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Perturbation theory in (2,2) signature

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Abstract

We identify a natural analytic continuation in four dimensions from Minkowski signature to a signature with two time-like momentum components. For two, three and four-point diagrams whose external momenta define a scattering plane, this continuation can be implemented as a contour deformation that leaves dependence on the momenta unchanged. For arbitrary ultraviolet-finite scalar diagrams it is possible to do two integrals per loop in terms of simple poles in the new signature. This results in a representation of any such diagram as a sum of terms, each with two remaining integrals per loop.

I. INTRODUCTION

Scattering amplitudes are to a large extent determined by their singularity structure in the complex planes of external momenta [1, 2]. This feature has been exploited, for example, to derive a recursive construction for tree amplitudes [3] from singularities at unphysical momenta, and enables the development of unitarity-based techniques [4, 5] for the evaluation of loop integrals [6, 7].

For the construction of scattering amplitudes, any diagram in perturbation theory can be thought of as a multidimensional complex integral, in the first instance by a Wick rotation from Euclidean space. The rotation effectively changes a free Euclidean Green function, $1/(-k_1^2 - \cdots - k_n^2 - m^2)$ to the causal propagator, $1/(k_0^2 - k_1^2 \cdots - k_{n-1}^2 - m^2 + i\epsilon)$. In this sense, the choice of contour corresponds to a change in the signature of the metric, from all minus (or plus) to $(1, 3)$.

Thus, the difference between Euclidean and Minkowski Green functions can be thought of as a difference in the choice of contour integration. It is therefore natural to study other signatures, corresponding to other choices of contour, in particular, a $(2, 2)$ signature, for which $k^2 = k_0^2 + k_1^2 - k_2^2 - k_3^2$. In this connection, it is of interest to ask how to construct a perturbation theory based on this signature as an analytic continuation of Minkowski, and therefore ultimately Euclidean, perturbation theory.

The symmetries characteristic of $(2, 2)$ signature help relate momentum to twistor spaces through a Fourier transform [8]. The relationship between perturbation theory in $(2, 2)$ and Minkowski formulations [9], however, appears to be subtle and not yet fully clarified. Toward this goal, we will show below that there exists a non-singular analytic continuation for scalar diagrams, analogous to Wick rotation, from Minkowski to $(2, 2)$ signature that crosses no singularities. Perhaps surprisingly, singularities in the rotated integrals are avoided by the same "$i\epsilon$" prescription as with Minkowski signature. Theories with 'two times' have also been studied for their own interest [10, 11], and most of our results below apply when the number of spatial dimensions is greater than two.

In the process of the transformation from $(1, 3)$ to $(2, 2)$, both internal loop integration contours and external momenta are continued in terms of a single angular variable. This naturally takes off shell any external momenta that are on the light cone for Minkowski signature, except for momenta with no components in the transverse direction that is rotated.

We observe that for such momenta overlapping collinear-infrared singularities survive the rotation, and clarify a subtlety in the use of light cone coordinates that can lead to an apparent vanishing of otherwise nonzero integrals. More generally, for two, three and four-point functions whose external momenta define a scattering plane, Lorentz invariance always allows us to choose a frame in which the Minkowski and $(2,2)$ functions are identical. This result holds for massive and massless internal and external lines, on shell and off shell.

We begin the explicit development of these results in Sec. II, where we show how to construct perturbation theory for $(2,2)$ signature by a Wick-like rotation from Minkowski space, and discuss similarities and differences in the singularity structure of diagrams evaluated in $(1,3)$ and $(2,2)$ signature. In $(2,2)$ signature it is natural to introduce two sets of light cone coordinates, and in Sec. III we use this approach to show that after integration over the two "minus" components of each loop, the remaining $2L$-dimensional integrals are over a finite region, dependent on the external momenta. We also observe that in $(2,2)$ signature, perturbative unitarity is realized in two different ways. Restricting ourselves to ultraviolet finite diagrams, in Sec. IV we derive a representation for an arbitrary $(2,2)$ scalar diagram as a $2L$-dimensional integral. We go on in Sec. V to derive a compact representation for one-loop integrals with arbitrary masses and external momenta, and illustrate how infrared singularities manifest themselves in $(2,2)$ signature, using our representation for the box diagram. We close with a summary of our results.

II. FROM MINKOWSKI TO $(2,2)$

As indicated above, our guiding criterion for the definition of $(2,2)$ integrals is that they be analytic continuations of corresponding integrals in Minkowski space, constructed so that the continuation manifestly encounters no singularities. In fact, such a construction can be carried out by a direct generalization of Wick rotation. In this discussion, we restrict ourselves to scalar integrals only. Like Wick rotation, the construction turns out to be completely general and rather simple. We give it below, followed by a few consequences.

A. Defining the integrals

We consider an arbitrary perturbative integral, written in covariant form, with L loops and N lines of arbitrary mass, possibly with positive imaginary parts, and with external momenta p_j, which may or may not be on shell,

$$I_{N,L}(p_j) = (-i)\, i^{L-1} \prod_{\text{loops } a=1}^{L} \int \frac{d^4 l_a}{(2\pi)^4} \prod_{\text{lines } i=1}^{N} \frac{1}{k_i^2(l_a, p_j) - m_i^2 + i\epsilon}\,. \tag{1}$$

We take $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$ to start. The first factor of $-i$ on the right hand side normalizes tree diagrams to be real whenever each vertex is associated with a factor $-i$ and each line with an i. Here and below, we set the coupling constant to unity. As indicated in Eq. (1), line momenta are themselves determined by the loop and external momenta, through linear combinations that can by summarized by matrices η_{ia} and ξ_{ij}, respectively,

$$k_i = \eta_{ia} l_a + \xi_{ij} p_j\,, \tag{2}$$

with η_{ia}, $\xi_{ij} = \pm 1$, 0. The integration contours are defined, as usual, by the "$i\epsilon$" prescription, in which energy integrals pass above the pole at the larger on shell energy for each propagator, and below the pole at the smaller on shell energy.

We now define a new parameter, θ, and a new function, $I_{N,L}(p_j, \theta)$, constructed so that it equals the original diagrammatic integral, (1) at $\theta = 0$,

$$I_{N,L}(p_j, \theta = 0) = I_{N,L}(p_j)\,. \tag{3}$$

The new function is defined in terms of momentum components, as a joint rotation of the 'one' components, p_j^1 of all external and l_a^1 of all loop momenta, as illustrated in Fig. 1,

$$\begin{aligned} I_{N,L}(p_j, \theta) \;=\; & (-i)\, i^{L-1} \prod_{\text{loops } a=1}^{L} \int \frac{dl_a^0 dl_a^3 dl_a^2}{(2\pi)^4} \int_{-\infty}^{\infty} d(l_a^1\, e^{-i\theta}) \\ & \times \prod_{\text{lines } i} \frac{1}{\left(k_i^0(l_a^0, p_j^0)\right)^2 - \left(\eta_{ia} l_a^1 e^{-i\theta} + \xi_{ij} p_j^1 e^{-i\theta}\right)^2 - \left(k_i^2(l_a^2, p_j^2)\right)^2 - \left(k_i^3(l_a^3, p_j^3)\right)^2 - m_i^2 + i\epsilon}\,. \end{aligned} \tag{4}$$

At finite θ, the real and imaginary parts of the denominator of propagator i are given by

$$\begin{aligned} \mathrm{Re}(k_i^2 + i\epsilon) \;&=\; (k_i^0)^2 - \left(\eta_{ia} l_a^1 + \xi_{ij} p_j^1\right)^2 \cos(2\theta) - (k_i^2)^2 - (k_i^3)^2 \\ \mathrm{Im}(k_i^2 + i\epsilon) \;&=\; \left(\eta_{ia} l_a^1 + \xi_{ij} p_j^1\right)^2 \sin(2\theta) \;+\; \epsilon\,. \end{aligned} \tag{5}$$

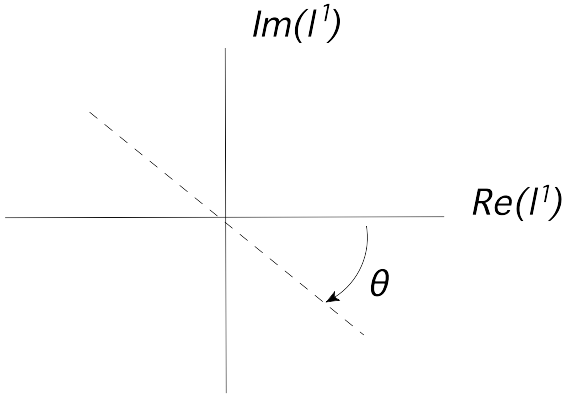

FIG. 1: Rotation of the l^1 contour.

As we vary θ from zero to $\frac{\pi}{2}$, the coefficient of the square of $(k_i^1)^2$ in the real part changes sign, while the imaginary part of each diagram starts at $+i\epsilon$, increases to a maximum at $\theta = \frac{\pi}{4}$, always staying positive, and decreases back down to $+i\epsilon$ at $\theta = \frac{\pi}{2}$. For fixed values of the original momenta, p_j, the integrand is thus finite over the entire continuation in θ, and crosses no singularities. The procedure works for any choice of masses, so long as their imaginary parts are negative.

The result of this procedure, continuation from $\theta = 0$ to $\theta = \frac{\pi}{2}$, is a smooth transition from Minkowski signature, with a single time-like momentum component, to a $(2,2)$ integral, in which the 1 component has joined the 0 component as a positive contribution to the invariant squares of the momenta. This fully-rotated integral is given explicitly by

$$I_{N,L}\left(p_j, \frac{\pi}{2}\right) = - \prod_{\text{loops } a=1}^{L+1} \int \frac{dl_a^0 dl_a^1\, dl_a^2 dl_a^3}{(2\pi)^4} \prod_{\text{lines } i=1}^{N} \frac{1}{\left(k_i^0\right)^2 + \left(k_i^1\right)^2 - \left(k_i^2\right)^2 - \left(k_i^3\right)^2 - m_i^2 + i\epsilon}\,, \tag{6}$$

where we have suppressed the linear dependence of line momenta on loops and external lines. We note that the integrals are defined by the same $i\epsilon$-prescription as in Minkowski space, a perhaps surprising result. This definition has (at least) two important consequences for the singularity structure of $(2,2)$ diagrams, which we develop in the following two subsections.

B. Signature-invariance of two, three and four-point functions

For two- three- and four-point functions in Minkowski space, we can often go to a frame where at least one component of spatial momentum is zero for all external lines. For $2 \to 2$ scattering, for example, this is the normal to the scattering plane. More generally, a scattering plane can be identified by solving $v \cdot p_i = 0$ for $i = 1 \dots 4$, which is always possible because at most three of the vectors are linearly independent. Of course, the vector v is orthogonal to any linear combination of the p_i, $L^\mu = \sum a_i p_i$. If we can find *any* such combination L^μ that is timelike, $L^2 > 0$, $v \cdot L = 0$ implies that v^μ is space-like. The subspace normal to v then generalizes the scattering plane defined by physical momenta to the full class of external momenta any of whose linear combinations is timeline. For any such set of momenta, if we transform to a frame where v^μ is in the 'one' direction above, all $p_j^1 = 0$, and the rotations of loop momenta can be carried out for fixed (Minkowskian) external momenta without crossing singularities. Indeed, if the momentum integrals are convergent, Cauchy's theorem ensures that the integrals are independent of θ, because the rotation can be treated as the change of a contour that can be closed at infinity. As a result, for such diagrams, we have

$$A_n^{(3,1)}(p_1 \dots p_n) = A_n^{(2,2)}(p_1 \dots p_n)\,, \tag{7}$$

for $n \leq 4$, so long as the extra time-like coordinate is chosen perpendicular to the space spanned by the p_i, which remain in a Minkowskian $(1,2)$ subspace. Such a choice is always possible for $n \leq 4$. This result applies to scalar diagrams of all orders, any choices of (real) masses, and for off-shell Green functions as well as on shell amplitudes. Indeed it applies to diagrams with any number of external lines so long as all $p_j^1 = 0$. We note that an analogous invariance applies to Wick rotation for diagrams with all $p_j^0 = 0$.

Although a simple consequence of analytic continuation, the relation (7) will enable us to give new representations for loop integrals in $(1,3)$ signature for two-, three- and four-point functions in Minkowski space, as special cases of general representations of n-point functions in $(2,2)$. These representations will follow from the introduction of double light cone coordinates in $(2,2)$ signature, which we will describe in Sec. III. We turn first, however, to a brief investigation of the singularity structure of $(2,2)$ integrals.

C. Singularities in $(2,2)$

Starting from the defining equation (6), we can make quite strong statements about the origin of the singularities of perturbative integrals in $(2,2)$ signature. In particular, because they share the same $i\epsilon$ prescription with their $(1,3)$ counterparts, the Landau equations [1, 12, 13] that help determine singularities in perturbative integrals take the same form for $(1,3)$ and $(2,2)$ signatures, Eqs. (1) and (6). This is most easily confirmed by reviewing the use of Feynman parameterization to identify possible pinches in loop integrals [1], to emphasize its signature independence. For the $(2,2)$ case, for example, we have simply

$$I_{N,L}\left(p_j, \frac{\pi}{2}\right) = -\,\Gamma(N-1) \prod_{\text{loops } a=1}^{L} \int \frac{dl_a^0 dl_a^1\, dl_a^2 dl_a^3}{(2\pi)^4} \prod_{\text{lines } i=1}^{N} \int_0^1 d\alpha_i\; \delta\left(1 - \sum_{i=1}^{N} \alpha_i\right) \times \frac{1}{\left[\sum_{i'=1}^{N} \alpha_{i'} \left[k_{i'}^2(l_a, p_j) - m_{i'}^2\right] + i\epsilon\right]^N}\,, \tag{8}$$

the difference from $(1,3)$ being entirely in the definition of the k_i^2 on the right hand side, and the argument on the left. Because line momenta $k_{i'}$ are linear in loop momenta l_a, the single, parameterized denominator is quadratic in every loop momentum component l_a^ν, while being linear in the parameters $\alpha_{i'}$. We note that the relative signs of the denominator terms in this expression are determined uniquely by requiring that the coefficient of the imaginary term $i\epsilon$ be α_i-independent. This ensures that whatever component integral we do first has one Nth order pole in the upper half plane, and one in the lower half plane.

Necessary conditions for the presence of a singularity in (8) are then that those line momenta $k_{i'}$ whose coefficients $\alpha_{i'}$ are nonzero must satisfy

$$\frac{\partial}{\partial l_a^\nu} \left[\sum_{i'=1}^{N} \alpha_{i'} \left(k_{i'}(l_a, p_j)\right)^2 + i\epsilon\right] = 2 \sum_{i'=1}^{N} \alpha_{i'} \eta_{ai'} k_{i'}^\nu(l_a, p_j) = 0\,, \tag{9}$$

for every component ν of every loop l_a, with η_{ai} the matrix that relates loop to line momenta in Eq. (2) above. These are the same (Landau) equations, whether in $(1,3)$ or $(2,2)$. A singularity also requires, of course, that $k_i^2 = m_i^2$ for the relevant lines. Thus, given the differences in the signatures that define k_i^2 for $(1,3)$ and $(2,2)$, there is no immediate correspondence between momentum configurations found in the two cases for the same diagrams. In particular, it is not obvious whether there is an analog in $(2,2)$ of the Coleman-Norton

criterion for singularities [14] in $(1,3)$, that on shell momenta at a singularity correspond to a physical scattering process. This would at least require us to develop intuition on what "physical scattering" means in $(2,2)$ signature. Nevertheless, although we do not have such a general criterion for singularities in $(2,2)$, we can make some significant observations, finding a wide range of both similarities and differences from $(1,3)$.

In this connection, we note a simple result on singularity surfaces for Green function integrals like $I_{N,L}(p_j,\theta)$, Eq. (8). When the external lines of a diagram are restricted to a subspace where one component vanishes for all lines,

$$p_j^\nu = 0\,, \quad \text{all } j\,, \tag{10}$$

the corresponding component of all *internal* on-shell lines must vanish at any pinch singularity. To see this, consider an arbitrary "candidate" pinch surface with a set of on shell lines, k_l, $k_l^2 = m_l^2$, some of which have nonzero component k_l^ν. Starting with any line momentum $k_i \in \{k_l\}$ with $k_i^\nu \neq 0$, we can follow the flow of positive (or negative) k_i^ν, from line i into some unique vertex of the diagram, which we label as, say, v_0. Let us consider the combination k_i, v_0 as the beginning of a path (a "chain") through the diagram. We continue the path by picking any line attached to vertex v_0 that carries positive ν component out of v_0 to some other vertex v_1. By momentum conservation, there must be at least one such line. In this way, we continue the path through the diagram. Because of our assumption (10), the ν component can never flow out of the diagram, and therefore the path will stay inside the diagram at each step. If the diagram is of finite order, the path will eventually intersect itself, by connecting a sequence of vertices,

$$v_0 \to v_1 \to \cdots \to v_n \to v_0\,. \tag{11}$$

In general, there is more than one such path if the diagram has more than one loop, but in any case we can pick a loop momentum l_a that flows precisely around the loop specified by the sequence of vertices (11). For this loop, all the factors η_{ia} and all the ν components of lines k_i are positive, and the Landau equations (9) cannot be satisfied for nonzero α_i. Therefore, this set of lines, and since they are arbitrary any set of lines with nonzero k_i^ν, cannot satisfy the Landau equations and cannot be pinched on shell.

This result shows us that a kinematic range where the two signatures give a similar singularity structure can be found for $2 \to 2$ on shell scattering amplitudes,

$$p_1 + p_2 \to p_3 + p_4\,, \quad p_i^1 \;=\; 0\,, \quad p_i^0 \;>\; 0\,. \tag{12}$$

For such a process, no pinch surface can have internal lines with a one component, and the classification of pinch singularities follows the same reasoning as in Minkowski space [16–18]. It is worth pointing out that in Minkowski space, because the scattering is planar in the center-of-mass, pinch surfaces are always restricted to a three-dimensional subspace here as well. Recall that we have observed above that the continuation can be carried out without changing external momenta in this frame. The only difference in $(2,2)$ compared to $(1,3)$ is that the "normal" to this subspace is now a time-like rather than a space-like variable. In particular, for fixed angle scattering in massless theories [16–18], pinch surfaces in $(2,2)$ reduce to the same "jet", "soft" and "hard" subdiagrams long known to characterize these amplitudes in Minkowski space. We will not pursue a further investigation of this case here, but only note that there is every reason to believe that for gauge theories the basic factorization properties and infrared structure of massless Minkowski $2 \to 2$ amplitudes [17, 18] are the same in $(2,2)$.

The fundamental similarity between $(1,3)$ and $(2,2)$ singularity structure for $2 \to 2$ amplitudes is certainly the exception, and we need not look far for fundamental differences, once we relax the condition $p_j^1 = 0$, for external lines. Indeed, once the number of external lines exceeds four, this condition restricts us to a subspace of their full momentum space. In the new signature, a general amplitude has many singularities that are qualitatively different from those found in Minkowski signature.

A fundamental property of light-like lines in Minkowski space is that the sum of two positive energy light-like momenta has a positive semi-definite invariant mass, which vanishes only when the momenta are proportional, that is to say, the lines are collinear. For $(2,2)$ signature, in contrast, every light-like momentum, v^μ, $v^0 > 0$ defines a one dimensional subspace of light-like vectors $\bar{v}^\mu$ with $\bar{v}^2 = \bar{v} \cdot v = 0$, found by making equal $SO(2)$ rotations on the pairs (v^0, v^1) and (v^2, v^3),

$$\bar{v}^\mu \;=\; \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \;,\; R \in SO(2)\,. \tag{13}$$

As a result, in $(2,2)$, the sum of two, non-collinear light-like momenta can also be light-like. This has consequences for the singularity structure even of tree diagrams, as illustrated by

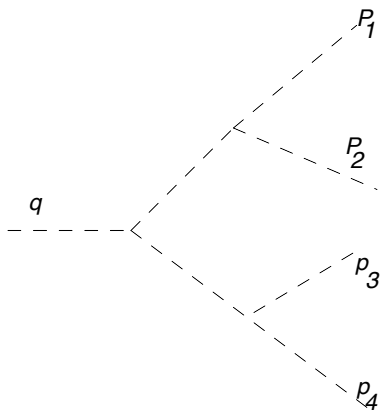

FIG. 2: One-to-four scalar process discussed in the text.

Fig. 2. Here we start with the generalized "rest" momentum, $q^\mu = (Q, Q', 0, 0)$ in $(2,2)$ signature, and we show a lowest-order diagram that produces four lines of momenta

$$\begin{aligned} p_1 &= \left(\frac{Q}{2}, 0, 0, \frac{Q}{2}\right), \\ p_2 &= \left(0, \frac{Q'}{2}, \frac{Q'}{2}, 0\right), \\ p_3 &= \left(\frac{Q}{2}, 0, 0, -\frac{Q}{2}\right), \\ p_4 &= \left(0, \frac{Q'}{2}, -\frac{Q'}{2}, 0\right). \end{aligned} \tag{14}$$

For this set of "outgoing particles", the virtual lines have $(p_1+p_2)^2 = (p_3+p_4)^2 = 0$, in sharp contrast to the corresponding diagrams of Minkowski space whenever the outgoing lines are noncollinear. This suggests that beyond the simplest amplitudes, the concept of "jets", for example, would have to be generalized in any complete picture of $(2,2)$ scattering.

III. LIGHT CONE VARIABLES

We now turn to another interesting feature of $(2,2)$ integrals that are ultraviolet convergent. In the rotated integral, $I_{N,L}(p_j, \frac{\pi}{2})$, Eq. (6), there is a nice symmetry between the pairs of components, $0,3$ and $1,2$, and it is natural to introduce two pairs of light cone loop momentum variables,

$$\begin{aligned} k^{\dot{\pm}} &\equiv k^1 \pm k^2, \\ k^{\pm} &= k^0 \pm k^3, \end{aligned} \tag{15}$$

where we have chosen a normalization for which

$$\begin{aligned} k^2 &= k^+k^- + k^{\tilde{+}}k^{\tilde{-}}\,, \\ 2k\cdot k' &= k^+k'^- + k^-k'^+ + k^{\tilde{+}}k'^{\tilde{-}} + k^{\tilde{-}}k'^{\tilde{+}}\,, \\ d^4k &= \frac{1}{4}dk^+dk^{\tilde{+}}dk^-dk^{\tilde{-}}\,. \end{aligned} \tag{16}$$

We use these variables below to develop a procedure for doing $2L$ integrals in an arbitrary ultraviolet finite L-loop diagram. Before doing so, we point out one subtle point in making such a change of variables. This observation applies as well to the use of light cone variables in $(1,3)$ to develop, for example, light cone ordered perturbation theory [19].

A. Convergence and light cone variables

Consider the manifestly finite two-dimensional integral, of a self-energy form,

$$\begin{aligned} I_2\left(p,m^2\right) &= -i\int_{-\infty}^{\infty}\frac{dk_1dk_0}{(2\pi)^2}\,\frac{1}{[(k_0+p)^2-(k_1-p)^2-m^2+i\epsilon]}\,\frac{1}{[k_0^2-k_1^2-m^2+i\epsilon]} \\ &= \frac{1}{4\pi\,m^2}\,. \end{aligned} \tag{17}$$

Here the two-dimensional "external" momentum is $P=(P_0,P_1)=(p,-p)$, with $p>0$. The result of this integral is independent of parameter p because $P^2=0$, and readily follows from standard formulas based on Feynman parameterization and Wick rotation. We can also evaluate (17) as a pair of complex integrals explicitly in terms of its poles. Each of the two variables, k_0 and k_1 encounters four poles, two in each half plane, and we can perform the integral by closing one contour in either the upper or lower half plane without Wick rotation.

Now let us try to re-express the integral, Eq. (17) in terms of light cone coordinates, $k^{\pm}\equiv k_0\pm k_1$, as in Eq. (15). In this notation, the vector P has $P^-=2p$ and $P^+=0$. This, however, gives

$$I_2\left(m^2\right)=\frac{-i}{2}\int_{-\infty}^{\infty}\frac{dk^+dk^-}{(2\pi)^2}\,\frac{1}{[k^+(k^-+2p)-m^2+i\epsilon]}\,\frac{1}{[k^+k^--m^2+i\epsilon]}\,, \tag{18}$$

which vanishes because the two poles in the k^- integral are always on the same side of the contour, regardless of the value of k^+. This would seem to imply that the self energy vanishes whenever $P^+=0$, a paradoxical result that would extend to four dimensions. On

the other hand, if we do the k^+ integral first, the result is nonzero, because the two poles in k^+ are on opposite sides of the contour for $-P^- < k^- < 0$.

The reason for this inconsistency is that the change from Cartesian to light cone variables involves an exchange of integrals that are not uniformly convergent in this case. To be specific, suppose we wish to do the k^- integral first at fixed k^+. We would then first change variables from (say) k_0 to k^+ in the original k_0, k_1 form, Eq. (17) at fixed k_1, giving

$$I_2\left(m^2\right) \;=\; \frac{-i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk^+ \; \frac{1}{[(k^+ - k_1 + p)^2 - (k_1 - p)^2 - m^2 + i\epsilon]} \times \frac{1}{[(k^+ - k_1)^2 - k_1^2 + i\epsilon]}\,. \tag{19}$$

The next step would be to exchange the k_1 and k^+ integrals, and then change variables from k_1 to k^- at fixed k^+, giving (18), but this is not possible because the unbounded k_1 integral diverges badly for $k^+ = 0$. We may note, however, that this pitfall does not prevent us from carrying out rotations in Cartesian coordinates from $(1,3)$ to $(2,2)$ as above. The transition to light cone coordinates is a separate issue.

B. Finite volume

Having pointed out a subtlety associated with the vanishing of external plus momenta, we can limit ourselves to all nonzero external plus momenta. In this case, we can do all the minus loop integrals in a given diagram, to get a sum of terms given by the rules of light cone ordered perturbation theory (LCOPT) [19]. This procedure does not depend at all on whether or not we have carried out the rotation that takes us from $(1,3)$ to $(2,2)$ signature. For a scalar diagram $\mathcal{G}$ (normalized as above so that tree graphs are real) the LCOPT expression found by integration over minus momenta is related to the covariant form by

$$\begin{aligned}\mathcal{G}(\{p_a\}) \;&\equiv\; (-i)\, i^{L-1} \int \prod_{\text{loops}\,\{l\}} \frac{d^4 l}{(2\pi)^4} \prod_{\text{lines}\,k} \frac{1}{k^2 - m_k^2 + i\epsilon} \\ &=\; -\sum_{\text{orderings}\;T} \int \prod_{\text{loops}\,\{l\}} \frac{dl^{\tilde{+}} dl^{\tilde{-}} dl^+}{4(2\pi)^3} \prod_{\text{lines}\,\{k\}} \frac{\theta(k^+)}{k^+} \prod_{\text{states}\,\{i\}\;\text{in}\;T} \frac{1}{P_i^- - s_i\left([k]\right) + i\epsilon}\;,\end{aligned} \tag{20}$$

where $P_i^- = \sum_{a\in i} p_a^-$ is the algebraic sum of total incoming and outgoing minus momenta up to state i, and where

$$\begin{aligned} s_i\left([k]\right) &= \sum_{\text{lines}\,\{k\}\,\in\,\text{state}\,i} [k]^- \\ &= \sum_{k\in i} \frac{-k^{\tilde{+}}k^{\tilde{-}} + m_k^2}{k^+} \\ &\equiv \sum_{k\in i} \Big(- k^{\tilde{-}} r_k + \mu_k \Big) \,, \end{aligned} \tag{21}$$

is the sum of all the on shell minus momenta in a specific state. We have written the result in terms of the $(2,2)$ signature transverse 'light cone' variables formed from $k_T = (k^1, k^2)$ in Eq. (15), and we define

$$\begin{aligned} r_k &\equiv \frac{k^{\tilde{+}}}{k^+} \\ \mu_k &\equiv \frac{m_k^2}{k^+} \,, \end{aligned} \tag{22}$$

where the label k identifies both the line momentum and the corresponding mass. The transition to $(2,2)$ signature can be carried out before the minus integrals that lead to the second equality in Eq. (20), or after.

We will first use the invariant integral representation of an arbitrary ultraviolet finite diagram in Eq. (20) to show that the volume of the l^+ integrals is finite after the l^- integrals at fixed $l^{\tilde{+}}$ and $l^{\tilde{-}}$. We will go on to use the light cone ordered form to show that the $l^{\tilde{+}}$ integrals also have a finite volume after the integration over the $l^{\tilde{-}}$ for diagrams that are ultraviolet finite.

Assume, then, that some plus loop momentum grows without bound in such a way that it is much larger than the corresponding components of all external momenta. As we shall see, it is then possible to find a minus loop integral such that all of its poles are in the same half-plane, either upper or lower. Such an integral gives zero, and because we assume that the diagrams are well-behaved at infinity, we can choose to do this minus integral first. We conclude that the integral is non-zero only in a bounded region in plus momentum. To be specific, let us provide an explicit construction of the loop in question, by an argument similar to that of Sec. II C above.

The construction begins by identifying the internal line with the largest plus momentum, which we may call $K_1^+ > 0$. We can choose the orientation of momentum flow so that this

quantity is positive. Momentum K_1^+ then flows into a unique vertex of the diagram, which we may call V_1, and out of a unique vertex V_0. Suppose that vertex V_1 is an a-point vertex. Since momentum K_1^+ flows in to V_1 at least one line must carry a momentum $K_2^+ \geq K_1^+/(a-1)$ out of V_1. If K_1^+ is sufficiently large, this line cannot flow out of the diagram, but must flow to another vertex, V_2, internal to the diagram. Assuming for simplicity that this is also an a-point vertex, at least one line must carry plus momentum $K_3^+ \geq K_1^+/(a-1)^2$ out of V_2. We repeat the process, following the largest flow of plus momentum, and in each case, we find a momentum that flows out of the next vertex that is proportional to K_1^+, and which therefore cannot carry momentum onto an external line when K_1^+ is large enough. For any diagram of finite order, we will eventually encounter a vertex $V_m = V_k$, with $k = 0 \ldots m-2$ ($m = 2$ is not possible for a diagram with no ultraviolet-divergent subdiagrams in four dimensions). This is the loop we are after.

Exactly the same reasoning would apply to show that the $l^{\tilde{+}}$ integrals also have a finite volume at fixed l^+ and l^-. We show next, however, that the $l^{\tilde{+}}$ integration regions are limited even *after* the l^- integrals are performed. For this, we apply a similar reasoning to the light cone ordered expression, the second equality in Eq. (20). That is, we assume that some set of loop momenta, $\{l_a^{\tilde{+}}\}$ become large enough that it is possible to find a loop around which every line carries plus tilde momentum in the direction of the loop. We claim that in this case, the momentum $l_b^{\tilde{-}}$ that flows around this loop sees poles only in the lower (or upper) half plane in Eq. (20), so that its integral vanishes. To show this, we consider the on shell momentum of the ith line in this loop, of momentum k_i. Neglecting external momenta and masses for large loop momenta, we have

$$[k_i]^- = -k_i^{\tilde{-}} \frac{k_i^{\tilde{+}}}{k^+} = -\left(\eta_{ib} l_b^{\tilde{-}} + \sum_{a\neq b} \eta_{ia} l_a^{\tilde{-}}\right) \frac{\left(\eta_{ib} l^{\tilde{+}} + \sum_{a\neq b} \eta_{ia} l_a^{\tilde{+}}\right)}{k_i^+}\,, \tag{23}$$

where as in Eq. (2), $\eta_{bi} = \pm 1$ around the loop, depending on whether loop l_b flows with or against the defining direction of line momentum k_i, and where the sum over a includes all loop momenta with the exception of l_b. To be definite, suppose $l_b^{\tilde{+}}$ is large and positive. The condition that each component $k_i^{\tilde{+}}$ flows in the same direction as loop momentum l_b can then be written as

$$\eta_{ib} l_b^{\tilde{+}} + \sum_{a\neq b} \eta_{ia} l_a^{\tilde{+}} = \eta_{ib} \left| l_b^{\tilde{+}} + \frac{1}{\eta_{ib}} \sum_{a\neq b} \eta_{ia} l_a^{\tilde{+}} \right| . \tag{24}$$

We then have

$$[k_i]^- = -\left(\eta_{ib}^2 \tilde{l_b^-} + \eta_{ib} \sum_{a\neq b} \eta_{ia} \tilde{l_a^-}\right) \frac{\left|\tilde{l_b^+} + \frac{1}{\eta_{ib}} \sum_{a\neq b} \eta_{ia} \tilde{l_a^+}\right|}{k_i^+}, \tag{25}$$

and the coefficient of $\tilde{l_b^-}$ is always positive for every term in which it appears in the LCOPT denominators of Eq. (20), since k_i^+ is also always positive. All $\tilde{l_b^-}$ poles are thus in the same half plane (lower for $\tilde{l_b^+}$ positive), and the integrals vanish so long as the loop appears in at least two states. This, however, is ensured by our assumption of an ultraviolet-finite scalar diagram.

C. Unitarity(ies)

The light cone ordered expression (20) for an arbitrary diagram implies the order-by-order unitarity of perturbation theory, a relation that has been used extensively in showing the cancellation of infrared divergences in inclusive cross sections [15, 20, 21]. Here we note only the fundamental identity at the basis of this connection. We consider an arbitrary diagram $\mathcal{G}^{(T)}$, with a specific light cone order T, and sum over the terms found by setting each state, s_i of T on shell in turn, replacing its light cone denominator by a delta function. Each such substitution we refer to as a "cut" of the diagram. All states before (to the left of) the cut retain a $+i\epsilon$ prescription, and those after the cut (to the right) are given a $-i\epsilon$ prescription. See the left hand side of Fig. 3.

Each cut in the figure splits the ordered diagram into two ordered sub-amplitudes, $\mathcal{G}_{j,l}^{(T)}$ and $\mathcal{G}_{j,r}^{(T)}$, at fixed loop momenta to the "left" and "right" of the cut, respectively. The fundamental identity, which holds at fixed values of the all loop momenta l_a^+, $\tilde{l_a^+}$ and $\tilde{l_a^-}$, is

$$\begin{aligned}
\mathcal{G}_{j,r}^{(T)*}\mathcal{G}_{j,l}^{(T)} &= \sum_{j=1}^{V_\mathcal{G}-1} \left(\prod_{i'=j+1}^{V_\mathcal{G}-1} \frac{1}{P_{i'}^- - s_{i'} - i\epsilon}\right) 2\pi\delta\left(P_j^- - s_j\right) \left(\prod_{i=1}^{j-1} \frac{1}{P_i^- - s_i + i\epsilon}\right) \\
&= -i\left[\left(\prod_{i'=1}^{V_\mathcal{G}-1} \frac{1}{P_{i'}^- - s_{i'} - i\epsilon}\right) - \left(\prod_{i=1}^{V_\mathcal{G}-1} \frac{1}{P_i^- - s_i + i\epsilon}\right)\right] \\
&= -i\left[\mathcal{G}^* - \mathcal{G}\right],
\end{aligned} \tag{26}$$

where $\mathcal{G}$ is the uncut diagram at fixed remaining components of loop momenta and $V_\mathcal{G}$ the number of vertices in $\mathcal{G}$. The on shell value of minus momentum for state i is s_i. The proof of this relation follows easily from repeated use of the distribution identity, $2\pi i\delta(x) =$

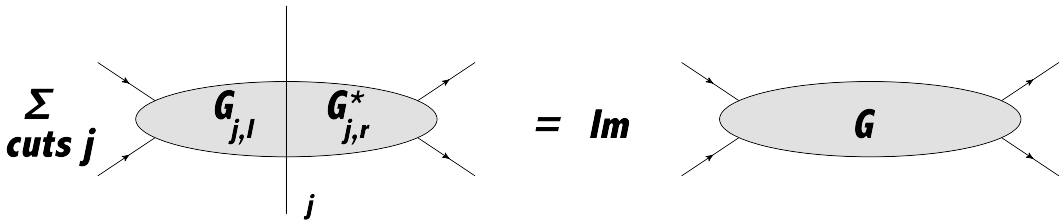

FIG. 3: A representation of perturbative unitarity, Eq. (26) for an arbitrary diagram G. As shown in the text, after an integral over loop l^- integrals, this relation holds for each light cone ordering of diagram G at fixed values of all loop l^+, and l^1, l^2 or $l^{\tilde{+}}$ and $l^{\tilde{-}}$. A similar result holds when all $l^{\tilde{-}}$ integrals are carried out at fixed $l^{\tilde{+}}$, l^+ and l^-.

$1/(x - i\epsilon) - 1/(x + i\epsilon)$. In this form the integrand of the sum of cut diagrams is related to the imaginary part of the integrand for the uncut diagram, a generalized form of the optical theorem, as illustrated by Fig. 3.

At the level of the fundamental identity, Eq. (26), then, unitarity is a property of perturbation theory in $(2, 2)$ signature as much as in Minkowski space. In fact, we can derive light cone ordered perturbation theory just as well by performing the $l^{\tilde{-}}$ integrals as the l^- integrals, deriving an identity of exactly the same form as Eq. (26) for an arbitrary diagram, but now at fixed loop momenta l^+, l^- and $l^{\tilde{+}}$. In a sense, then, there is an extra unitarity relation for $(2, 2)$ compared to $(1, 3)$. We do not have a practical application of this result to propose at this time.

IV. $2L$-DIMENSIONAL REPRESENTATION

The double set of light cone coordinates of Eq. (15) can be used to derive a new representation for diagrammatic integrals, based on the linearity of all propagators in the minus and minus tilde variables. We start with the general scalar integral, Eq. (6), in $(2, 2)$ signature for an arbitrary diagram with L loops and N lines, assuming that $N_g > 2L_g$ for any subgraph, g, so that all subintegrals are convergent,

$$I_{N,L}(p_j) \equiv - \int \prod_{i=1}^{L} \frac{dl_i^+ \, dl_i^{\tilde{+}}}{2(2\pi)^2} \, \frac{dl_i^{\tilde{-}} \, dl_i^-}{2(2\pi)^2} \, \prod_{\alpha=1}^{N} \frac{1}{D_\alpha} \, . \tag{27}$$

In the defining normalizations of Eq. (15), the denominators are given by

$$\begin{aligned} D_\alpha &= (l_\alpha - p_\alpha)^2 - m_\alpha^2 + i\epsilon \\ &\equiv \left(l_\alpha^+ - p_\alpha^+\right)\left(l_\alpha^- - p_\alpha^-\right) + \left(l_\alpha^{\tilde{+}} - p_\alpha^{\tilde{+}}\right)\left(l_\alpha^{\tilde{-}} - p_\alpha^{\tilde{-}}\right) - m_\alpha^2 + i\epsilon\,. \end{aligned} \tag{28}$$

Here l_α and p_α are the combinations of loop momenta l_i and external momenta p_j, respectively, flowing along internal line line α, with momentum k_α. In the notation of Eq. (2),

$$l_\alpha = \eta_{\alpha i} l_i\,, \quad p_\alpha = \xi_{\alpha j} p_j\,, \tag{29}$$

with $\eta_{\alpha i},\ \xi_{\alpha j} = \pm 1, 0$. Making the minus and minus-tilde loop momentum dependence explicit, we write the denominators as

$$D_\alpha = A_{\alpha i}^+ l_i^- \ + \ A_{\alpha i}^{\tilde{+}} l_i^{\tilde{-}} + B_\alpha\,, \tag{30}$$

in terms of coefficients A and B, defined by

$$\begin{aligned} A_{\alpha i}^+ &= (l_\alpha^+ - p_\alpha^+)\eta_{\alpha i}\,, \\ A_{\alpha i}^{\tilde{+}} &= (l_\alpha^{\tilde{+}} - p_\alpha^{\tilde{+}})\eta_{\alpha i}\,, \\ B_\alpha &= (p_\alpha^+ - l_\alpha^+)p_\alpha^- \ + \ (p_\alpha^{\tilde{+}} - l_\alpha^{\tilde{+}})p_\alpha^{\tilde{-}} - m_\alpha^2 \\ &= p_\alpha^2 - m_\alpha^2 - 2\hat{p}_\alpha \cdot l\,, \end{aligned} \tag{31}$$

where in the second relation for B_α, we define a vector with only minus and minus tilde components,

$$\hat{p}_\alpha^\mu \equiv \left(0^+, p_\alpha^-, 0^{\tilde{+}}, p_\alpha^{\tilde{-}}\right)\,. \tag{32}$$

The linearity of all denominators, (28) in both sets of integration variables $\{l_i^-\}$ and $\{l_i^{\tilde{-}}\}$ will allow us to derive an explicit form for each integral $I_{N,L}$ as a sum over choices of $2L$ on shell ('cut') lines.

Our integrals can be put into a more compact form by introducing a single index to cover the sum over components *and* loops,

$$I_{N,L}(p_j) = -\ \left(\frac{1}{4(2\pi)^4}\right)^L \int \prod_{k=1}^{2L} dy_k \int \frac{\prod_{j=1}^{2L} dx_j}{\prod_{\alpha=1}^{N}(\sum_{j=1}^{2L} A_{\alpha j} x_j + B_\alpha + i\epsilon)}\,, \tag{33}$$

where $\{x_j\} \equiv \{l_i^-, l_i^{\tilde{-}}\}$, runs over the minus and minus tilde components of all loops and α over the set of lines. To make our result as explicit as possible, we are free to define

$$\begin{aligned} x_{2i-1} &= l_i^-\,, \\ x_{2i} &= l_i^{\tilde{-}}\,, \end{aligned} \tag{34}$$

where i runs from 1 to L. Correspondingly, we may define the remaining $2L$ integration variables as

$$\begin{aligned} y_{2i-1} &= l_i^{\tilde{+}}\,, \\ y_{2i} &= l_i^{+}\,, \end{aligned} \tag{35}$$

for the set y_k. The relabeled coefficients $A_{\alpha i}$ are then linear functions of parameters y and can be thought of as defining a matrix. To be explicit, in terms of the coefficients of Eq. (31), we define

$$\begin{aligned} A_{\alpha,2i-1} &\equiv A_{\alpha i}^{+}\,, \\ A_{\alpha,2i} &\equiv A_{\alpha i}^{\tilde{+}}\,. \end{aligned} \tag{36}$$

We may choose to do the integrals in the order $y_1 \cdots y_{2L}$, and as we will see, individual terms in our results depend in a structured manner on the order of integration. The final result, however, cannot depend on the order.

The essential observation regarding the integral in Eq. (33) is that the singularity structure of the integrand for each x_j is simple poles at every step in the integration procedure, and that closing on these poles does not affect the limits of the remaining x_j, only the y_j. We will choose to perform these integrals by closing contours in each lower half complex x_j-plane. The choice of each pole sets one line on shell, and at the end of $2L$ integrations we have a sum of terms in which $2L$ lines are "cut" in this fashion. Let an arbitrary sequence of k lines found in this way be labelled $\mathcal{A}_k$, where $k = 1$ labels the first line set on shell, and $\mathcal{A}_{2L}$ the full set for the sequence. Each set $\mathcal{A}_k$ must be such that: (i) its lines carry k linearly independent loop momenta, and (ii) after any m integrals $x_1 \ldots x_m$, $m \leq k-1$, there must be a lower half-plane pole in the next integration variable, x_{m+1}. Let us denote by $A^{(\mathcal{A}_k)}$ the $k \times k$ matrix whose elements are $A_{\alpha j}$, such that $j = 1 \ldots k$ and $\alpha \in \mathcal{A}_k$.

The result we are after clearly depends on the values of the x_j when k, $k = 1 \ldots 2L$, lines are set on shell, that is on solutions to a system of $2L$ linear equations in $2L$ variables. For any choice of k lines, where k need not be an even number, these equations are

$$A_\alpha^{(\mathcal{A}_k)} \cdot x + B_\alpha^{(\mathcal{A}_k)} + i\epsilon \equiv \sum_{j=1}^{k} A_{\alpha j}^{(\mathcal{A}_k)} x_j + B_\alpha^{(\mathcal{A}_k)} + i\epsilon = 0\,, \quad \alpha \in \mathcal{A}_k\,, \tag{37}$$

where, again, the superscripts identify $A^{(\mathcal{A}_k)}$ as a $k \times k$ matrix and $B^{(\mathcal{A}_k)}$ as a k-component vector. The matrix, of course, must be non-singular, which is to say that we will find k

independent poles only if the momenta of these lines are linearly independent. The solution to Eq. (37) can be represented in terms of its real and imaginary parts $x_j = X_j^{(\mathcal{A}_k)} + i\epsilon Y_j^{(\mathcal{A}_k)}$, $j = 1 \ldots k$ as [1]

$$\begin{aligned} X_j^{(\mathcal{A}_k)} &= -\sum_{\alpha'} \left(A^{(\mathcal{A}_k)}\right)^{-1}_{j\alpha'} B_{\alpha'}^{(\mathcal{A}_k)}\,, \\ Y_j^{(\mathcal{A}_k)} &= -\sum_{\alpha'} \left(A^{(\mathcal{A}_k)}\right)^{-1}_{j\alpha'}\,, \end{aligned} \tag{38}$$

in terms of the inverse of matrix $A^{(\mathcal{A}_k)}$. Note the sum over unrepeated index α' in the expression for the imaginary part. The solutions in (38) determine the values of the remaining denominators when all $k \to 2L$ denominators are replaced by delta functions. This result alone does not determine the integral, however, because of theta functions that result from closing each contour in the lower half-plane in turn. The arguments of these step functions depend, in general, on the order in which the integrals are carried out.

We will now show that in the notation of Eq. (38), the result of doing the $2L$ x_j integrals in (33) is given by

$$\begin{aligned} I_{N,L} = -\left(\frac{-1}{4(2\pi)^2}\right)^L \sum_{\mathcal{A}_{2L}} \int \prod_{k=1}^{2L} dy_k\; \theta\left(\frac{\det A^{(\mathcal{A}_{k-1})}\; F_{\alpha_k}^{(\mathcal{A}_k)}(y_1 \ldots y_k)}{\det A^{(\mathcal{A}_k)}(y_1 \ldots y_k)}\right) \\ \times \frac{1}{\det(A^{(\mathcal{A}_{2L})})} \frac{1}{\prod_{\beta \notin \mathcal{A}_{2L}} (A_\beta \cdot X^{(\mathcal{A}_{2L})} + B_\beta + i\epsilon(1 + A_\beta \cdot Y^{(\mathcal{A}_{2L})}))}\,. \end{aligned} \tag{39}$$

The product of theta functions depends, as suggested above, on the order of integration. For the kth integration, we find

$$F_{\alpha_k}^{(\mathcal{A}_k)} = 1 + \sum_{j=1}^{k-1} A_{\alpha_k j}^{(\mathcal{A}_k)}\, Y_j^{(\mathcal{A}_{k-1})}\,, \tag{40}$$

where α_k is the index of the kth line put on shell, as above $A^{(\mathcal{A}_k)}$ is the $k \times k$ matrix associated with the first k lines, and where $Y_j^{(\mathcal{A}_{k-1})}$ is the solution for the imaginary part of x_j given in (38) when the first $k-1$ lines are put on shell. It should be noted that in the sum over sequences $\mathcal{A}_{2L}$ there are many terms that differ only in sign and integration region. The sign comes from the determinant of $A^{(\mathcal{A}_{2L})}$. Note the response of the imaginary parts to the selection of poles, as analyzed in the context of "loop-tree" dualities for Minkowski integrals [22–24].

[1] Here we assume that all masses are real. The generalization to masses with positive imaginary parts is immediate.

For an inductive proof of Eq. (39), we start by noting that that the role of the y_j is entirely passive. We need therefore only consider the proof of

$$\begin{aligned} J_{N,l}\left(A_{\alpha i}, B_\alpha\right) \;&\equiv\; \int \frac{\prod_{j=1}^{l} dx_j}{\prod_{\alpha=1}^{N}(\sum_{j=1}^{l} A_{\alpha j}x_j + B_\alpha + i\epsilon)} \\ &=\; -\,(-2\pi i)^l \sum_{\mathcal{A}_l}\prod_{k=1}^{l}\,\theta\left(\frac{\det A^{(\mathcal{A}_{k-1})}\; F_{\alpha_k}^{(\mathcal{A}_k)}(y_1\ldots y_l)}{\det A^{(\mathcal{A}_k)}(y_1\ldots y_l)}\right) \\ &\qquad \times \frac{1}{\det(A^{(\mathcal{A}_l)})}\frac{1}{\prod_{\beta\notin\mathcal{A}_l}(A_\beta\cdot X^{(\mathcal{A}_l)} + B_\beta + i\epsilon(1 + A_\beta\cdot Y^{(\mathcal{A}_l)}))}\,, \end{aligned} \tag{41}$$

for arbitrary l. The case of $l = 1$, $J_{N,1}$ is easily verified, and for any l, we can use the relation

$$J_{N,l}\left(A_{\alpha i}, B_\alpha\right) = \int dx_l\; J_{N,l-1}\left(A_{\alpha i}, B_\alpha + A_{\alpha l}x_l\right)\,, \tag{42}$$

in which the x_l integral of $J_{N,l}$ is absorbed into the B's for $J_{N,l-1}$. Now assuming the result (41) for $l-1$, and using (38), we have

$$\begin{aligned} J_{N,l}\left(A_{\alpha i}, B_\alpha\right) \;=\; -\,(-2\pi i)^{l-1}\; \int dx_l\; \sum_{\mathcal{A}_l}\prod_{k=1}^{l-1}\,\theta\left(\frac{detA^{(\mathcal{A}_{k-1})}\; F_{\alpha_k}(y_1\ldots y_k)}{\det A^{(\mathcal{A}_k)}(y_1\ldots y_k)}\right)\frac{1}{\det(A^{(\mathcal{A}_l)})} \\ \times \prod_{\beta\notin\mathcal{A}_{l-1}}\Bigg[\left(A_{\beta l}\;-\;A_{\beta j}\left(A^{(\mathcal{A}_{l-1})}\right)^{-1}_{j\alpha'}A_{\alpha' l}\right)x_l \\ + B_\beta - \;A_{\beta j}\left(A^{(\mathcal{A}_{l-1})}\right)^{-1}_{j\alpha'}B^{(\mathcal{A}_{l-1})}_{\alpha'} + i\epsilon\left(1\;-\;A_{\beta j}\sum_{\alpha'}\left(A^{(\mathcal{A}_{l-1})}\right)^{-1}_{j\alpha'}\right)\Bigg]^{-1}. \end{aligned} \tag{43}$$

To this expression, we apply an elementary identity, applicable to any nonsingular, $(n+1)\times(n+1)$ matrix, $M^{(n+1)}$ defined by $M_{i,j}$, $i,j = 1\ldots n+1$ in terms of its submatrix $M^{(n)}_{a,b} \equiv M_{a,b}$, $a,b = 1\ldots n$,

$$\frac{\det M^{(n+1)}}{\det M^{(n)}} = M_{n+1,n+1}\;-\;\sum_{i=1}^{n}\sum_{j=1}^{n}M_{n+1,i}\left(M^{(n)}\right)^{-1}_{i,j}M_{j,n+1}\,. \tag{44}$$

This is readily proved using the relation of the inverse of a matrix to minors of its determinant. Applying Eq. (44) to the coefficient of x_l in (43), the form of Eq. (41) for $J_{N,l}$ is then simply the sum of residues found by closing the x_l integral in the lower half plane. By identifying l with $2L$, Eq. (39) follows directly.

In fact, the identity (44) can be applied again, to the imaginary and real parts of (39), to provide an alternative expression for the integrand in eq. (39) entirely in turns of the matrices $A_{\alpha i}$ and vectors B_α. For each sequence $\mathcal{A}_k$, we find in the remaining denominators, β,

$$A_\beta \cdot X^{(\mathcal{A}_k)} + B_\beta = \frac{1}{\det A^{(\mathcal{A}_k)}} \begin{vmatrix} A^{(\mathcal{A}_k)}_{\alpha_1 1} & \cdots A^{(\mathcal{A}_k)}_{\alpha_1 n} & B_{\alpha_1} \\ \vdots & \vdots & \vdots \\ A^{(\mathcal{A}_k)}_{\alpha_k 1} & \cdots A^{(\mathcal{A}_k)}_{\alpha_k k} & B_{\alpha_n} \\ A_{\beta 1} & \cdots A_{\beta k} & B_\beta \end{vmatrix} \equiv \frac{G^{(\mathcal{A}_{k+1})}_\beta}{\det A^{(\mathcal{A}_k)}} \,. \tag{45}$$

We have a similar form for the arguments of the theta functions in Eq. (39),

$$F^{(\mathcal{A}_{k+1})}_\beta = \frac{1}{\det A^{(\mathcal{A}_k)}} \begin{vmatrix} A^{(\mathcal{A}_k)}_{\alpha_1 1} & \cdots A^{(\mathcal{A}_k)}_{\alpha_1 k-1} & 1 \\ \vdots & \vdots & \vdots \\ A^{(\mathcal{A}_k)}_{\alpha_k 1} & \cdots A^{(\mathcal{A}_k)}_{\alpha_k k} & 1 \\ A_{\beta 1} & \cdots A_{\beta k} & 1 \end{vmatrix} \equiv \frac{H^{(\mathcal{A}_{k+1})}_\beta}{\det A^{(\mathcal{A}_k)}} \,. \tag{46}$$

We can thus reinterpret the result of the x_i integrals, Eq. (39) as

$$\begin{aligned} I_{N,L} \;=\; -\left(\frac{-1}{4(2\pi)^2}\right)^L \sum_{\mathcal{A}_{2L}} \int \prod_{k=1}^{2L} dy_k \; \theta\left(\frac{H^{(\mathcal{A}_k)}_{\alpha_k}}{\det A^{(\mathcal{A}_k)}}\right) \left(\det A^{(\mathcal{A}_{2L})}\right)^{N-2L-1} \\ \times \prod_{\beta \notin \mathcal{A}_{2L}} \frac{1}{G^{(\mathcal{A}_{2L+1})}_\beta + i\epsilon H^{(\mathcal{A}_{2L+1})}_\beta} \,, \end{aligned} \tag{47}$$

where, as the notation indicates, the determinants G and H are of $(2L+1) \times (2L+1)$ matrices, determined in each case by the coefficients of on shell lines, and of each remaining, uncut line β. In this expression the entire integrand is specified by determinants of elements $A_{\alpha i}$ and B_α. These coefficients, in turn, given in (31), are linear functions of the plus and plus tilde loop momentum components in addition to external momenta and masses. Note that for $k = 1$, the theta function corresponds to the condition that the pole in the first integral, over loop momentum l_1^-, be in the lower half-plane, so that, because the set $\mathcal{A}_1$ consists of one line only, say i, we have

$$\begin{aligned} H^{(A_1)}_{\alpha_1} \;&\equiv\; 1 \,, \\ \det A^{(A_1)} \;&=\; \left(l_i^+ - p_i^+\right) \eta_{i1} \,, \end{aligned} \tag{48}$$

with no sum on i in the second expression. The integrand in Eq. (47) is a rational function of the remaining $2L$ components, y_j. Individual denominators labelled by index β may involve

powers of up to order $2L+1$ in these variables, although by examining the one-loop case below, we will see that the power can be lower.

Eq. (47) is our final result for ultraviolet finite scalar integrals in $(2,2)$. For any such diagram, $2L < N-1$, so that the number of integrations remaining is fewer than the number of Feynman parameter integrals for the corresponding diagram, at the price of having a sum of terms. In these expressions, the finiteness of the remaining integration regions, shown in Sec. III B above, is not manifest. It results from cancellations between different terms at each stage in the integration. We will give an example in the next section, where we study the one-loop case.

V. ONE LOOP DIAGRAMS

We now turn to the application of our basic $(2,2)$ result, (47) to one loop diagrams. We begin with a one loop diagram of any order, with completely arbitrary real masses and external momenta. We will not attempt to perform the remaining two integrals, but will be able to identify certain interesting general features. Following this, we confirm the presence of double-logarithmic behavior in a sample $(2,2)$ box diagram.

A. The general one loop diagram in (2,2) notation

For the case $L=1$ in Eq. (47), the sum over sets of cut lines, $\mathcal{A}_1$ and $\mathcal{A}_2$ is simply a sum of ordered choices of lines, say $\alpha_1 = i$ and $\alpha_2 = j$, which we will denote by $\mathcal{A}_1 = \mathcal{A}_i$ and $\mathcal{A}_2 = \mathcal{A}_{(ij)}$. With the labeling of momenta specified in Eq. (31), the first index, $\alpha_1 = i$ denotes the line set on shell by the integral over loop component $x_1 = l^-$, while $\alpha_2 = j$ identifies the line set on shell by the integral over $x_2 = \tilde{l^-}$, in the notation of Eq. (34). In these terms, we find, using (48), for $L=1$,

$$I_{N,1} = \frac{1}{4(2\pi)^2} \sum_{i,j} \int dl^+ \, \theta\left(\frac{1}{l^+ - p_i^+}\right) \int d\tilde{l^+} \theta\left(\frac{H_{\alpha_j}^{(\mathcal{A}_{(ij)})}}{\det A^{(\mathcal{A}_{ij})}}\right) \times \left(\det A^{(\mathcal{A}_{(ij)})}\right)^{N-3} \prod_{\beta \neq i,j} \frac{1}{G_\beta^{(\mathcal{A}_{(ij\beta)})} + i\epsilon H_\beta^{(\mathcal{A}_{(ij\beta)})}} \,, \tag{49}$$

where $\mathcal{A}_{(ij\beta)}$ in the superscripts of determinants G and H corresponds to $\mathcal{A}_{2L+1}$ in (47). To illustrate the method, we evaluate the remaining determinants in the expression. These are

from the 2×2 matrices, $A^{(\mathcal{A}_{(ij)})}$,

$$\det A^{(\mathcal{A}_{(ij)})} \;=\; \begin{vmatrix} l^+ - p_i^+ & l^{\tilde{+}} - p_i^{\tilde{+}} \\ l^+ - p_j^+ & l^{\tilde{+}} - p_j^{\tilde{+}} \end{vmatrix} \;= (l^+ - p_i^+)(p_i^{\tilde{+}} - p_j^{\tilde{+}}) - (l^{\tilde{+}} - p_i^{\tilde{+}})(p_i^+ - p_j^+)\,, \quad (50)$$

and $H_{\alpha_j}^{(\mathcal{A}_{(ij)})}$,

$$H_{\alpha_j}^{(\mathcal{A}_{(ij)})} \;=\; \begin{vmatrix} l^+ - p_i^+ & 1 \\ l^+ - p_j^+ & 1 \end{vmatrix} \;=\; p_j^+ - p_i^+\,, \quad (51)$$

and the two 3×3 matrices, $G_\beta^{(\mathcal{A}_{(ij)})}$,

$$\begin{aligned} \det G_\beta^{(\mathcal{A}_{(ij)})} \;&=\; \begin{vmatrix} l^+ - p_i^+ & l^{\tilde{+}} - p_i^{\tilde{+}} & B_i \\ l^+ - p_j^+ & l^{\tilde{+}} - p_j^{\tilde{+}} & B_j \\ l^+ - p_\beta^+ & l^{\tilde{+}} - p_\beta^{\tilde{+}} & B_\beta \end{vmatrix} \\ &=\; B_i \,\det A^{(\mathcal{A}_{(j\beta)})} \;- B_j \,\det A^{(\mathcal{A}_{(i\beta)})} \;+\; B_\beta \,\det A^{(\mathcal{A}_{(ij)})}\,, \end{aligned} \quad (52)$$

and $H_\beta^{(\mathcal{A}_{(ij)})}$,

$$\det H_\beta^{(\mathcal{A}_{(ij)})} \;=\; \begin{vmatrix} l^+ - p_i^+ & l^{\tilde{+}} - p_i^{\tilde{+}} & 1 \\ l^+ - p_j^+ & l^{\tilde{+}} - p_j^{\tilde{+}} & 1 \\ l^+ - p_\beta^+ & l^{\tilde{+}} - p_\beta^{\tilde{+}} & 1 \end{vmatrix} \;=\; (p_\beta^+ - p_i^+)(p_\beta^{\tilde{+}} - p_j^{\tilde{+}}) - (p_\beta^+ - p_j^+)(p_\beta^{\tilde{+}} - p_i^{\tilde{+}})\,. \quad (53)$$

Recalling that the B_i are linear in loop momenta, we see that the denominators β in Eq. (49) are of power two jointly in l^+ and $l^{\tilde{+}}$, rather than three.

In order to write our result in a more compact form, we introduce an antisymmetric product

$$\{v, w\} \;\equiv\; v^+ w^{\tilde{+}} - w^+ v^{\tilde{+}}\,. \quad (54)$$

In this notation, the general one-loop scalar integral becomes

$$\begin{aligned} I_{N,1} \;&=\; \frac{1}{4(2\pi)^2} \sum_{i,j} \int dl^+ \;\theta\left(l^+ - p_i^+\right) \int dl^{\tilde{+}} \theta\left(\frac{\{l, p_i - p_j\} + \{p_i, p_j\}}{p_j^+ - p_i^+}\right) \;\left(\{l, p_i - p_j\} + \{p_i, p_j\}\right)^{N-3} \\ &\qquad \times \prod_{\beta \neq i,j} \frac{1}{\frac{1}{2} \sum_{\{a,b,c\}=\{i,j,\beta\}} \epsilon_{abc} B_a \left(\{l, p_b - p_c\} + \{p_b, p_c\}\right) + i\epsilon\{p_\beta - p_i, p_\beta - p_j\}} \\ &\equiv\; \frac{1}{4(2\pi)^2} \sum_{i,j} \int dl^+ \;\theta\left(l^+ - p_i^+\right) \int dl^{\tilde{+}} \theta\left(l^{\tilde{+}} - l^+ r_{p_i - p_j} + \frac{\{p_i, p_j\}}{p_j^+ - p_i^+}\right) \;\omega_{ji}(l^+, l^{\tilde{+}})\,, \end{aligned} \quad (55)$$

where in the second equality we have evaluated the theta function for the $l^{\tilde{+}}$ integral, using the notation of Eq. (22), and have defined ω_{ji} as the integrand that results from taking the ith pole for the l^- integral, and the jth pole for $l^{\tilde{-}}$. The ϵ symbol reflects the antisymmetry of determinant $G_\beta^{(\mathcal{A}_{(ij)})}$, Eq. (52), in indices i, j and β, corresponding to line momenta k_i, k_j and k_β. We now note that for fixed i, the subsequent $l^{\tilde{+}}$ integral vanishes if all its poles are in the lower half plane, which leads to the identity,

$$\sum_{j\neq i} \omega_{ji}(l^+, l^{\tilde{+}}) = 0\,. \tag{56}$$

This enables us to rewrite $I_{N,1}$, (49) as

$$\begin{aligned} I_{N,1} &= \frac{1}{4(2\pi)^2} \sum_i \int_{p_i^+}^\infty dl^+ \sum_{j\neq i} \left(\int_{l^+ r_{p_i-p_j} - \frac{\{p_i,p_j\}}{p_i^+ - p_j^+}}^\infty dl^{\tilde{+}}\ \omega_{ji}(l^+, l^{\tilde{+}}) \ - \ \int_{\sigma(l^+)}^\infty dl^{\tilde{+}} \omega_{ji}(l^+, l^{\tilde{+}}) \right) \\ &= \frac{1}{4(2\pi)^2} \sum_i \sum_{j\neq i} \int_{p_i^+}^\infty dl^+ \int_{l^+ r_{p_i-p_j} - \frac{\{p_i,p_j\}}{p_i^+ - p_j^+}}^{\sigma(l^+)} dl^{\tilde{+}}\ \omega_{ji}(l^+, l^{\tilde{+}})\,, \end{aligned} \tag{57}$$

where $\sigma(l^+)$ is a completely arbitrary function of l^+ (possibly a constant), which must be chosen the same for every pair i, j.

We can simplify this expression further by using that in Eq. (57), the integrand ω_{ji} is fully antisymmetric under the exchange of p_i and p_j, that is,

$$\omega_{ji}(l^+, l^{\tilde{+}}) = -\omega_{ij}(l^+, l^{\tilde{+}})\,. \tag{58}$$

Equation (57) can thus be rewritten as a sum over $(1/2)N(N-1)$ ordered pairs of terms, with fixed limits on the l^+ integrals, and linear one-sided limits for the $l^{\tilde{+}}$ integrals,

$$\begin{aligned} I_{N,1} &= \frac{1}{4(2\pi)^2} \sum_i \sum_{j\neq i} \theta(p_j^+ - p_i^+) \int_{p_i^+}^{p_j^+} dl^+ \ \int_{\sigma(l^+)}^{l^+ r_{p_i-p_j} - \frac{\{p_i,p_j\}}{p_i^+ - p_j^+}} dl^{\tilde{+}} \\ &\quad \times \frac{\left(\{l, p_i - p_j\} + \{p_i, p_j\}\right)^{N-3}}{\prod_{\beta\neq i,j} \left[\frac{1}{2} \sum_{\{a,b,c\}=\{i,j,\beta\}} \epsilon_{abc} B_a \left(\{l, p_b - p_c\} + \{p_b, p_\beta\}\right) + i\epsilon\{p_\beta - p_i, p_\beta - p_j\}\right]}\,, \end{aligned} \tag{59}$$

where we observe again that because of the identity (56), the result is independent of our choice of $\sigma(l^+)$. The integration region is illustrated in Fig. 4.

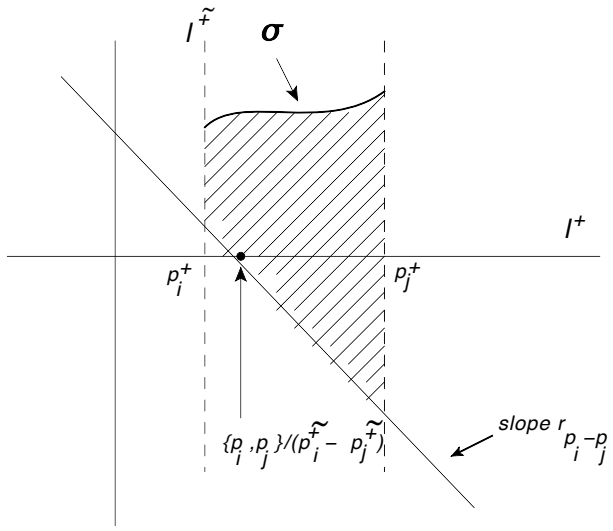

FIG. 4: Region of integration (shaded) corresponding to Eq. (59) in the $l^+, l^{\tilde{+}}$ plane. For the case shown, the parameter $r_{p_i - p_j}$ is negative, corresponding to a negative slope in the lower limit of the $l^{\tilde{+}}$ integral. Positive slopes and negative intercepts are also possible. As explained in the text, the boundary σ is arbitrary.

In principle, Eq. (59) could be the starting point of an explicit calculation, but in any case an arbitrary one-loop diagram can be reduced to box diagrams [25]-[29], which are known for any choices of masses [30]-[38]. Our emphasis here is rather on the extension of the formalism to the new signature.

B. Double logs in a (2,2) box

We have already argued that four-point amplitudes are insensitive to the choice of Minkowski or $(2,2)$ signature. To illustrate this point, let us show how double-logarithmic integrals arise in the one-loop box with a suitable choice of massless internal and external lines, directly from the $(2,2)$ result, Eq. (59) with $N = 4$.

We consider the scalar box, Fig. 5 describing a pair production process in "deep-inelastic scattering" kinematics,

$$p + q \to K_1 + K_2 \,, \tag{60}$$

where incoming line p is massless, two outgoing lines are massive,

$$\begin{aligned} p^2 &= 0\,, \\ q^2 &< 0\,, \\ K_1^2 &= K_2^2 = M^2, \end{aligned} \tag{61}$$

and where the process is initiated by a space-like momentum transfer, q. In the notation of Eq. (28) and Fig. 5, we have we have four line momenta, $l - p_i$, with

$$\begin{aligned} p_1 &= 0\,, \\ p_2 &= p\,, \\ p_3 &= p + q\,, \\ p_4 &= K_1\,. \end{aligned} \tag{62}$$

We assign a mass M to the propagator carrying momentum $l - K_1$, while other propagators are taken as massless,

$$I_{4,1}(\{p_i\}, M) = -i \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 + i\epsilon} \frac{1}{(l-p)^2 + i\epsilon} \frac{1}{(l-p-q)^2 + i\epsilon} \frac{1}{(l-K_1)^2 - M^2 + i\epsilon}\,. \tag{63}$$

In Minkowski space and with the momenta chosen as above, this integral has a double-logarithmic infrared behavior when the loop momentum l becomes proportional to p (collinear singularity) with vanishing energy (soft singularity), and no other sources of double logarithms. Without fully evaluating the diagram, Fig. 5, let us see how a double-logarithmic behavior emerges in the $(2,2)$ integral.

The term that has double-logarithmic behavior in Eq. (59) for this diagram in $(2,2)$ signature is the choice $k_i = l$, $k_j = l - p$, that is, the term with the mass shell poles of the two lines that become parallel. To be definite, we label $k_{\beta_1} = l - p - q$, $k_{\beta_2} = l - K_1$. With the routing of momenta shown in the figure, $B_i = 0$ and $p_i = 0$, so that the relevant term in (59) is

$$\begin{aligned} I_{4,1}^{(l,l-p)} = \frac{1}{4(2\pi)^2} \int_0^{l_1^+} dl^+ \int d\tilde{l^-}\theta\left(\{l,p\}\right) \{l,-p\} \\ \times \frac{1}{-B_{l-p}\{l,l-K_1\} + B_{l-K_1}\{l,l-p\} + i\epsilon\{p,K_1\}} \\ \times \frac{1}{-B_{l-p}\{l,l+q\} + B_{l-q-p}\{l,q-p\} + i\epsilon\{p,p+q\}}\,, \end{aligned} \tag{64}$$

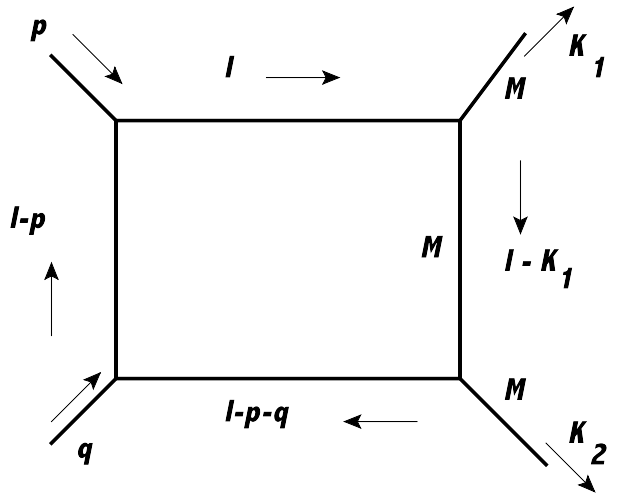

FIG. 5: Box diagram

where we have replaced indices d on the B_d by the corresponding momenta, k_d. The coefficients of the B_{k_d} are given by

$$\begin{aligned}\{l, p_i - p_j\} &= \{l, -p\} \\ &= p^+ l^{\tilde{+}} - l^+ p^{\tilde{+}} \\ &= l^+ p^+ (r_l - r_p) \,, \end{aligned} \tag{65}$$

where we have have used the notation of Eq. (22) for r_l and r_p. This antisymmetric combination vanishes both when loop momentum l is proportional to the massless momentum p, so that $r_l = r_p$, and when l^+ vanishes. These are the collinear and soft limits from Minkowski analysis, and the limits for l^+ and $l^{\tilde{+}}$ are just at these points. The numerator factor vanishes linearly in both the collinear and soft limits, but the denominators with momenta $l - K_1$ and $l - p - q$ behave as

$$\begin{aligned} - B_{l-p}\{l, -K_1\} + B_{l-K_1}\{l, -p\} &= - u_1 (l^+)^2 (r_l - r_p) + \cdots \\ - B_{l-p}\{l, -p-q\} + B_{l,-p-q}\{l, l-p\} &= s\, p^+ l^+ (r_l - r_p) + \cdots \,, \end{aligned} \tag{66}$$

respectively, with $s \equiv (p+q)^2$ and $u_1 \equiv 2p \cdot K_1$, where neglected terms are higher order in l^+ and/or $r_l - r_p$. In deriving these results, we have used that $p^2 = 0$ implies $p^{\tilde{+}}/p^+ = -p^-/p^{\tilde{-}}$. Now changing variables from $l^{\tilde{+}}$ to r_l, we find near the end-points a double-logarithmic integral,

$$I_{4,1}^{(l,l-p)} = - \frac{1}{4(2\pi)^2} \frac{1}{u_1\, s} \int_0 \frac{dl^+}{l^+} \int_{r_p} \frac{dr_l}{r_l - r_p} \,. \tag{67}$$

It is straightforward to check that no other term in the sum over poles has an end-point singularity at $r_l = r_p$, and hence a collinear singularity.

We can compare the result (67) to the double-logarithmic integral in Minkowski signature, which appears by taking the energy pole at $l^0 = \sqrt{|\vec{l}|^2}$ in Fig. 5. In that case, in the limit that $\cos\theta_{pl} \to 1$, where θ_{pl} is the angle between $\vec{l}$ and $\vec{p}$, we find

$$I_{DL} = -\frac{1}{4(2\pi)^2}\,\frac{1}{u_1\,s}\int_0 \frac{d|\vec{l}|}{|\vec{l}|}\int^1 \frac{d\cos\theta_{pl}}{1-\cos\theta_{pl}}\,, \tag{68}$$

with the same double-logarithmic behavior as (67) up to a change of variables.

In the above calculation, we have not discussed regulation of infrared-divergent integrals. The simplest regulation for the example above is to take $p_1^2 < 0$, but with gauge theories in mind it is natural to ask whether dimensional regularization is possible. Although our approach to $(2,2)$ signature is closely linked to four dimensions, there is in fact nothing to keep us from dimensionally regulating. The interpretation is particularly straightforward for infrared regulation, which requires $\varepsilon = 2 - D/2 < 0$, with D the number of dimensions, taken greater than four. We thus imagine adding -2ε dimensions to the four dimensions spanned by our coordinates $l^\pm$ and $l^{\tilde{\pm}}$.

While a full discussion of dimensional regularization for multi loop diagrams would take an extensive analysis, we will content ourselves here with the observation that if we label the momenta of the extra dimensions as $l_\perp$, and keep the external momenta in four dimensions, all of the analysis leading to our one loop result, Eq. (59), for example, is unchanged. The effect of dimensional regularization is simply to add a term $-l_\perp^2$ to every squared mass term in the denominators of (59), $B_\alpha \to B_\alpha - l_\perp^2$ in Eq. (31), and to introduce an overall integration over the "extra" dimensions of the form

$$\frac{2\pi^\varepsilon}{\Gamma(\varepsilon)}\int_0^\infty dl_\perp\, l_\perp^{-2\varepsilon-1}\,, \tag{69}$$

acting on the modified integrand, where the prefactor represents the angular volume. In the limit $\varepsilon \to 0$, the zero of the angular integration is balanced by the (infrared) pole from the radial integral. For infrared finite integrands, the net result is unity for $\varepsilon = 0$, but for divergent integrals as in Eq. (67), the result is infrared regulated after the $l_\perp$ integration.

VI. SUMMARY AND CONCLUSIONS

We have studied scalar perturbation theory in $(2,2)$ signature, and have identified a natural analytic continuation from Minkowski signature, which crosses no singularities and can be used to define diagrams with arbitrary external momenta. The resulting integrals have a standard "$i\epsilon$" prescription for the definition of contours in the presence of propagator singularities. This enables us to appeal to standard Landau analysis to identify pinches of momentum integrals, and singularities in external momenta. The singularities in $(2,2)$ are in general quite different than those in $(1,3)$ signature. An exception is when external momenta are restricted to a plane in Minkowski space; in this case the contour rotation to $(2,2)$ signature does not change the integral.

For diagrams that are fully ultraviolet finite (in all subdiagrams), we can introduce two sets of light cone variables, all four of which are linear in all denominators. We have derived a general expression for such an L-loop N-line integral as the sum of $2L$-dimensional integrals using $(2,2)$ integration. Whether these expressions can be of use in the practical evaluation of higher-loop scalar integrals is a subject for further investigation.

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