: the "scale anomaly" of F cancels against $2\log s$. So we can simplify by putting $s = 1$ and expressing F in terms of L:

$$\langle \left[\bar{\phi}_s(u) - \bar{\phi}_s(v)\right]^2 \rangle \equiv \rho^2(u, v)$$
$$= L(u - v - 1) + L(u - v + 1) - 2L(u - v) \quad (4.9)$$

as was claimed.

B. Continuity of Brownian Paths

In using the Dudley integral, it is useful to begin with a well-known example. The most familiar example of a Gaussian process is Wiener's model of Brownian motion, for which $d(x, y) = \sqrt{|x - y|}$. If an interval $[0, 1]$ is divided into N equal parts, each part is contained in a $d-$ball of radius $\epsilon = \sqrt{\frac{1}{2N}}$. Thus $N(\epsilon) = 1 + \text{Floor}\left(\frac{1}{2\epsilon^2}\right)$ and for small δ, (where $\text{Floor}(a)$ is the integer part of the real number a)

$$J(\delta) \approx \delta\sqrt{-2\log \delta} \quad (4.10)$$

Thus Brownian sample paths B are almost surely continuous. More quantitatively, we may construct

$$\omega(r) = J(d(r)) = \sqrt{r \log(1/r)} \quad (4.11)$$

to obtain the result of Lévy that, with probability one,

$$\frac{|B(x) - B(y)|}{\sqrt{|x - y|\log\frac{1}{|x - y||}}} < C \quad (4.12)$$

as $x \to y$ for some constant C.

C. Continuity of $\bar{\phi}_s$ for logarithmically correlated fields

We can now show that the sample paths $\bar{\phi}_s(u)$ are continuous with probability one. Again, if $[0,1]$ is divided into N intervals, each will have radius $\epsilon = \rho\left(\frac{1}{N}\right)$. To get small ϵ, we must choose a large N; using the asymptotic behavior

$$\rho(r) \approx r\sqrt{-\log r} \qquad (4.13)$$

for small r,

$$\epsilon \approx \frac{1}{N}\sqrt{-\log\left[\frac{1}{N}\right]} \implies N(\epsilon) \approx \frac{1}{\epsilon}\sqrt{-\log \epsilon} \quad (4.14)$$

The Dudley integral converges:

$$J(\delta) \approx \delta\sqrt{\log\frac{1}{\delta}}, \quad \delta \to 0 \qquad (4.15)$$

$$J\left(\rho(r)\right) \equiv \omega(r) \approx r\log(1/r) \qquad (4.16)$$

which yields the claimed modulus of continuity.

D. Additional Examples for Comparison

1. Moving Average of Brownian Paths

It is informative to apply the moving average procedure to the Brownian case, where the paths $B(x)$ which we average over are continuous functions to begin with. Proceeding analogously, consider two intervals with width s with centers su and sv respectively. Then we can define, analogous to 4.1 but with some added foresight,

$$F(su-\frac{s}{2}, su+\frac{s}{2}, sv-\frac{s}{2}, sv+\frac{s}{2})$$
$$\equiv -\int_{su-s/2}^{su+s/2} dx \int_{sv-s/2}^{sv+s/2} dy|x-y| \qquad (4.17)$$

$$= \begin{cases} \frac{1}{3}\left(r^3\left(-3s^2+3s-1\right)+3r^2s^3+s^3\right) & 0<r<s \\ s^3r & r>s \end{cases}$$
$$\qquad (4.18)$$

Where $r = |u-v|$. We then have

$$\left\langle\left[\bar{B}_s(u)-\bar{B}_s(v)\right]^2\right\rangle \equiv \rho^2(r) \qquad (4.19)$$

$$= \frac{2s}{3} - \frac{2}{s^2}F\left(su-\frac{s}{2}, su+\frac{s}{2}, sv-\frac{s}{2}, sv+\frac{s}{2}\right) \quad (4.20)$$

It is easily seen that $\rho^2(\lambda r) = \lambda\rho^2(r)$, breaking scale invariance. Still for comparison purposes, we consider averaging over intervals of width $s=1$, noting that the scaling behavior will only change $\omega(r)$ by a constant factor.

As before, ρ^2 does not define a metric, but its square root ρ does. In the large-r limit we have

$$\rho(r) \sim \sqrt{r} \qquad (4.21)$$

While for small r,

$$\rho(r) \sim r \qquad (4.22)$$

This short distance behavior suggests by dimensional analysis that $\bar{B}_s(x)$ might be Lipschitz in the usual metric $|u-v|$, but the Dudley integral yields a weaker limit

$$\omega(r) \approx r\log(\frac{1}{r}) \qquad (4.23)$$

$$\frac{|\bar{B}_s(u)-\bar{B}_s(v)|}{r\log(\frac{1}{r})} < C \qquad (4.24)$$

Thus $\bar{B}_s$ is just shy of being Lipschitz in the usual metric, but is a.s. Lipschitz with respect to the metric $\omega(u,v) \approx |u-v|\log\frac{1}{|u-v|}$. Interestingly, this is the same ω we obtained in 4.16 for the log-correlated case, even though the short distance behavior of ρ is not quite the same (the difference in the Dudley integral vanishes for small δ). However ω for the Brownian sample paths prior to averaging 4.11 contains a square root not present here.

2. Power Law Correlations in 1D

We can use the same method as with Brownian motion to consider the moving average of a more general power-law correlated field such that

$$\langle\phi[h]\phi[h']\rangle = \text{sign}(\alpha)\int|x-y|^\alpha h(x)h'(y)dxdy, \quad (4.25)$$

$$\int h(x)dx = 0 \qquad (4.26)$$

When $\alpha > 0$ this is related to fractional Brownian motion [10]. When $\alpha = -1$ it is the restriction to one dimension of a massless scalar quantum field in $2+1$ dimensions. The moving average is no longer independent of the width of the intervals. Still, for purposes of comparison, we consider the average on intervals of fixed width $s=1$.

It is not difficult to evaluate the integrals to find that, in the small r limit,

$$\rho^2(r) \sim \begin{cases} r^2 & \alpha > 0 \\ r^{\alpha+2} & -2 < \alpha < 0,\ \alpha \neq -1 \\ r\log r & \alpha = -1 \end{cases} \quad (4.27)$$

The moving average is Lipschitz with respect to the modulus

$$\omega(r) = \begin{cases} r\log(1/r) & \alpha > 0 \\ r^{\frac{\alpha}{2}+1}\log(1/r) & -2 < \alpha < 0, \end{cases} \quad (4.28)$$

When $\alpha \leq -2$ the divergences are such that the moving average is an insufficient tool to smooth the quantum field. Note that $\omega(r)$ is the same in the logarithmic case as the case where $\alpha > 0$. The logarithmic case can be thought of as the critical case where the smoothness implied by Dudley's criterion starts to lessen.

3. Log Correlated Scalar Field in 3D

It is useful to work out a case in higher dimensions as well. The massless scalar field in $n+1$ space-time dimensions has correlation

$$\langle \phi(x)\phi(y) \rangle \propto \frac{1}{|x-y|^{n-1}} \quad (4.29)$$

Thus for $n > 1$ will we get power law, instead of logarithmic correlations. Yet a logarithmically correlated, non-relativistic, scalar field in 3 space dimensions is still of interest in cosmology [6, 11]. As with the log-correlated scalar field in 1D we must average it over a test function

$$\langle \phi[h]\phi[h'] \rangle = -\int \log|x-y| h(x)h'(y)dx^3 dy^3, \\ \int h(x)dx^3 = 0 \quad (4.30)$$

Recall that

$$-\log|x| + \text{const} = c\int e^{ik\cdot x}\frac{1}{|k|^3}\frac{d^3k}{(2\pi)^3} \quad (4.31)$$

where the constant $c = 2\pi^2$. The integral is not absolutely convergent, so we define it through zeta regularization.

We perform our moving average over the interior of a sphere with radius t, centered at tu

$$\bar{\phi}_t(u) \equiv \int_{|w|\leq 1}\{\phi(t[u-w]) - \phi(t[0-w])\}dw \quad (4.32)$$

$$\langle \bar{\phi}_t(u)\bar{\phi}_t(v) \rangle = \int_{|w|\leq 1}\langle \phi(t[u-w_1])\phi(t[v-w_1])\rangle dw_1 dw_2 \quad (4.33)$$

$$= c^2\int \frac{1}{|k|^3}e^{ik\cdot(u-v)t}\frac{d^3k}{(2\pi)^3}\int_{|w|\leq 1}e^{-ik\cdot(w_1-w_2)t}dw_1 dw_2 \quad (4.34)$$

Taking $t = 1$, this can be reduced to the form

$$\langle [\bar{\phi}_t(u) - \bar{\phi}_t(v)]^2 \rangle = 2^6\pi^4 G(r) \quad (4.35)$$

where

$$G(r) = \int_0^\infty dk \frac{1}{k^7}[\sin k - k\cos k]^2\left[1 - \frac{\sin kr}{kr}\right] \quad (4.36)$$

and $r = |u-v|$. We are not able to evaluate the integral analytically, but its convergence is clear, justifying the independence on t. In the large r limit, the integral is dominated by small k contribution. We then have

$$G(r) \approx \int_0^\infty dk \frac{1}{3k}\left[1 - \frac{\sin kr}{kr}\right] \quad (4.37)$$

$$\sim \log(r) + O(1) \quad (4.38)$$

In the case of small r, the dominant contribution comes from the first peak of $\frac{1}{k^7}[\sin k - k\cos k]^2$, which must occur for $k < 2\pi$ (i.e., $k \approx 5.678$). This allows us to treat kr as small, yielding the behavior

$$G(r) \approx r^2 \cdot \int_0^\infty dk \frac{1}{3k^5}[\sin k - k\cos k]^2 \quad (4.39)$$

$$\sim r^2 \quad (4.40)$$

The approximation can be verified numerically. This small r behavior dictates the continuity modulus discussed above. Namely, we have that for the 3D Log-correlated scalar field,

$$\rho(r) \sim r \quad (4.41)$$

$$\omega(r) \sim r\log(1/r) \quad (4.42)$$

A similar metric can be obtained for a log correlated field in other dimensions. Note that, once we have $\rho(r) \sim r$ for small r, the logarithm in the Dudley integral ensures that ω will not depend on the dimensionality (up to proportionality). This is not true if $\rho(r)$ has some other short distance behavior.

V. OUTLOOK

Gaussian processes correspond to free fields. The most elegant way to introduce interactions into a scalar field theory is to let it take values in a curved Riemannian manifold. This is the nonlinear sigma model in physics language, or the wave map in the mathematical literature. In 1+1 dimensions, such a theory, with a target space of a sphere or a compact Lie group, is well-studied in the physics literature. The short distance behavior is approximated by free fields with corrections computable in perturbation theory (asymptotic freedom). The only case for which mathematically rigorous results are known is that of the Wess-Zumino-Witten model, which has non-gaussian behavior at short distances; i.e., a "non-trivial fixed point" for the renormalization group. The related measure for the ground state of the quantum field has been constructed by Pickrell. (For a review, see [12]). It is natural to ask for regularity results analogous to ours in this case.

Looking further out, it would be interesting to quantify the regularity of quantum fields of the non-linear sigma model in two dimensional space time; and even further out, $\lambda\phi^4$ theory in four dimensions. It is possible that the "naturalness problem" of the standard model of particle physics has a resolution in terms of such a deeper understanding of the regularity of scalar quantum field theory. The "modern" theory [3] of regularity of non-gaussian processes ought to help with this daunting task. Even harder is the case of Yang-Mills fields. An analogue of our moving average is the Wilson loop. The measure of integration over the space of gauge fields is only known rigorously for the two dimensional case [13]. Regularity of Yang-Mills fields satisfying classical evolution equations (let alone random processes) is already a formidable problem under active investigation (see for example [14]).

VI. ACKNOWLEDGEMENTS

Appendix A: The Resistance Metric as the Variance of Potential Fluctuations

Without being aware of the "modern" theory [2–5] of random processes, we argued in an earlier paper [1] that the two point function (for space-like separations) of a quantum scalar field

$$\sqrt{\langle [\phi(x) - \phi(y)]^2 \rangle} \qquad (A1)$$

be used as the metric on spacetime. Since quantities such as $< \phi^2(x) >$ are divergent in a quantum field theory, the metric was defined with a regularization. With the lattice regularization of a free massless scalar field, our proposal for the metric fitted well with the idea of a resistance metric [7–9] popular in network theory.

In this appendix we show that the resistance metric (more precisely its square root) is simply a finite dimensional special case of the metric d appearing in the theory of Gaussian processes. This connection can be thought of as a particular case of the fluctuation-dissipation theorem of statistical mechanics: the potential difference across a resistor has thermal fluctuations with variance proportional to dissipation.

Imagine each edge of a network as a unit resistor connecting two vertices. Then, if a unit potential difference is applied across two vertices (k, l), the reciprocal of the power dissipated defines the effective resistance R_{kl} between them. Kirchhoff's laws imply a variational principle for this quantity [7]

$$R_{kl} = \frac{1}{\inf_\phi \left\{ \sum_{ij} A_{ij}(\phi_i - \phi_j)^2 \mid \phi_k - \phi_l = 1 \right\}} \qquad (A2)$$

where A is the adjacency matrix of the network. It is well known that this R_{kl} satisfies the triangle inequality, and is used as a metric in network theory.

It is convenient to introduce another symmetric matrix K by

$$\sum_{ij} A_{ij}(\phi_i - \phi_j)^2 = \sum_{ij} K_{ij}\phi_i\phi_j \qquad (A3)$$

Some Linear Algebra

Note that K is degenerate; it vanishes on the vector whose components are all equal to one:

$$\sum_j K_{ij}c_j = 0, \quad c \equiv (1, 1, , \cdots 1) \qquad (A4)$$

In particular, the equation

$$\sum_j K_{ij}\phi_j = J_i \qquad (A5)$$

has a solution only if

$$\sum_i J_i = 0. \qquad (A6)$$

But the solution is not unique because if ϕ_i is a solution, so is $\phi_i + ac_i$ .We can construct an inverse for K by restricting the potentials to the subspace satisfying

$$\sum_i \phi_i = 0. \tag{A7}$$

This fixes the overall constant ("ground potential") in ϕ_i. Now, K is an invertible map of this $n-1$ dimensional subspace to itself; there is a matrix G satisfying

$$\phi_i = \sum_j G_{ij} J_j \tag{A8}$$

Equivalently, we can define G by the equations

$$\sum_i G_{ij} = 0 \tag{A9}$$

$$\sum_j K_{ij} G_{jk} = \delta_{ik} - \frac{1}{n} c_i c_k \tag{A10}$$

Variational Principle

We can solve this variational problem for effective resistance using a Lagrange multiplier:

$$S = \sum_{ij} A_{ij} (\phi_i - \phi_j)^2 + \lambda (\phi_k - \phi_l) \tag{A11}$$

$$\frac{\partial S}{\partial \phi_i} = 0 \implies 2 \sum_j K_{ij} \phi_j + \lambda [\delta_{ik} - \delta_{il}] = 0. \tag{A12}$$

The solution is

$$\phi_i = -\frac{\lambda}{2} [G_{ik} - G_{il}] \tag{A13}$$

The constraint $\phi_k - \phi_l = 1$ determines λ:

$$\lambda = -\frac{2}{[G_{kk} + G_{ll} - G_{kl}]} \tag{A14}$$

Then

$$\sum K_{ij} \phi_i \phi_j = -\frac{\lambda}{2} \sum_i \phi_i (\delta_{ik} - \delta_{il}) = -\frac{\lambda}{2} (\phi_k - \phi_l) \tag{A15}$$

$$= \frac{1}{[G_{kk} + G_{ll} - G_{kl}]} \tag{A16}$$

Thus

$$R_{kl} = G_{kk} + G_{ll} - G_{kl}. \tag{A17}$$

Gaussian Integral

Given a matrix K with all positive eigenvalues except for one zero eigenvalue (with eigenvector c) we can define a Gaussian integral

$$Z(J) = \frac{1}{Z} \int_V e^{-\frac{1}{2} \sum_{ij} K_{ij} \phi_i \phi_j + \sum_i J_i \phi_i} d\phi \equiv \langle e^{J \cdot \phi} \rangle \tag{A18}$$

where $\sum_i J_i = 0$. The normalization factor Z is chosen such that $Z(0) = 1$.

Also, the range of integration is $V \equiv \mathbb{R}^n / \mathbb{R}$; the quotient of $\mathbb{R}^n$ by the translation $\phi_i \mapsto \phi_i + a c_i$. From each such orbit we can pick a representative that satisfies

$$\sum_i \phi_i = 0 \tag{A19}$$

This is an elementary example of "gauge fixing".

On this $n-1$ dimensional subspace K is invertible with the inverse G defined earlier. So

$$Z(J) = e^{\frac{1}{2} \sum_{ij} G_{ij} J_i J_J} \tag{A20}$$

In particular

$$\langle \phi_k \phi_l \rangle = G_{kl} \tag{A21}$$

and

$$\langle (\phi_k - \phi_k)^2 \rangle = G_{kk} + G_{ll} - 2 G_{kl} \tag{A22}$$

Thus, the effective resistance is equal to the variance of the voltage fluctuations:

$$R_{kl} = \langle (\phi_k - \phi_k)^2 \rangle. \tag{A23}$$

This point of view on the resistance is especially convenient if we average over K (e.g., percolation). We hope to return to this issue in another publication.

This procedure for deriving a formula for variance breaks down in the continuum limit. We need to work not with the potential itself, but an average of it over a small region.

Appendix B: Abstract Wiener Spaces

There is another point of view on the regularity of random processes, based on function spaces. Given an orthonormal basis e_n in an infinite dimensional Hilbert space H we can try to define a random variable

$$\phi = \sum_n g_n e_n \tag{B1}$$

where g_n are independent Gaussian random variables of zero mean and variance one. But the probability of this series converging in the norm of H is zero. For convergence, we need a weaker norm. More precisely, we seek a Banach space B and an embedding $i : H \to B$ such that the sum converges to a random variable valued in B. Such a triple (i, H, B) is the Abstract Wiener Space of Gross [15]. There is no "best possible" B; the choice is usually motivated by physics or geometry.

Recall that the Sobolev space H^s is the Hilbert space equipped with inner product $(f, \Delta^s g)$. For Brownian motion, the Hilbert space H defined above is the Sobolev space H^1 of functions whose derivatives are square integrable. One choice for B is the space of continuous functions. A more refined choice would be the space of functions with norm

$$\sup_{x,y} \frac{|f(x) - f(y)|}{\omega(x, y)}. \tag{B2}$$

where ω is the Lévy modulus described above. What is the Abstract Wiener Space for a massless scalar quantum field? Note that the ground state wave function of such a field is (in the notation preferred by physicists)

$$\psi(\phi) \propto e^{-\frac{1}{2} \int |k| |\tilde{\phi}(k)|^2 \frac{dk}{2\pi}} \tag{B3}$$

The quadratic form in the exponent can be written as

$$(\phi, \sqrt{\Delta}\phi) \tag{B4}$$

where Δ is the Laplacian and $(f, f) = \int |f(x)|^2 dx$. Thus, in more mathematical language, the log-correlated scalar field is the Gaussian process modeled on the Sobolev space $H^{\frac{1}{2}}(\mathbb{R})$.

Gross [16] has shown that any choice of B must fit within a small band of Hilbert spaces: $L^2 \subset B \subset H^{-\epsilon}$ for $\epsilon > 0$. We can make a proposal for the Abstract Wiener Space for the massless scalar field on the real line, based on the modulus of continuity: the completion of the space of continuous functions (modulo constants) by the norm

$$||f||_\omega = \sup_{u,v,s} \frac{|\bar{f}_s(u) - \bar{f}_s(v)|}{\omega(|u - v|)}, \quad \omega(u, v) \approx |u-v| \log \frac{1}{|u - v|} \tag{B5}$$

[1] A. Kar and S.G. Rajeev, Phys. Rev. D **86**, 065022 (2012).
[2] R. M. Dudley, in *The Annals of Probability*, Vol. 1 (1973) pp. 66–103.
[3] M. Ledoux and M. Talagrand, *Probability in Banach Spaces: isoperimetry and processes*, Vol. 23 (Springer, 2013).
[4] R. J. Adler, Lecture Notes-Monograph Series , i (1990).
[5] V. I. Bogachev, *Gaussian measures*, 62 (American Mathematical Soc., 1998).
[6] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas, [arXiv:1407.5605] (2014).
[7] J. Kigami, *Resistance Forms, Quasisymmetric Maps, and Heat Kernel Estimates* (American Mathematical Soc., 2012).
[8] R. Strichartz, *Differential Equations on Fractals* (Princeton, 2006).
[9] P. E. T. Jorgensen and E. P. J. Pearse, [arXiv:0906.2535] (2010).
[10] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang, *Stochastic calculus for fractional Brownian motion and applications* (Springer, 2008).
[11] S. Dodelson, *Modern cosmology* (Academic press, 2003).
[12] D. Pickrell, *Invariant measures for unitary groups associated to Kac-Moody Lie algebras*, Vol. 693 (American Mathematical Soc., 2000).
[13] A. Sengupta, J funct. Analysis **108**, 231 (1992).
[14] N. Charalambous and L. Gross, Comm. Math. Phys. **317**, 727 (2013).
[15] L. Gross *et al.*, in *Berkeley Symp. on Math. Statist. and Prob. Proc. Fifth Berkeley Symp. on Math. Statist. and Prob.*, Vol. 2, Pt. 1 (Univ. of Calif. Press, 1967), 31-42 (The Regents of the University of California, 1967).
[16] L. Gross, Private Communication (2015).