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Total Angular Momentum Waves for Scalar, Vector, and Tensor Fields

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Most calculations in cosmological perturbation theory, including those dealing with the inflationary generation of perturbations, their time evolution, and their observational consequences, decompose those perturbations into plane waves (Fourier modes). However, for some calculations, particularly those involving observations performed on a spherical sky, a decomposition into waves of fixed total angular momentum (TAM) may be more appropriate. Here we introduce TAM waves—solutions of fixed total angular momentum to the Helmholtz equation —for three-dimensional scalar, vector, and tensor fields. The vector TAM waves of given total angular momentum can be decomposed further into a set of three basis functions of fixed orbital angular momentum (OAM), a set of fixed helicity, or a basis consisting of a longitudinal (L) and two transverse (E and B) TAM waves. The symmetric traceless rank-2 tensor TAM waves can be similarly decomposed into a basis of fixed OAM or fixed helicity, or a basis that consists of a longitudinal (L), two vector (VE and VB, of opposite parity), and two tensor (TE and TB, of opposite parity) waves. We show how all of the vector and tensor TAM waves can be obtained by applying derivative operators to scalar TAM waves. This operator approach then allows one to decompose a vector field into three covariant scalar fields for the L, E, and B components and symmetric-traceless-tensor fields into five covariant scalar fields for the L, VE, VB, TE, and TB components. We provide projections of the vector and tensor TAM waves onto vector and tensor spherical harmonics. We provide calculational detail to facilitate the assimilation of this formalism into cosmological calculations. As an example, we calculate the power spectra of the deflection angle for gravitational lensing by density perturbations and by gravitational waves. We comment on an alternative approach to CMB fluctuations based on TAM waves. An accompanying paper will work out three-point functions in terms of TAM waves and their relation to the usual Fourier-space bispectra. Our work may have applications elsewhere in field theory and in general relativity.

I. INTRODUCTION

Much of modern cosmology involves the study of the origin and evolution of scalar, vector, and tensor fields. Examples of scalar fields include the inflaton [1] and the quintessence field [2]. Perturbations in the inflaton are considered as seeds for primordial perturbations to the curvature, and there is active investigation of the effects of quintessence perturbations on the evolution of density perturbations at late times. Magnetic fields provide an example of vector fields in cosmology [3, 4], and vector fields have appeared elsewhere as well [5]. The most general perturbation to the spacetime metric involves a rank-2 tensor field [6], the six components of which, as is well known, can be decomposed into a trace, a longitudinal mode, two vector modes, and two transverse-traceless modes, the latter of which propagate in general relativity as gravitational waves. A stochastic background of cosmological gravitational waves may have been produced during inflation [7] and are now being actively sought through the curl component (B mode) they induce in the anisotropy of the cosmic microwave background (CMB) polarization [8].

The vast majority of the literature on such fields and perturbations proceeds by decomposing the perturbations into Fourier modes (or plane waves) $e^{i\mathbf{k}\cdot\mathbf{x}}$, each of which then evolves independently to first order in perturbation theory. A primordial random field, such as that produced by inflation, is then assembled by adding all such plane waves with the Fourier amplitude for each wavevector $\mathbf{k}$ selected from a Gaussian distribution with a variance given by the power spectrum $P(k)$.

However, observations of the Universe are performed on a spherical sky. It may thus be advantageous, in some cases, to consider decomposition of the scalar/vector/tensor fields under consideration into a basis that reflects the rotational symmetry of the spherical sky. With this motivation in mind, we introduce here total-angular-momentum (TAM) waves for scalar, vector, and tensor fields. We provide a complete orthonormal set of basis functions for scalar, vector, and tensor fields on three-dimensional Euclidean space, of fixed orbital angular momentum. These basis functions are eigenfunctions of the Laplacian operator. There are three vector TAM waves for each total angular momentum, and we decompose these three into a basis of fixed orbital angular momentum (OAM), a basis of fixed helicity, and a basis that separates the longitudinal (L) and two transverse modes, E and B, of opposite parity. There are similarly five sets of TAM basis functions for traceless rank-2 symmetric tensors, and we provide similar decompositions into three sets of bases: an OAM basis, a helicity basis, and a basis that decomposes into a longitudinal (L) TAM wave, two vector waves, VE and VB, of opposite parity, and two transverse waves, TE and TB, of opposite parity. A random field can then be assembled by adding all such TAM waves, with the amplitude for each TAM wave of wavenumber k selected from a Gaussian distribution with variance $P(k)$, as will be detailed below.

The $L/E/B$ basis for vectors, and the $L/VE/VB/TE/TB$ basis for traceless symmetric rank-2 tensors, are first derived from the OAM basis. The helicity bases are then simply related to these $L/E/B$ and $L/VE/VB/TE/TB$ bases. We then present an alternative derivation of these TAM-wave bases by introducing sets of vector and traceless-tensor differential operators that, when applied to the scalar TAM waves, yield the $L/E/B$ and $L/VE/VB/TE/TB$ TAM waves. This operator approach then allows one to write an arbitrary vector field $V_a(\mathbf{x})$ in terms of three covariant scalar functions $V^L(\mathbf{x})$, $V^E(\mathbf{x})$, and $V^B(\mathbf{x})$ and an arbitrary tensor field $h_{ab}(\mathbf{x})$ in terms of five covariant scalar functions $h^L(\mathbf{x})$, $h^{VE}(\mathbf{x})$, $h^{VB}(\mathbf{x})$, $h^{TE}(\mathbf{x})$, $h^{TB}(\mathbf{x})$. The operator approach also allows one to obtain sets of vector and tensor basis functions from any other set of scalar basis functions.

We also provide the projections of the TAM vector and tensor waves onto vector and tensor spherical harmonics. This is equivalent, as will be seen below, to providing the $\hat{\mathbf{n}}$, $\hat{\theta}$, and $\hat{\phi}$ components of the TAM vector and tensor waves. The utility of TAM waves, as well as these projections, is illustrated with a re-derivation of the power spectra for weak lensing by density perturbations and gravitational waves.

The bases we provide here for three-dimensional fields are to be contrasted with earlier work [8], developed for CMB polarization and weak-lensing shear, on bases for tensor fields on the two-sphere $\mathbb{S}^2$, and with bases for vector fields (the weak-lensing deflection angle) [9]; we will show below, though, how the $\hat{\theta}$-$\hat{\phi}$ components of the three-dimensional TAM waves map onto the familiar vector/tensor spherical harmonics. Ref. [10–12] present the E/B modes of three-dimensional vector and tensor harmonics in open and closed Friedmann-Robertson-Walker space. The TAM-wave basis for scalar fields has already been employed in cosmology [13–17] (sometimes referred to as a "spherical-wave" or "Fourier-Bessel" expansion). The vector TAM waves are familiar from electromagnetic theory [19, 20]. Some steps along these lines for tensor fields were taken in Ref. [21], although they retained plane waves for the spatial dependence. There are some resemblances to Ref. [22], who were considering classical cosmological tests. Refs. [23, 24] have taken significant steps in the direction we pursue here for the description of weak lensing by density perturbations, and there are some analogues to this work in the gravitational-wave literature (see, e.g., Ref. [25]).

Below we begin in Section II with a brief discussion of our notation. Section III presents TAM waves for scalar fields beginning, by way of introduction, with a review in Section III A of the Fourier expansion of scalar fields. Section IV discusses vector fields, beginning in IV A with plane waves and moving on in IV B to TAM waves with vector fields. There the TAM waves of fixed total angular momentum are decomposed into OAM, $L/E/B$, and helicity bases. We introduce in Section IV B 3 a set of derivative operators that, when applied to the scalar TAM waves, provide TAM vector waves in the $L/E/B$ basis. We also show here how this operator approach can be used to find scalar functions associated with the L, E, and B components. The rest of Section IV discusses the projection of the TAM vector waves onto vector spherical harmonics (IV B 5), results that are useful, e.g., for observational quantities like the lensing deflection field that are represented as vectors on the two-sphere; the transformation between vector plane waves and vector TAM waves (IV B 6); and the expansion of vector fields in terms of TAM waves and the relation between the TAM-wave power spectra and the more familiar plane-wave power spectra (IV B 7). Section V provides a discussion of tensor TAM waves with an organization that parallels precisely that for vector waves in Section IV. Section VI presents, as an example of the utility of the TAM-wave formalism, a calculation of the power spectra for the deflection angle from gravitational lensing by density (scalar) perturbations and gravitational waves (transverse-traceless tensor perturbations). Section VII discusses the prospects for writing the Boltzmann equations for the evolution of CMB fluctuations using the TAM-wave formalism. Section VIII provides closing remarks. Appendixes A and B provide calculational details, and Appendix C provides a proof that the functions obtained by the action of irreducible-tensor operators on TAM waves are TAM waves of the same total-angular-momentum quantum numbers JM.

II. NOTATION

Throughout this paper we use the symbol $\Psi^k(\mathbf{x})$ to denote solutions to the Helmholtz equation,

$$(\nabla^2 + k^2)\Psi^k(\mathbf{x}) = 0, \tag{1}$$

where $\mathbf{x}$ is the spatial position, and k is the magnitude of the wavevector. In order to reduce clutter, we will often suppress the k superscript. We will be dealing with solutions $\Psi^k_{(JM)}(\mathbf{x})$ that are eigenstates of total angular momentum and its z component labeled by eigenvalues J and M, respectively. We will also obtain scalar, vector, and tensor solutions to the Helmholtz equation, and we will denote those solutions (actually, the components of those solutions) by $\Psi_{(JM)}(\mathbf{x})$, $\Psi_{(JM)a}(\mathbf{x})$, and $\Psi_{(JM)ab}(\mathbf{x})$ (where we have suppressed the k superscript, as we will do frequently throughout), respectively. The number of indices in the subscript, outside the parentheses, indicates whether the quantity is a scalar, vector, or tensor. As we will see, the vector and tensor eigenfunctions of fixed JM can be decomposed into states of fixed orbital angular momentum, fixed helicity, or a longitudinal/transverse decomposition. These will be labeled by a superscript. For example, the vector eigenstate of total angular momentum JM for wavevector k

with orbital angular momentum l will be $\Psi^{l,k}_{(JM)a}(\mathbf{x})$, and the vector TAM waves in the transverse/longitudinal basis will be referred to by $\Psi^{\alpha,k}_{(JM)a}(\mathbf{x})$, for $\alpha = L, E, B$, and in the helicity basis by $\Psi^{\lambda,k}_{(JM)a}(\mathbf{x})$, for $\lambda = 0, \pm 1$. Again, the k superscript will often be suppressed. We often refer to V_a as a "vector," although strictly speaking, it is the dual vector associated with the vector V^a; there should never be any confusion given that the dual vector has a lowered index and the vector a raised index. The indices are raised and lowered with a metric g_{ab}, and the antisymmetric tensor is ϵ_{abc}. Since we are dealing with flat three-dimensional space, the metric may be taken to be a Kronecker delta with Cartesian coordinates, in which case the raising and lowering of indices is trivial. However, we will at times work in spherical coordinates r, θ, ϕ in which case g_{ab} is not trivial. In some places we will deal with functions on the two-sphere $\mathbb{S}^2$, and in these cases we denote the metric and antisymmetric tensor for the two-sphere by g_{AB} and ϵ_{AB}, respectively, with capital indices.

We will also, by way of introduction, deal with plane-wave solutions to the Helmholtz equation. We will label the scalar, vector, and tensor solutions by $\Psi^{\mathbf{k}}(\mathbf{x})$, $\Psi^{\mathbf{k}}_a(\mathbf{x})$, and $\Psi^{\mathbf{k}}_{ab}(\mathbf{x})$, respectively. An additional superscript will denote the decomposition into OAM, helicity, or longitudinal/transverse eigenstates.[1] For reference, we list in Table I the symbols used in this paper.

III. SCALAR FIELDS

A. Plane waves

We begin with scalar fields to provide a simple introduction. Our aim is to find solutions $\phi(\mathbf{x})$ to the scalar Helmholtz equation, $(\nabla^2 + k^2)\phi(\mathbf{x}) = 0$. The most general solution can be written in terms of plane waves $\Psi^{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$, eigenfunctions of the momentum operator $-i\nabla$. The set of solutions for all $\mathbf{k}$ constitute a complete orthonormal basis for scalar functions $\phi(\mathbf{x})$, normalized so that

$$\int d^3x\, \Psi^{\mathbf{k}}(\mathbf{x}) \left[\Psi^{\mathbf{k}'}(\mathbf{x})\right]^* = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}'), \tag{2}$$

where $\delta_D(\mathbf{k} - \mathbf{k}')$ is a Dirac delta function. The most general scalar function can then be expanded,

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(\mathbf{k}) \Psi^{\mathbf{k}}(\mathbf{x}), \qquad \text{where} \quad \tilde{\phi}(\mathbf{k}) = \int d^3x \phi(\mathbf{x}) \left[\Psi^{\mathbf{k}}(\mathbf{x})\right]^*. \tag{3}$$

The power spectrum $P(k)$ for a scalar field is then defined by

$$\left\langle \tilde{\phi}(\mathbf{k}) \tilde{\phi}^*(\mathbf{k}') \right\rangle = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P(k), \tag{4}$$

where the angle brackets denote an expectation value over all realizations of the random field.

B. Total-angular-momentum waves

Our aim here, though, is to find solutions that are eigenfunctions of angular momentum. This is easily done with the plane-wave expansion,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{lm} 4\pi i^l j_l(kr) Y^*_{(lm)}(\hat{\mathbf{k}}) Y_{(lm)}(\hat{\mathbf{n}}), \tag{5}$$

where $j_l(x)$ is a spherical Bessel function, and $Y_{(lm)}(\hat{\mathbf{n}})$ are (scalar) spherical harmonics.[2] We then find that if we choose total-angular-momentum (TAM) basis functions,

$$\Psi^k_{(lm)}(\mathbf{x}) \equiv j_l(kr) Y_{(lm)}(\hat{\mathbf{n}}), \tag{6}$$

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[1] Note that there is no such superscript for scalar waves, as the OAM and TAM waves coincide for scalar fields, since they have no spin.

[2] We choose the spherical Bessel function of the first kind $j_l(kr)$ rather than the second kind $n_l(kr)$, so that the TAM waves are regular at the origin. There may be cases, for example in application of this formalism to emission or scattering of gravitational radiation, in which the second function $n_l(kr)$ may need to be introduced.

$\mathbf{x} = x, y, z$, $r = \lvert\mathbf{x}\rvert$, and $\hat{\mathbf{n}}$	a point in $\mathbb{R}^3$, its norm, and a unit vector in its direction	
$\mathbf{k}$, k_a, and $k = \lvert\mathbf{k}\rvert$	Fourier wavevector, its components, and its magnitude	
$\hat{\mathbf{k}}$, $\hat{\theta}$, etc. and $\hat{k}_a$, $\hat{\theta}_a$, etc.	unit vectors in the $\mathbf{k}$, θ, etc. directions and their components	
∇, ∇_a	covariant derivative wrt $\mathbf{x}$ and its components	
$*$ and $\dagger$	as superscripts represent complex conjugation and hermitian conjugate	
$\langle X \rangle$	average over all realizations of random variable X	(4)
$\delta_D(k-k')$	the one-dimensional Dirac delta function	(9)
$\delta_D(\mathbf{k}-\mathbf{k}')$	the three-dimensional Dirac delta function	(2)
$a, b, c, \ldots$	three-dimensional tensor indices	
$A, B, C, \ldots$	two-dimensional tensor indices	
$\hat{\varepsilon}_a$	polarization vector	
δ_{ij}	Kronecker delta	
g_{ab} and ϵ_{abc}	metric and antisymmetric tensor in $\mathbb{R}^3$	
g_{AB} and ϵ_{AB}	metric an antisymmetric tensor on $\mathbb{S}^2$	(89)
ε^s_{ab}	polarization tensors	(67)
$\hat{\varepsilon}_{ab}(\mathbf{k})$	polarization tensor for tensor plane wave	(96)
$\tilde{h}_{ab}(\mathbf{k})$	Fourier transform of $h_{ab}(\mathbf{x})$	(67)
$h_s(\mathbf{k})$	amplitudes for tensor plane-wave components	(67)
$\mathbf{L}$ and L_a	orbital-angular-momentum operator and its components	
$\mathbf{S}$	spin operator	
$\mathbf{J}$ and J_a	total-angular-momentum operator and its components	
J and M	Quantum numbers for total angular momentum and its z component	
l and m	Quantum numbers for orbital angular momentum and its z component	
$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$	Wigner-3j symbol	(A7)
$\begin{Bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{Bmatrix}$	Wigner-6j symbol	(B4)
$\langle l_1 m_1 l_2 m_2 \vert JM \rangle$	Clebsch-Gordan coefficient	(20)
$V_a(\mathbf{x})$	vector field	(15)
$h_{ab}(\mathbf{x})$	tensor field	(67)
$\tilde{V}^\alpha(\mathbf{k})$	Fourier coefficients for vector field	(15)
$V^\alpha(\mathbf{x})$	scalar fields for $\alpha = L, E, B$ components of vector field	(64)
$h^\alpha(\mathbf{x})$	scalar fields for $\alpha = L, VE, VB, TE, TB$ components of traceless tensor field	(114)
$P(k)$	power spectrum for density perturbations	(4)
$P_L(k)$,$P_T(k)$	power spectra for longitudinal and transverse modes of vector field	(17)
$P_+(k)$,$P_-(k)$	power spectra for left- and right-circularly polarized vector fields	(18)
$P_h(k)$	power spectrum for transverse-traceless mode of tensor field	(68)
$\Psi^k(\mathbf{x})$	Solutions to the Helmholtz equation for wavenumber k	(1)
$\phi(\mathbf{x})$	scalar functions	(3)
$\tilde{\phi}(\mathbf{k})$	the Fourier transform of $\phi(\mathbf{x})$	(3)
$\phi_{(lm)}(k)$	TAM-wave transform of $\phi(\mathbf{x})$	(8)
$\xi(\mathbf{x})$	scalar for longitudinal component of h_{ab}	(67)
$w_a(\mathbf{x})$	transverse-vector field for vector component of h_{ab}	(67)
$h^{TT}_{ab}(\mathbf{x})$	transverse-traceless part of h_{ab}	(67)
$\Psi^{\mathbf{k}}(\mathbf{x})$	scalar plane-wave mode	(2)
$\Psi^{\alpha,\mathbf{k}}_a(\mathbf{x})$	vector plane-wave mode for polarization $\alpha = L, 1, 2$	(13)
$\Psi^{\lambda,\mathbf{k}}_a(\mathbf{x})$	circularly-polarized vector plane-wave mode for helicity $\lambda = \pm 1$	(18)
$j_l(x)$ and $n_l(x)$	spherical Bessel functions of the first and second kind	(9)
$Y_{(lm)}(\hat{\mathbf{n}})$	scalar spherical harmonic	(9)
e^m_a	spherical basis for vector	(19)
t^m_{ab}	spherical basis for tensor	(70)
$Y^l_{(JM)a}(\hat{\mathbf{n}})$	vector spherical harmonic of OAM l, for $l = J-1, J, J+1$	(20)
$Y^\alpha_{(JM)a}(\hat{\mathbf{n}})$	vector spherical harmonic in the longitudinal/transverse basis for $\alpha = L, E, B$	(44)
$Y^\lambda_{(JM)a}(\hat{\mathbf{n}})$	vector spherical harmonic of helicity $\lambda = 0, \pm 1$	(51)
$Y^l_{(JM)ab}(\hat{\mathbf{n}})$	tensor spherical harmonic of OAM l, for $l = J-2, \ldots, J+2$	(71)
$Y^\alpha_{(JM)ab}(\hat{\mathbf{n}})$	tensor spherical harmonic for $\alpha = L, VE, VB, TE, TB$	(92)
$Y^\lambda_{(JM)ab}(\hat{\mathbf{n}})$	tensor spherical harmonic of helicity $\lambda = 0, \pm 1, \pm 2$	(99)

$\Psi_{(lm)}(\mathbf{x})$	scalar TAM wave	(6)
$\Psi^{l}_{(JM)a}(\mathbf{x})$	vector TAM wave of OAM l, for $l = J-1, J, J+1$	(20)
$\Psi^{\alpha}_{(JM)a}(\mathbf{x})$	vector TAM wave in the longitudinal/transverse basis for $\alpha = L, E, B$	(35)
$\Psi^{\lambda}_{(JM)a}(\mathbf{x})$	vector TAM wave of helicity $\lambda = 0, \pm 1$	(36)
$V^{l}_{(JM)}$, $V^{\alpha}_{(JM)}$, $V^{\lambda}_{(JM)}$	vector TAM-wave coefficients	(54)
$\Psi^{l}_{(JM)ab}(\mathbf{x})$	tensor TAM wave of OAM l, for $l = J-2, \ldots, J+2$	(71)
$\Psi^{\alpha}_{(JM)ab}(\mathbf{x})$	tensor TAM wave for $\alpha = L, VE, VB, TE, TB$	(85)
$\Psi^{\lambda}_{(JM)ab}(\mathbf{x})$	tensor TAM wave of helicity $\lambda = 0, \pm 1, \pm 2$	(87)
$h^{l}_{(JM)}$, $h^{\alpha}_{(JM)}$, $h^{\lambda}_{(JM)}$	tensor TAM-wave expansion coefficients	(102)
$A^{l}_{(JM)}$, $A^{\alpha}_{(JM)}$, $A^{\lambda}_{(JM)}$,	plane-wave expansion coefficients for vectors	(47)
$B^{l}_{(JM)}$, $B^{\alpha}_{(JM)}$, $B^{\lambda}_{(JM)}$,	plane-wave expansion coefficients for tensors	(96)
D_a, M_a, and K_a	differential operators	(30)
N_a, $M_{\perp a}$, and K_a	operators normal (N_a) tangential ($M_{\perp a}$, K_a) to $\mathbb{S}^2$	(39)
T^{α}_{ab}	differential operators generating $\Psi^{\alpha}_{(JM)ab}$ from $\Psi_{(JM)}$	(82)
W^{α}_{ab}	differential operators generating $Y^{\alpha}_{(JM)ab}$ from $Y_{(JM)}$	(90)
$T^{J}_{\alpha l}$, $T^{J}_{\lambda\alpha}$, $T^{J}_{\lambda l}$	transformation matrices between different vector bases	(37)
$U^{J}_{\alpha l}$, $U^{J}_{\lambda\alpha}$, $U^{J}_{\lambda l}$	transformation matrices between different tensor bases	(88)
Δ_a	lensing deflection field	(118)
Π_{ab}	projection tensor	(38)
η and η_0	conformal time and its value today	(118)
$\varphi(\hat{\mathbf{n}})$ and $\Omega(\hat{\mathbf{n}})$	projected lensing potentials	(119)
$\varphi_{(JM)}$ and $\Omega_{(JM)}$	lensing-potential spherical-harmonic coefficients	(120)
$C^{\varphi\varphi}_{J}$ and $C^{\Omega\Omega}_{J}$	lensing angular power spectra	(120)
$T^{\rm sca}(k)$	density-perturbation transfer function	(122)
$a(\eta)$ and $D_1(\eta)$	scale factor and density-perturbation growth factor	(122)
$\Phi^{k,p}_{(JM)}$	primordial amplitude for TAM wave of density perturbation	(122)
$F^{\rm sca,GW}_{J}(k)$	lensing transfer function for density perturbations and gravitational waves	(124)
C^{EE}_{J} and C^{BB}_{J}	lensing angular power spectra	(126)
$T(k,\eta)$	time evolution for gravitational wave	(127)
$f^{X}_{J}(kr)$	radial eigenfunctions for gravitational-wave transfer functions	(133)
$\Theta(\mathbf{x}, \hat{\mathbf{q}}; \eta)$	radiation perturbation	(144)
$\Theta^{k,JM}_{ll'}$	expansion coefficients for radiation perturbation	(144)
$\Xi^{k,JM}_{ll'}(\mathbf{x}, \hat{\mathbf{q}})$	TAM eigenfunctions of $\mathbf{x}$ and $\hat{\mathbf{q}}$	(144)
$\hat{\mathbf{q}}$	direction of photon momentum	(144)
$\mathcal{O}^{l}_{m}$	mth component of irreducible-tensor operator of rank l	(C2)
$\mathcal{O}^{\alpha}_{a}$	vector operator for $\alpha = L, E, B$	(57)
$\mathcal{D}^{l}_{mm'}$	Wigner rotation matrix	(C4)
$\mathcal{R}$	an $O(3)$ rotation	(C2)

TABLE I: A list of mathematical symbols used. The number in the right-most column indicates the equation in or near which the symbol is first defined or used.

where $r \equiv |\mathbf{x}|$ and $\hat{\mathbf{n}} \equiv \mathbf{x}/r$, then an arbitrary scalar function can be expanded as

$$\phi(\mathbf{x}) = \sum_{lm} \int \frac{k^2\, dk}{(2\pi)^3} \phi_{(lm)}(k) 4\pi i^l \Psi^{k}_{(lm)}(\mathbf{x}), \tag{7}$$

with

$$\phi_{(lm)}(k) = \int d^3\mathbf{x} \left[4\pi i^l \Psi^{k}_{(lm)}(\mathbf{x})\right]^* \phi(\mathbf{x}) = \int d^2\hat{\mathbf{k}}\, \tilde{\phi}(\mathbf{k}) Y^{*}_{(lm)}(\hat{\mathbf{k}}). \tag{8}$$

Here we have used the relations,

$$\int k^2\, dk j_l(kr) j_l(kr') = \frac{\pi}{2r^2}\delta_D(r-r'), \qquad \sum_{lm} Y_{(lm)}(\hat{\mathbf{n}}) Y^{*}_{(lm)}(\hat{\mathbf{n}}') = \delta_D(\hat{\mathbf{n}} - \hat{\mathbf{n}}'). \tag{9}$$

The orthonormality relation for the basis functions is

$$16\pi^2 \int d^3x \left[\Psi^{k}_{(lm)}(\mathbf{x})\right]^* \Psi^{k'}_{(l'm')}(\mathbf{x}) = \delta_{ll'}\delta_{mm'} \frac{(2\pi)^3}{k^2} \delta_D(k-k'), \tag{10}$$

where δ_{ij} is the Kronecker delta. The basis functions also satisfy,

$$\sum_{lm}\int \frac{k^2\,dk}{(2\pi)^3}\left[4\pi i^l \Psi^k_{(lm)}(\mathbf{x})\right]^* \left[4\pi i^l \Psi^k_{(lm)}(\mathbf{x}')\right] = \delta_D(\mathbf{x}-\mathbf{x}'), \tag{11}$$

which demonstrates that the $\Psi^k_{(lm)}(\mathbf{x})$ constitute a complete basis for scalar functions on $\mathbb{R}^3$. The products of TAM-wave coefficients have expectation values,

$$\left\langle \phi_{(lm)}(k)\phi^*_{(l'm')}(k')\right\rangle = \frac{(2\pi)^3}{k^2}\delta_D(k-k')\delta_{ll'}\delta_{mm'}P(k). \tag{12}$$

IV. VECTOR FIELDS

A. Plane waves

We now generalize to vector fields. Again we begin by reviewing plane-wave vector solutions to the Helmholtz equation. Three solutions to the vector Helmholtz equation, $(\nabla^2+k^2)\Psi_a(\mathbf{x})=0$ can be obtained, for each Fourier wavevector $\mathbf{k}$, as

$$\begin{aligned}
\Psi^{L,\mathbf{k}}_a(\mathbf{x}) &= (1/k)\nabla_a\Psi^{\mathbf{k}}(\mathbf{x}) = i\hat{k}_a e^{i\mathbf{k}\cdot\mathbf{x}},\\
\Psi^{1,\mathbf{k}}_a(\mathbf{x}) &= \frac{1}{|\mathbf{k}\times\hat{\mathbf{z}}|}\left(\nabla\times\hat{\mathbf{z}}\Psi^{\mathbf{k}}(\mathbf{x})\right)_a = \frac{1}{|\mathbf{k}\times\hat{\mathbf{z}}|}\epsilon_{abc}\nabla^b\hat{z}^c\Psi^{\mathbf{k}}(\mathbf{x}) = \frac{i\left[\mathbf{k}\times\hat{\mathbf{z}}\right]_a}{|\mathbf{k}\times\hat{\mathbf{z}}|}e^{i\mathbf{k}\cdot\mathbf{x}},\\
\Psi^{2,\mathbf{k}}_a(\mathbf{x}) &= \frac{-i}{k}\epsilon_{ab}{}^{c}\,\nabla^b\Psi^{1,\mathbf{k}}_c(\mathbf{x}) = \frac{-i}{k|\mathbf{k}\times\hat{\mathbf{z}}|}(\nabla_a\nabla_b - g_{ab}\nabla^2)\hat{z}^b\Psi^{\mathbf{k}}(\mathbf{x}) = \frac{i\left[\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{z}})\right]_a}{k|\mathbf{k}\times\hat{\mathbf{z}}|}e^{i\mathbf{k}\cdot\mathbf{x}},
\end{aligned} \tag{13}$$

where $\hat{\mathbf{z}}$ is a unit vector in the z direction, and ϵ_{abc} is the totally antisymmetric tensor. Here $\Psi^{L,\mathbf{k}}_a(\mathbf{x})$ is a longitudinal vector field, and $\Psi^{1,\mathbf{k}}_a(\mathbf{x})$ and $\Psi^{2,\mathbf{k}}_a(\mathbf{x})$ are the two linear polarizations for the transverse part of the vector field. We could have written Eq. (13) more simply as $\Psi^{X,\mathbf{k}}_a(\mathbf{x}) = i\hat{\varepsilon}^X_a(\hat{\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{x}}$, with $\hat{\varepsilon}^L_a = k_a$ and $\hat{\varepsilon}^{1,2}_a$ two other unit vectors orthogonal to k_a and to each other. We have written in Eq. (13) one choice for these polarization vectors explicitly in terms of a fixed unit vector $\hat{z}$ to motivate a choice of polarization vectors for the TAM waves later.

These mode functions are normalized so that they constitute a complete orthonormal set,

$$\int d^3x\,\Psi^{\alpha,\mathbf{k}\,a}(\mathbf{x})\left[\Psi^{\beta,\mathbf{k}}_a(\mathbf{x})\right]^* = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')\delta_{\alpha\beta}, \tag{14}$$

where $\alpha,\beta=\{L,1,2\}$. The three mode functions are, furthermore, orthogonal at each point. An arbitrary vector field $V_a(\mathbf{x})$ can then be expanded as

$$V_a(\mathbf{x}) = \int\frac{d^3k}{(2\pi)^3}\left[\tilde{V}^L(\mathbf{k})\Psi^{L,\mathbf{k}}_a(\mathbf{x}) + \tilde{V}^1(\mathbf{k})\Psi^{1,\mathbf{k}}_a(\mathbf{x}) + \tilde{V}^2(\mathbf{k})\Psi^{2,\mathbf{k}}_a(\mathbf{x})\right], \tag{15}$$

in terms of Fourier expansion coefficients,

$$\begin{aligned}
\tilde{V}^L(\mathbf{k}) &= \int d^3x\,V^a(\mathbf{x})\left[\Psi^{L,\mathbf{k}}_a(\mathbf{x})\right]^* = -\int d^3x\,\left[\Psi^{\mathbf{k}}(\mathbf{x})\right]^*\frac{1}{k}\nabla^a V_a(\mathbf{x}),\\
\tilde{V}^1(\mathbf{k}) &= \int d^3x\,V^a(\mathbf{x})\left[\Psi^{1,\mathbf{k}}_a(\mathbf{x})\right]^* = \int d^3x\,\left[\Psi^{\mathbf{k}}(\mathbf{x})\right]^*\frac{1}{|\mathbf{k}\times\hat{\mathbf{z}}|}\epsilon_{abc}\hat{z}^a\nabla^b V^c(\mathbf{x}),\\
\tilde{V}^2(\mathbf{k}) &= \int d^3x\,V^a(\mathbf{x})\left[\Psi^{2,\mathbf{k}}_a(\mathbf{x})\right]^* = \int d^3x\,\left[\Psi^{\mathbf{k}}(\mathbf{x})\right]^*\frac{-i}{k|\mathbf{k}\times\hat{\mathbf{z}}|}\hat{z}^a(\nabla_a\nabla_b - g_{ab}\nabla^2)V^b(\mathbf{x}).
\end{aligned} \tag{16}$$

We obtain the last equality in each line by integrating by parts.

The statistics of the vector field are given in terms of power spectra $P_L(k)$ and $P_T(k)$ for the longitudinal and transverse components, respectively, that satisfy

$$\begin{aligned}
\left\langle\tilde{V}^L(\mathbf{k})\tilde{V}^{L\,*}(\mathbf{k}')\right\rangle &= (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')P_L(k),\\
\left\langle\tilde{V}^1(\mathbf{k})\tilde{V}^{1\,*}(\mathbf{k}')\right\rangle &= (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')P_T(k),\\
\left\langle\tilde{V}^2(\mathbf{k})\tilde{V}^{2\,*}(\mathbf{k}')\right\rangle &= (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')P_T(k).
\end{aligned} \tag{17}$$

The decomposition of the transverse component into the two modes V^α ($\alpha = 1, 2$) is not rotationally invariant—the decomposition would be different if we had chosen a different direction for $\hat{\mathbf{z}}$—so the power spectra for the two must be the same. However, we can alternatively decompose the transverse-vector modes into plane waves of right and left circular polarization, or positive and negative helicity,

$$\Psi_a^{\pm,\mathbf{k}}(\mathbf{x}) = \frac{1}{\sqrt{2}}\left(\Psi_a^{1,\mathbf{k}}(\mathbf{x}) \pm i\Psi_a^{2,\mathbf{k}}(\mathbf{x})\right). \tag{18}$$

Since $\Psi_a^{2,\mathbf{k}}(\mathbf{x}) = -(i/k)\epsilon_{abc}\nabla^b\Psi^{1,\mathbf{k},c}(\mathbf{x})$, these modes are invariant under rotations about the $\hat{\mathbf{k}}$ direction and thus in some sense more "physical" than the 1 and 2 linear polarizations. It is possible (although it would require parity breaking) that $P_+(k)$ and $P_-(k)$ could differ. In the absence of parity breaking $P_+(k) = P_-(k) = P_T(k)$.

B. TAM Waves

The aim now is to find vector-valued functions $V_a(\mathbf{x})$ that satisfy the vector Helmholtz equation, $\left(\nabla^2 + k^2\right)V_a(\mathbf{x}) = 0$, for definite wavenumber magnitude k, and that transform under spatial rotation as representations of order J. In other words, we seek eigenfunctions of total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $L_a = -i\epsilon_{abc}x^b\nabla^c$ is the orbital angular momentum and $\mathbf{S}$ is the $S = 1$ spin associated with the vector space spanned by a set of basis vectors at each spatial point. This differs from the case of scalar fields where, with spin $S = 0$, total-angular-momentum eigenstates coincide with orbital-angular-momentum states.

Our strategy will be to first construct vector-valued eigenfunctions of total angular momentum that are also eigenfunctions of orbital angular momentum l. We will then construct linear combinations of states of definite total angular momentum JM that are curl-free (the longitudinal component) and divergence-free (the two transverse components). We will then decompose the TAM waves also into helicity eigenstates.

1. *The orbital-angular-momentum basis*

From the usual set of Cartesian basis unit vectors $e_a^\mu = \delta_a^\mu$, for $\mu = x, y, z$, one can construct a spherical basis $e_a^{\bar{m}}$ for $\bar{m} = +1, 0, -1$, through[3]

$$e_a^0 = e_a^z, \quad e_a^\pm = \mp\left(e_a^x \pm ie_a^y\right)/\sqrt{2}. \tag{19}$$

These constitute a complex but global basis, so these unit vectors commute with differential operators. Under spatial rotations, they transform as an $l = 1$ representation. We know for the spatial part of the eigenfunction that the conventional scalar-valued spherical harmonics $Y_{(lm)}(\hat{\mathbf{n}})$ form a representation of order l of the spatial rotation group. Vector-valued eigenfunctions of total angular momentum are therefore constructed via the usual scheme for adding two angular momenta [20],

$$\Psi^{l,k}_{(JM)a}(\mathbf{x}) = j_l(kr)Y^l_{(JM)a}(\hat{\mathbf{n}}) \equiv \sum_{m\bar{m}}\langle 1\bar{m}lm|JM\rangle j_l(kr)Y_{(lm)}(\hat{\mathbf{n}})e_a^{\bar{m}}, \tag{20}$$

where $\langle l_1m_1l_2m_2|JM\rangle$ are Clebsch-Gordan coefficients. Here, $J = 0, 1, 2, \ldots$, $M = -J, -J+1, \ldots, J-1, J$, and $l = J-1, J, J+1$. The TAM waves $\Psi^{l,k}_{(JM)a}(\mathbf{x})$ are also eigenfunctions of orbital angular momentum squared $\mathbf{L}^2 = L^aL_a$ with eigenvalue $l(l+1)$. The angular parts $Y^l_{(JM)a}(\hat{\mathbf{n}})$ (three-dimensional vector spherical harmonics of given total and orbital angular momentum) are normalized to

$$\int d^2\hat{\mathbf{n}}\left[Y^{l'\,a}_{(J'M')}(\hat{\mathbf{n}})\right]^* Y^l_{(JM)a}(\hat{\mathbf{n}}) = \delta_{ll'}\delta_{JJ'}\delta'_{MM}. \tag{21}$$

There are three eigenfunctions for given total angular momentum JM distinguished by their orbital angular momentum l. The TAM waves are normalized so that

$$\int d^3x\,\left[4\pi i^J\Psi^{l,k}_{(JM)a}(\mathbf{x})\right]^* 4\pi i^{J'}\Psi^{l',k'\,a}_{(J'M')}(\mathbf{x}) = \delta_{ll'}\delta_{JJ'}\delta'_{MM}\frac{(2\pi)^3}{k^2}\delta_D(k-k'). \tag{22}$$

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[3] Barred indices like $\bar{m}$ are reserved for the order-1 spherical basis.

We also have that

$$\sum_{JMl}\int\frac{k^2dk}{(2\pi)^3}\left[4\pi i^l\Psi^{l,k\ a}_{(JM)}(\mathbf{x}')\right]^*\left[4\pi i^l\Psi^{l,k}_{(JM)a}(\mathbf{x})\right]=\delta_D(\mathbf{x}-\mathbf{x}'),\tag{23}$$

which demonstrates that the $\Psi^{l,k}_{(JM)a}(\mathbf{x})$ constitute a complete basis for vector functions on $\mathbb{R}^3$. To show this, we use the definition in Eq. (20) to rewrite the left-hand side as

$$\sum_{JMl}\int\frac{k^2dk}{(2\pi)^3}(4\pi)^2j_l(kr)j_l(kr')\left[Y^{l\ a}_{(JM)}(\hat{\mathbf{n}}')\right]^*Y^l_{(JM)a}(\hat{\mathbf{n}})=\left[\frac{2}{\pi}\int k^2dkj_l(kr)j_l(kr')\right]\sum_{JMl}\left[Y^{l\ a}_{(JM)}(\hat{\mathbf{n}}')\right]^*Y^l_{(JM)a}(\hat{\mathbf{n}}).\tag{24}$$

The k integral is

$$\frac{2}{\pi}\int k^2dkj_l(kr)j_l(kr')=\frac{1}{r^2}\delta_D(r-r'),\tag{25}$$

and the sum then becomes

$$\begin{aligned}\sum_{JMl}\left[Y^{l\ a}_{(JM)}(\hat{\mathbf{n}}')\right]^*Y^l_{(JM)a}(\hat{\mathbf{n}}) &= \sum_{JMl}\sum_{m\bar{m}}\sum_{m'\bar{m}'}\langle 1\bar{m}lm|JM\rangle\langle 1\bar{m}'lm'|JM\rangle\left[Y_{(lm)}(\hat{\mathbf{n}}')e^{\bar{m}\ a}\right]^*Y_{(lm')}(\hat{\mathbf{n}})e^{\bar{m}'}_a\\ &= \sum_{JMl}\sum_{m\bar{m}}\sum_{m'\bar{m}'}\langle 1\bar{m}lm|JM\rangle\langle 1\bar{m}'lm'|JM\rangle\left[Y_{(lm)}(\hat{\mathbf{n}}')\right]^*Y_{(lm')}(\hat{\mathbf{n}})\delta_{\bar{m}\bar{m}'}\\ &= \sum_{lm\bar{m}}\sum_{m'\bar{m}'}\left[Y_{(lm)}(\hat{\mathbf{n}}')\right]^*Y_{(lm')}(\hat{\mathbf{n}})\delta_{\bar{m}\bar{m}'}\delta_{mm'}\\ &= \sum_{lm}\left[Y_{(lm)}(\hat{\mathbf{n}}')\right]^*Y_{(lm')}(\hat{\mathbf{n}})=\delta_D(\hat{\mathbf{n}}-\hat{\mathbf{n}}'),\end{aligned}\tag{26}$$

from which Eq. (23) follows. Note that this also demonstrates that the $Y^l_{(JM)a}(\hat{\mathbf{n}})$ constitute a complete basis for three-dimensional vectors on the two-sphere.

2. *The longitudinal/transverse basis*

The next step will be to construct linear combinations of these OAM waves that are longitudinal and transverse. To do so, we must calculate the divergence and curl of $\Psi^{l,k}_{(JM)a}(\mathbf{x})$. The result for the divergence, detailed in Appendix A, is

$$r\nabla^a\Psi^l_{(JM)a}(\mathbf{x})=\begin{cases}-\sqrt{\frac{J}{2J+1}}\,(kr)\,j_J(kr)Y_{(JM)}(\hat{\mathbf{n}}), & l=J-1,\\ 0, & l=J,\\ -\sqrt{\frac{J+1}{2J+1}}\,(kr)\,j_J(kr)Y_{(JM)}(\hat{\mathbf{n}}), & l=J+1.\end{cases}\tag{27}$$

Since the parity of a given OAM state is $(-1)^l$, and the basis vectors in Eq. (20) of odd parity, we choose transverse-vector fields of parity $(-1)^J$ and $(-1)^{J+1}$ to be, respectively,[4]

$$\Psi^B_{(JM)a}(\mathbf{x})=i\Psi^J_{(JM)a}(\mathbf{x}),\qquad\text{and}\qquad\Psi^E_{(JM)a}(\mathbf{x})=i\left[\sqrt{\frac{J+1}{2J+1}}\Psi^{J-1}_{(JM)a}(\mathbf{x})-\sqrt{\frac{J}{2J+1}}\Psi^{J+1}_{(JM)a}(\mathbf{x})\right].\tag{28}$$

The basis functions for the longitudinal field may then be taken to be

$$\Psi^L_{(JM)a}(\mathbf{x})=i\left[\sqrt{\frac{J}{2J+1}}\Psi^{J-1}_{(JM)a}(\mathbf{x})+\sqrt{\frac{J+1}{2J+1}}\Psi^{J+1}_{(JM)a}(\mathbf{x})\right].\tag{29}$$

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[4] The parity of $(-1)^J$ for the vector B mode and $(-1)^{J+1}$ for the vector E mode differ by -1 from the parities of the E/B tensor spherical harmonics. However, the expansion coefficients for the vector E and B modes have, as we will see below, parities $(-1)^{J+1}$ and $(-1)^J$, as do the tensor-spherical-harmonic expansion coefficients. The reason traces back to the transformation property of the vector field under a parity inversion.

The prefactors have been chosen so that the three sets of eigenfunctions are normalized as in Eq. (22). We thus have a complete orthonormal set of basis functions, of fixed total angular momentum, for the transverse and longitudinal components of a vector field.

3. *The longitudinal/transverse basis in terms of derivative operators*

There is, however, an alternative and useful route to these longitudinal and transverse basis functions. In Appendix C it is proved that if an operator $\mathcal{O}$ is an irreducible tensor under rotations, then $J^2\mathcal{O}Y_{(JM)} = J(J+1)\mathcal{O}Y_{(JM)}$ and $J_z\mathcal{O}Y_{(JM)} = M\mathcal{O}Y_{(JM)}$. We can therefore construct vector TAM waves by applying appropriately defined vector operators to scalar TAM waves.

Consider three vector operators

$$D_a \equiv \frac{i}{k}\nabla_a, \quad K_a \equiv -iL_a, \quad M_a \equiv \epsilon_{abc}D^bK^c. \tag{30}$$

These are irreducible-vector operators, and they all commute with ∇^2. They therefore yield, when acting on scalar TAM waves, TAM vector waves of total angular momentum JM that are also solutions of the vector Helmholtz equation. These three sets of vector fields must be linear combinations of $\Psi^{l,k}_{(JM)}(\mathbf{x})$, for $l = J-1, J, J+1$ or $\Psi^{\alpha,k}_{(JM)}(\mathbf{x})$ for $\alpha = L, E, B$. Since the three operators satisfy

$$D^aK_a = K^aD_a = 0, \qquad D^aM_a = 0, \qquad M^aD_a = 2, \qquad K^aM_a = M^aK_a = 0, \tag{31}$$

it follows that D_a generates the longitudinal vector field $D_a\Psi^k_{(JM)}(\mathbf{x}) \propto \Psi^{L,k}_{(JM)a}(\mathbf{x})$, while K_a and M_a generate divergence-free vector fields. Since K_a is axial-vector-like and M_a vector-like, parity considerations tell us that K_a, generates the B mode, $K_a\Psi^k_{(JM)}(\mathbf{x}) \propto \Psi^{B,k}_{(JM)a}(\mathbf{x})$, while M_a generates the E mode, $M_a\Psi^k_{(JM)}(\mathbf{x}) \propto \Psi^{E,k}_{(JM)a}(\mathbf{x})$.

The operators D_a, K_a, and M_a are also operators in the Hilbert space of vector-valued fields, so we can calculate their hermitian conjugates to be

$$(D_a)^\dagger = D_a, \qquad (K_a)^\dagger = -K_a, \qquad (M_a)^\dagger = -M_a + 2D_a. \tag{32}$$

Thus, when acting on $\Psi^k_{(JM)}$, the three operators have norms

$$(D^a)^\dagger D_a = 1, \qquad (K^a)^\dagger K_a = L^aL_a = J(J+1), \qquad (M^a)^\dagger M_a = L^aL_a = J(J+1). \tag{33}$$

These results enable us to normalize the vector TAM waves and to reproduce the longitudinal/transverse basis. This operator approach has the advantage that many calculations involving vector or higher-spin TAM waves can be reduced to the algebra of operators that act on scalar spherical waves. The following properties of the three operators will be useful in calculations:

$$\begin{aligned}
&[D_a, D_b] = 0, \qquad [K_a, D_b] = \epsilon_{abc}D^c, \qquad [M_a, D_b] = g_{ab} - D_aD_b, \\
&[K_a, K_b] = \epsilon_{abc}K^c, \qquad [K_a, M_b] = \epsilon_{abc}M^c, \qquad [M_a, M_b] = -\epsilon_{abc}K^c.
\end{aligned} \tag{34}$$

We can gain insight into the operators D_a, K_a, and M_a from the far-field limit $kr \to \infty$, where D_a is approximated by an ordinary vector in the radial direction, and K_a and M_a asymptote to two orthogonal vectors in the plane perpendicular to the radial direction, when they act on a scalar TAM wave. The factor of i in the definition of D_a is chosen so that $D_a = -\hat{k}_a$ in this limit. The sign convention for the E/B vector TAM waves is chosen so that if we rotate the E mode by $+90°$ about the direction of wave propagation we obtain a B mode.

To summarize, the decomposition into longitudinal and transverse modes is

$$\begin{aligned}
\text{B mode}: \quad & \Psi^B_{(JM)a}(\mathbf{x}) = \frac{K_a}{\sqrt{J(J+1)}}\Psi_{(JM)}(\mathbf{x}) = \frac{-i}{k}\epsilon_{abc}\nabla^b\Psi^{E\ c}_{(JM)}(\mathbf{x}) = i\Psi^J_{(JM)a}(\mathbf{x}), \\
\text{E mode}: \quad & \Psi^E_{(JM)a}(\mathbf{x}) = \frac{M_a}{\sqrt{J(J+1)}}\Psi_{(JM)}(\mathbf{x}) = \frac{i}{k}\epsilon_{abc}\nabla^b\Psi^{B\ c}_{(JM)}(\mathbf{x}) \\
& \qquad = i\left[\left(\frac{J+1}{2J+1}\right)^{1/2}\Psi^{J-1}_{(JM)a}(\mathbf{x}) - \left(\frac{J}{2J+1}\right)^{1/2}\Psi^{J+1}_{(JM)a}(\mathbf{x})\right], \\
\text{longitudinal mode}: \quad & \Psi^L_{(JM)a}(\mathbf{x}) = D_a\Psi_{(JM)}(\mathbf{x}) = i\left[\left(\frac{J}{2J+1}\right)^{1/2}\Psi^{J-1}_{(JM)a}(\mathbf{x}) + \left(\frac{J+1}{2J+1}\right)^{1/2}\Psi^{J+1}_{(JM)a}(\mathbf{x})\right].
\end{aligned} \tag{35}$$

4. The helicity basis

We can define another basis, denoted by the helicity $\lambda = 0, \pm 1$, by

$$\Psi^{\pm 1}_{(JM)a}(\mathbf{x}) = \frac{1}{\sqrt{2}}\left[\Psi^{E}_{(JM)a}(\mathbf{x}) \pm i\Psi^{B}_{(JM)a}(\mathbf{x})\right], \qquad \Psi^{0}_{(JM)a}(\mathbf{x}) = \Psi^{L}_{(JM)a}(\mathbf{x}). \tag{36}$$

These are eigenstates of the helicity operator $H = \mathbf{S}\cdot\hat{\mathbf{p}}$, where $(S_b)_{ac} = i\epsilon_{abc}$ is the spin operator and $\hat{p}_a = -i\nabla_a/k$ the normalized momentum operator, with eigenvalues λ.

We may summarize the transformation between the three bases—labeled by $l = J, J-1, J+1$ for the orbital-angular-momentum basis, $\alpha = B, E, L$ for the longitudinal/transverse basis, and $\lambda = 1, 0, -1$ for the helicity basis—by the transformation matrices,

$$T^J_{\alpha l} = i\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{J+1}{2J+1}} & -\sqrt{\frac{J}{2J+1}} \\ 0 & \sqrt{\frac{J}{2J+1}} & \sqrt{\frac{J+1}{2J+1}} \end{pmatrix}, \qquad T^J_{\lambda\alpha} = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \qquad T^J_{\lambda l} = i\begin{pmatrix} \frac{i}{\sqrt{2}} & \sqrt{\frac{J+1}{2(2J+1)}} & -\sqrt{\frac{J}{2(2J+1)}} \\ 0 & \sqrt{\frac{J}{2J+1}} & \sqrt{\frac{J+1}{2J+1}} \\ \frac{-i}{\sqrt{2}} & \sqrt{\frac{J+1}{2(2J+1)}} & -\sqrt{\frac{J}{2(2J+1)}} \end{pmatrix}. \tag{37}$$

5. Projection onto Vector Spherical Harmonics

Here we have constructed three different bases for three-dimensional vectors on $\mathbb{R}^3$. We now show how the angular components project onto the more familiar vector spherical harmonics $Y^E_{(JM)a}(\hat{\mathbf{n}})$ and $Y^B_{(JM)a}(\hat{\mathbf{n}})$, for two-dimensional vectors that live on the two-sphere. These vector spherical harmonics are given by

$$Y^E_{(JM)a}(\hat{\mathbf{n}}) = \frac{-r}{\sqrt{J(J+1)}}\nabla_{\perp\, a} Y_{(JM)}(\hat{\mathbf{n}}), \qquad Y^B_{(JM)a} = \frac{-r}{\sqrt{J(J+1)}}\epsilon_{abc}\hat{n}^b\nabla^c Y_{(JM)}(\hat{\mathbf{n}}), \tag{38}$$

both of which have $\hat{n}^a Y^E_{(JM)a}(\hat{\mathbf{n}}) = 0 = \hat{n}^a Y^B_{(JM)a}(\hat{\mathbf{n}})$. Here $\nabla_{\perp\, a} = \Pi_a{}^b\nabla_b$ is the gradient operator in the $\hat{\theta}$-$\hat{\phi}$ space, and $\Pi_{ab} = g_{ab} - \hat{n}_a\hat{n}_b$ projects onto that space. In addition, we can define a third vector spherical harmonic $Y^L_{(JM)a}(\hat{\mathbf{n}}) = -\hat{n}_a Y_{(JM)}(\hat{\mathbf{n}})$ to account for the component of a three-dimensional vector in the normal direction.

This set of three vector spherical harmonics provides a complete set of orthonormal basis functions for three-dimensional vectors that live on the two-sphere. We can obtain these vector spherical harmonics using an operator approach that parallels that we developed for TAM waves. Define three dimensionless irreducible-vector operators,

$$N_a = -\hat{n}_a, \qquad K_a = -iL_a, \qquad M_{\perp a} = \epsilon_{abc}N^b K^c. \tag{39}$$

These are analogues of the three operators D_a, K_a, and M_a we defined to derive vector TAM waves, but they act on the Hilbert space of all functions of $\hat{\mathbf{n}}$; i.e. they do not act on the radial coordinate r. This new set of operators satisfies precisely the same algebra as the set $\{D_a, K_a, M_a\}$. They are orthogonal to each other,

$$N_a K^a = K_a N^a = 0, \qquad N_a M^a_\perp = 0, \quad M_{\perp a}N^a = 2, \qquad K_a M^a_\perp = M_{\perp a}K^a = 0, \tag{40}$$

and they are normalized to

$$(N_a)^\dagger N^a = N_a N^a = 1, \qquad (K_a)^\dagger K^a = -K_a K^a = \mathbf{L}^2, \qquad (M_{\perp a})^\dagger M^a_\perp = -M_{\perp a}M^a_\perp = \mathbf{L}^2. \tag{41}$$

As operators in the Hilbert space, their hermitian conjugates are

$$(N_a)^\dagger = N_a, \qquad (K_a)^\dagger = -K_a, \qquad (M_{\perp a})^\dagger = -M_{\perp a} + 2N_a. \tag{42}$$

Furthermore, they satisfy the algebraic relations,

$$\begin{aligned} &[N_a, N_b] = 0, \qquad [M_{\perp a}, N_b] = (g_{ab} - N_a N_b), \qquad [M_{\perp a}, M_{\perp b}] = -\epsilon_{abc}K^c, \\ &[K_a, K_b] = \epsilon_{abc}K^c, \qquad [K_a, N_b] = \epsilon_{abc}N^c, \qquad [K_a, M_{\perp b}] = \epsilon_{abc}M^c_\perp. \end{aligned} \tag{43}$$

The two operators K_a and $M_{\perp a}$ generate the two transverse-vector spherical harmonics $Y^E_{(JM)a}(\hat{\mathbf{n}})$ and $Y^B_{(JM)a}(\hat{\mathbf{n}})$, in terms of E/B modes, while N_a generates the longitudinal vector spherical harmonic $Y^L_{(JM)a}(\hat{\mathbf{n}})$ in the normal direction. In summary,

$$Y^B_{(JM)a}(\hat{\mathbf{n}}) = \frac{1}{\sqrt{J(J+1)}} K_a Y_{(JM)}(\hat{\mathbf{n}}) = i Y^J_{(JM)a}(\hat{\mathbf{n}}),$$
$$Y^E_{(JM)a}(\hat{\mathbf{n}}) = \frac{1}{\sqrt{J(J+1)}} M_{\perp a} Y_{(JM)}(\hat{\mathbf{n}}) = -\sqrt{\frac{J+1}{2J+1}} Y^{J-1}_{(JM)a}(\hat{\mathbf{n}}) - \sqrt{\frac{J}{2J+1}} Y^{J+1}_{(JM)a}(\hat{\mathbf{n}}),$$
$$Y^L_{(JM)a}(\hat{\mathbf{n}}) = N_a Y_{(JM)}(\hat{\mathbf{n}}) = -\sqrt{\frac{J}{2J+1}} Y^{J-1}_{(JM)a}(\hat{\mathbf{n}}) + \sqrt{\frac{J+1}{2J+1}} Y^{J+1}_{(JM)a}(\hat{\mathbf{n}}), \tag{44}$$

normalized so that

$$\int d^2\hat{\mathbf{n}} \left[Y^{\alpha,a}_{(JM)}(\hat{\mathbf{n}}) \right]^* Y^{\beta}_{(J'M')a}(\hat{\mathbf{n}}) = \delta_{JJ'}\delta_{MM'}\delta_{\alpha\beta}, \tag{45}$$

for $\alpha, \beta = \{E, B, L\}$. Note that although the relations between the OAM and $E/B/L$ vector-spherical-harmonic bases in Eq. (44) resemble those between the OAM and $E/B/L$ bases for TAM waves in Eqs. (35) and (37), there are subtle, and important, sign differences. The minus sign in the definition of N_a is chosen to match $D_a = -\hat{k}_a$. The sign convention for E/B vector spherical harmonics is chosen so that a rotation of the E mode by $+90^\circ$ about the outward normal direction (i.e. the direction of $\hat{\mathbf{n}}$) yields the B mode.

Finally, we can write the $E/B/L$ TAM waves in terms of the $E/B/L$ spherical harmonics:

$$\begin{aligned}
\Psi^{k,B}_{(JM)a}(\mathbf{x}) &= j_J(kr) Y^B_{(JM)a}(\hat{\mathbf{n}}),\\
\Psi^{k,E}_{(JM)a}(\mathbf{x}) &= -i\left[j'_J(kr) + \frac{j_J(kr)}{kr}\right] Y^E_{(JM)a}(\hat{\mathbf{n}}) - i\sqrt{J(J+1)}\frac{j_J(kr)}{kr} Y^L_{(JM)a}(\hat{\mathbf{n}}),\\
\Psi^{k,L}_{(JM)a}(\mathbf{x}) &= -i\sqrt{J(J+1)}\frac{j_J(kr)}{kr} Y^E_{(JM)a}(\hat{\mathbf{n}}) - i j'_J(kr) Y^L_{(JM)a}(\hat{\mathbf{n}}).
\end{aligned} \tag{46}$$

Although the mode functions are orthonormal, we now see that they are *not* orthogonal at each point. Although the L and B modes are everywhere perpendicular and the E and B modes everywhere perpendicular, the L and E vector TAM waves are not always perpendicular. The B mode has components only in the θ-ϕ plane; i.e., $n^a \Psi^B_{(JM)a}(\mathbf{x}) = 0$. The E and L modes most generally have components in the tangential plane and along the normal n^a. In the far-field limit $kr \to \infty$, however, the three modes are asymptotically perpendicular to each other.

6. *The plane wave expansion for vector fields*

We now determine the transformation between the vector plane-wave basis and the vector TAM-wave bases. We start with the OAM basis. Since the $\Psi^{l,k}_{(JM)a}(\mathbf{x})$ constitute a complete basis, we may write,

$$\hat{\varepsilon}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{lJM} 4\pi i^l A^l_{(JM)}(\hat{\mathbf{k}}) \Psi^{l,k}_{(JM)a}(\mathbf{x}) = \sum_{lJM} 4\pi i^l A^l_{(JM)}(\hat{\mathbf{k}}) j_l(kr) Y^l_{(JM)a}(\hat{\mathbf{n}}). \tag{47}$$

Here $\hat{\varepsilon}_a$ is a (unit) polarization vector for the wave. The coefficients $A^l_{(JM)}(\mathbf{k})$ may be obtained by writing $e^{i\mathbf{k}\cdot\mathbf{x}}$ in the usual scalar plane-wave expansion,

$$\hat{\varepsilon}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{lm} 4\pi i^l j_l(kr) Y^*_{(lm)}(\hat{\mathbf{k}}) Y_{(lm)}(\hat{\mathbf{n}}) \hat{\varepsilon}_a(\mathbf{k}). \tag{48}$$

We then use orthonormality of the $Y^l_{(JM)a}(\hat{\mathbf{n}})$ to infer that

$$A^l_{(JM)}(\hat{\mathbf{k}}) = \hat{\varepsilon}^a(\mathbf{k}) Y^{l\,*}_{(JM)a}(\hat{\mathbf{k}}). \tag{49}$$

We can similarly expand in terms of L, E, B modes, or helicity modes, as

$$\hat{\varepsilon}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\alpha=L,E,B} \sum_{JM} 4\pi i^J A^{\alpha}_{(JM)}(\hat{\mathbf{k}}) \Psi^{k,\alpha}_{(JM)a}(\mathbf{x}), \qquad \hat{\varepsilon}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\lambda=-1,0,1} \sum_{JM} 4\pi i^J A^{\lambda}_{(JM)}(\hat{\mathbf{k}}) \Psi^{k,\lambda}_{(JM)a}(\mathbf{x}), \tag{50}$$

in terms of expansion coefficients

$$A^{\alpha}_{(JM)}(\hat{\mathbf{k}}) = \hat{\varepsilon}^a(\mathbf{k}) Y^{\alpha\,*}_{(JM)a}(\hat{\mathbf{k}}), \qquad A^{\lambda}_{(JM)}(\hat{\mathbf{k}}) = \hat{\varepsilon}^a(\mathbf{k}) Y^{\lambda\,*}_{(JM)a}(\hat{\mathbf{k}}). \tag{51}$$

Here, the spin-1 vector spherical harmonics are $Y^{\lambda=\pm1}_{(JM)a} = 2^{-1/2}\left[Y^E_{(JM)a} \pm iY^B_{(JM)a}\right]$. These are related to the usual spin-1 spherical harmonics ${}_\lambda Y_{(JM)}(\hat{\mathbf{n}})$ [26, 27] by

$$\hat{\varepsilon}^a_{\lambda'}(\mathbf{k}) Y^{\lambda}_{(JM)a}(\hat{\mathbf{k}}) = {}_{-\lambda}Y_{(JM)}(\hat{\mathbf{k}})\delta_{\lambda\lambda'}, \qquad \lambda, \lambda' = 0, \pm 1. \tag{52}$$

Here, $\hat{\varepsilon}^a_\lambda(\mathbf{k})$, for $\lambda = 0, \pm1$, are the polarization vectors for a vector plane wave with wavevector $\mathbf{k}$ and helicity λ. This equation defines our phase convention for $\hat{\varepsilon}_\lambda$. In terms of basis vectors in spherical coordinates, these are defined as

$$\hat{\varepsilon}^a_0 = \hat{n}^a, \qquad \hat{\varepsilon}^a_{\pm1} = \mp\frac{1}{\sqrt{2}}(\hat{\theta}^a \mp i\hat{\phi}^a). \tag{53}$$

7. *Expansion of vector fields and power spectra*

An arbitrary vector field $V_a(\mathbf{x})$ can be expanded in the OAM basis by

$$V_a(\mathbf{x}) = \sum_{JM} \sum_{l=J-1,J,J+1} \int \frac{k^2 dk}{(2\pi)^3} V^l_{(JM)}(k) 4\pi i^l \Psi^{l,k}_{(JM)a}(\mathbf{x}), \tag{54}$$

in terms of expansion coefficients,

$$V^l_{(JM)}(k) = \int d^3\mathbf{x}\, V^a(\mathbf{x}) \left[4\pi i^l \Psi^{l,k}_{(JM)a}(\mathbf{x})\right]^*. \tag{55}$$

As we show in Ref. [30], these can also be written as vector-spherical-harmonic transforms,

$$V^l_{(JM)}(k) = \int d^2\hat{\mathbf{k}}\, \tilde{V}^a(\mathbf{k}) Y^{l\,*}_{(JM)a}(\hat{\mathbf{k}}), \tag{56}$$

of the vector Fourier coefficients. Analogous relations hold for the $L/E/B$ and helicity bases as well.

For the $L/E/B$ basis, we may use the operator approach discussed above to rewrite the expansion coefficients in terms of scalar TAM waves by integrating by parts:

$$\begin{aligned} V^{\alpha}_{(JM)}(k) &= \int d^3\mathbf{x}\, V^a(\mathbf{x}) \left[4\pi i^J \Psi^{\alpha,k}_{(JM)a}(\mathbf{x})\right]^* = \int d^3\mathbf{x}\, V^a(\mathbf{x}) \left[4\pi i^J \mathcal{O}^\alpha_a \Psi^k_{(JM)}(\mathbf{x})\right]^* \\ &= \int d^3\mathbf{x} \left[(\mathcal{O}^\alpha_a)^\dagger V^a(\mathbf{x})\right] \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x})\right]^*, \end{aligned} \tag{57}$$

where $\mathcal{O}^\alpha_a = \{D_a, K_a, M_a\}$, and the hermitian conjugates of each operator are as given in Eq. (32). Explicit expressions for the expansion coefficients are

$$V^L_{(JM)}(k) = \int d^3\mathbf{x} \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x})\right]^* D_a V^a(\mathbf{x}), \tag{58}$$

$$V^B_{(JM)}(k) = \int d^3\mathbf{x} \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x})\right]^* (-K_a)\, V^a(\mathbf{x}), \tag{59}$$

$$V^E_{(JM)}(k) = \int d^3\mathbf{x} \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x})\right]^* (-M_a + 2D_a)\, V^a(\mathbf{x}), \tag{60}$$

Likewise the coefficients for the helicity basis are

$$V^{\lambda=0}_{(JM)}(k) = V^L_{(JM)}(k), \qquad V^{\lambda=\pm1}_{(JM)}(k) = \frac{1}{\sqrt{2}}\left[V^E_{(JM)}(k) \mp iV^B_{(JM)}\right]. \tag{61}$$

In other words, the expansion coefficients for vector TAM waves are the same as the coefficients of the scalar TAM waves for following three scalar functions:

$$V^L(\mathbf{x}) \;=\; D_a V^a(\mathbf{x}) = \frac{i}{k}\nabla_a V^a(\mathbf{x}) = \frac{i}{k}\nabla\cdot\mathbf{V}(\mathbf{x}), \tag{62}$$

$$V^B(\mathbf{x}) \;=\; -K_a V^a(\mathbf{x}) = \epsilon_{abc}x^b\nabla^c V^a(\mathbf{x}) = [\mathbf{x}\times\nabla]\cdot\mathbf{V}(\mathbf{x}), \tag{63}$$

$$V^E(\mathbf{x}) \;=\; (-M_a + 2D_a)V^a(\mathbf{x}) = -\frac{i}{k}\epsilon_{abc}\nabla^b K^c V^a(\mathbf{x}) + 2\frac{i}{k}\nabla_a V^a(\mathbf{x}) = \frac{i}{k}\left[\{\nabla\times(\mathbf{x}\times\nabla)\} + 2\nabla\right]\cdot\mathbf{V}(\mathbf{x}). \tag{64}$$

These scalars may be useful to calculate the theoretical expectation for TAM-wave coefficients.

Suppose now that $V_a(\mathbf{x})$ is written in terms of its longitudinal and transverse parts and that these have power spectra $P_L(k)$ and $P_T(k)$, as defined in Section IV A. It follows then that

$$\left\langle \left[V^{\alpha}_{(JM)}(k)\right]^* V^{\beta}_{(J'M')}(k')\right\rangle = P_T(k)\delta_{JJ'}\delta_{MM'}\delta_{\alpha\beta}\frac{(2\pi)^3}{k^2}\delta_D(k-k'), \tag{65}$$

for $\{\alpha,\beta\}=\{E,B\}$. Similarly,

$$\left\langle \left[V^{L}_{(JM)}(k)\right]^* V^{L}_{(J'M')}(k')\right\rangle = P_L(k)\delta_{JJ'}\delta_{MM'}\frac{(2\pi)^3}{k^2}\delta_D(k-k'), \tag{66}$$

for the longitudinal modes. The projections of a linearly-polarized transverse-vector plane wave onto the E and B vector TAM waves have equal amplitudes. Therefore, the power spectra for the E and B modes must always be the same for a realization of a statistically homogeneous random field.

V. SYMMETRIC TENSOR FIELDS

A. Introduction and Plane Waves

We now consider solutions to the Helmholtz equation, $\left(\nabla^2+k^2\right)h_{ab}(\mathbf{x})=0$, for a symmetric tensor field $h_{ab}(\mathbf{x}) = h_{(ab)}(\mathbf{x}) \equiv \left[h_{ab}(\mathbf{x})+h_{ba}(\mathbf{x})\right]/2$. The most general such tensor field can be decomposed into a trace component $h(\mathbf{x})$, a longitudinal component $\xi(\mathbf{x})$, two vector components w_a (with $\nabla^a w_a = 0$), and two transverse-traceless tensor components h^{TT}_{ab} (which satisfy $\nabla^a h^{TT}_{ab}=0$ and $h^a{}_a=0$), as

$$h_{ab} = h g_{ab} + \left(\nabla_a\nabla_b - \frac{1}{3}g_{ab}\nabla^2\right)\xi + \nabla_{(a}w_{b)} + h^{TT}_{ab}. \tag{67}$$

Our goal is to derive rank-2 tensor solutions to the Helmholtz equation, of definite total angular momentum, for these different components.

We begin, though, by reviewing the Fourier decomposition of the rank-2 tensor field. Each Fourier component of the tensor field can be expanded as $\tilde{h}_{ab}(\mathbf{k}) = \sum_s \varepsilon^s_{ab}(\hat{\mathbf{k}})h_s(\mathbf{k})$ in terms of six polarization states $\varepsilon^s_{ab}(\mathbf{k})$, where $s=\{0,z,x,y,+,\times\}$, for the trace, longitudinal, two vector, and two transverse-traceless polarizations, respectively, with amplitudes $h_s(\mathbf{k})$ [28]. The polarization tensors satisfy $\varepsilon^{s\,ab}\varepsilon^{s'}_{ab} = 2\delta_{ss'}$. The trace polarization tensor is $\varepsilon^0_{ab}\propto\delta_{ab}$, and the longitudinal is $\varepsilon^z_{ab}\propto(k^ak^b - k^2\delta_{ab}/3)k^{-2}$. The two vector-mode polarization tensors satisfy $\varepsilon^{x,y}_{ab}\propto k_{(a}w^{x,y}_{b)}$ where $w^{x,y}_a$ are two orthogonal ($w^{x\,a}w^y_a=0$) and transverse ($k^a w^{x,y}_a=0$) vectors. The two transverse-traceless polarization states have $k^a\varepsilon^{+,\times}_{ab}=0$.

The two vector (spin-1) modes x,y can alternatively be written in terms of a helicity basis by defining two helicity-1 polarization tensors $\varepsilon^{\pm1}_{ab} = (\varepsilon^x_{ab}\pm i\varepsilon^y_{ab})/\sqrt{2}$. Similarly, the two transverse-traceless (spin-2) modes $+,\times$ can alternatively be written in terms of a helicity basis by defining two helicity-2 polarization tensors $\varepsilon^{\pm2}_{ab} = (\varepsilon^+_{ab}\pm i\varepsilon^\times_{ab})/\sqrt{2}$.

In general relativity, power spectra $P_h(k)$ for gravitational waves (transverse-traceless tensor fields) h_{ab} are defined, for example, by

$$\langle h_s(\mathbf{k})h_{s'}(\mathbf{k}')\rangle = \delta_{ss'}(2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')\frac{P_h(k)}{4}, \tag{68}$$

for $s,s'=\{+,\times\}$, so that

$$\left\langle \tilde{h}_{ab}(\mathbf{k})\tilde{h}^{ab}(\mathbf{k})\right\rangle = (2\pi)^3 P_h(k)\delta_{ss'}\delta_D(\mathbf{k}-\mathbf{k}'). \tag{69}$$

B. TAM waves

Our aim now is to find tensor-valued functions $h_{ab}(\mathbf{x})$, solutions to the tensor Helmholtz equation for wavenumber k, that transform under spatial rotation as representations of order J. These will be eigenfunctions of total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Here the spin can be either $S = 0$ for the trace of h_{ab} or $S = 2$ for the trace-free part. The expansion of the trace is simply in terms of scalar TAM waves. We will therefore focus our attention in the following on trace-free rank-2 tensors. Therefore, $\mathbf{S}$ is now the $S = 2$ spin associated with the vector space spanned by a set of basis tensors at each spatial point.

We start by constructing a rank-2 spherical basis $t_{ab}^{\tilde{m}}$, for $\tilde{m} = \pm 2, \pm 1, 0$, that transforms under rotations as a representation of order 2 by taking direct products of the order-1 spherical basis,[5]

$$t_{ab}^{\tilde{m}} \equiv \sum_{\bar{m}_1 \bar{m}_2} \langle 1\bar{m}_1 1\bar{m}_2 | 2\tilde{m} \rangle e_a^{\bar{m}_1} e_b^{\bar{m}_2}. \tag{70}$$

Using orthonormality of Clebsch-Gordan coefficients, these are normalized to $\left(t^{\tilde{m}_1}\right)^{ab} t_{ab}^{\tilde{m}_2\,*} = \delta_{\tilde{m}_1 \tilde{m}_2}$.

1. *The orbital-angular-momentum basis*

We begin by expanding the five components of the rank-2 traceless tensor in terms of five tensor TAM waves of definite orbital-angular-momentum-squared $\mathbf{L}^2$ for each total angular momentum JM, as

$$\Psi_{(JM)ab}^{l,k}(\mathbf{x}) \equiv j_l\,(kr)\, Y_{(JM)ab}^{l,k}(\hat{\mathbf{n}}) = \sum_{\tilde{m}m} \langle 2\tilde{m} l m | JM \rangle j_l\,(kr)\, Y_{(lm)}(\hat{\mathbf{n}}) t_{ab}^{\tilde{m}}, \qquad l = J-2, J-1, J, J+1, J+2, \tag{71}$$

an equation that also defines the OAM tensor spherical harmonics $Y_{(JM)ab}^{l,k}(\hat{\mathbf{n}})$. These OAM tensor spherical harmonics of fixed orbital angular momentum satisfy the orthonormality relation,

$$\int d^2\hat{\mathbf{n}}\, Y_{(JM)}^{l\;ab}(\hat{\mathbf{n}}) Y_{(J'M')ab}^{l'\,*}(\hat{\mathbf{n}}) = \delta_{ll'} \delta_{JJ'} \delta_{MM'}. \tag{72}$$

The demonstration that the $\Psi_{(JM)ab}^{l,k}(\mathbf{x})$ constitute a complete basis for traceless symmetric tensors on $\mathbb{R}^3$, and that $Y_{(JM)ab}^{l}(\hat{\mathbf{n}})$ constitute a complete basis for three-dimensional traceless tensors on $\mathbb{S}^2$, are straightforward and similar to the analogous proofs for vector harmonics presented in Section IV B 1.

2. *The longitudinal/vector/transverse-traceless basis*

We now proceed to write the five traceless tensor harmonics for each JM in terms of a longitudinal (L) component, two vector components (VE and VB), and two transverse-traceless components (TE and TB). In Appendix B we derive the divergence of the tensor spherical waves of fixed orbital angular momentum in terms of vector spherical waves to be

$$\frac{1}{k} \nabla^a \Psi_{(JM)ab}^{l,k}(\mathbf{x}) = -\begin{cases} \sqrt{\frac{J-1}{2J-1}}\, \Psi_{(JM)b}^{k,J-1}(\mathbf{x}), & l = J-2, \\ \sqrt{\frac{J-1}{2(2J+1)}}\, \Psi_{(JM)b}^{k,J}(\mathbf{x}), & l = J-1, \\ \sqrt{\frac{(J+1)(2J+3)}{6(2J-1)(2J+1)}}\, \Psi_{(JM)b}^{k,J-1}(\mathbf{x}) + \sqrt{\frac{J(2J-1)}{6(2J+1)(2J+3)}}\, \Psi_{(JM)b}^{k,J+1}(\mathbf{x}), & l = J, \\ \sqrt{\frac{J+2}{2(2J+1)}}\, \Psi_{(JM)b}^{k,J}(\mathbf{x}), & l = J+1, \\ \sqrt{\frac{J+2}{2J+3}}\, \Psi_{(JM)b}^{k,J+1}(\mathbf{x}), & l = J+2. \end{cases} \tag{73}$$

Note that the divergence of a tensor of fixed total angular momentum JM yields a vector of the same JM, since we have acted with ∇_a, an irreducible-vector operator.

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[5] Tilded indices like $\tilde{m}$ are reserved for the order-2 spherical basis.

The transverse-traceless modes. We can, from these results, immediately construct two linear combinations of $\Psi^{k,l}_{(JM)ab}(\mathbf{x})$, of different parity, with vanishing divergence,

$$\begin{aligned}\Psi^{TE}_{(JM)ab}(\mathbf{x}) \equiv& \left(\frac{(J+1)(J+2)}{2(2J-1)(2J+1)}\right)^{1/2}\Psi^{J-2}_{(JM)ab}(\mathbf{x}) - \left(\frac{3(J-1)(J+2)}{(2J-1)(2J+3)}\right)^{1/2}\Psi^{J}_{(JM)ab}(\mathbf{x})\\ &+\left(\frac{J(J-1)}{2(2J+1)(2J+3)}\right)^{1/2}\Psi^{J+2}_{(JM)ab}(\mathbf{x}),\\ \Psi^{TB}_{(JM)ab}(\mathbf{x}) \equiv& \left(\frac{J+2}{2J+1}\right)^{1/2}\Psi^{J-1}_{(JM)ab}(\mathbf{x}) - \left(\frac{J-1}{2J+1}\right)^{1/2}\Psi^{J+1}_{(JM)ab}(\mathbf{x}).\end{aligned} \tag{74}$$

These two spherical waves form a basis for the transverse-traceless (TT) part of the tensor. We label them E and B according to their parity, $(-1)^J$ or $(-1)^{J+1}$, respectively.

The vector modes. The divergence of the vector component of the rank-2 tensor yields a divergence-free vector field. Recalling that the vector harmonics $i\Psi^{J}_{(JM)a}(\mathbf{x}) = \Psi^{B}_{(JM)a}(\mathbf{x})$ are divergence-free, we can construct one vector mode of the tensor field by taking the other orthogonal linear combination of $\Psi^{J-1}_{(JM)ab}(\mathbf{x})$ and $\Psi^{J+1}_{(JM)ab}(\mathbf{x})$. Likewise, the other vector mode should have a divergence that is proportional to $\Psi^{E}_{(JM)a}(\mathbf{x})$, but it should also be orthogonal to the transverse-traceless modes we already obtained. After some algebra, we find the two vector modes of the tensor to be

$$\begin{aligned}\Psi^{VB}_{(JM)ab}(\mathbf{x}) =& \left(\frac{J-1}{2J+1}\right)^{1/2}\Psi^{J-1}_{(JM)ab}(\mathbf{x}) + \left(\frac{J+2}{2J+1}\right)^{1/2}\Psi^{J+1}_{(JM)ab}(\mathbf{x}),\\ \Psi^{VE}_{(JM)ab}(\mathbf{x}) =& \left(\frac{2(J-1)(J+1)}{(2J-1)(2J+1)}\right)^{1/2}\Psi^{J-2}_{(JM)ab}(\mathbf{x}) + \left(\frac{3}{(2J-1)(2J+3)}\right)^{1/2}\Psi^{J}_{(JM)ab}(\mathbf{x})\\ &-\left(\frac{2J(J+2)}{(2J+1)(2J+3)}\right)^{1/2}\Psi^{J+2}_{(JM)ab}(\mathbf{x}).\end{aligned} \tag{75}$$

These form a basis for the vector part of the tensor field. The divergences of these basis functions are,

$$\frac{1}{k}\nabla^a\Psi^{VB}_{(JM)ab}(\mathbf{x}) = \frac{i}{\sqrt{2}}\Psi^{B}_{(JM)b}(\mathbf{x}), \qquad \frac{1}{k}\nabla^a\Psi^{VE}_{(JM)ab}(\mathbf{x}) = \frac{i}{\sqrt{2}}\Psi^{E}_{(JM)b}(\mathbf{x}). \tag{76}$$

It then follows that the vector TAM waves of the tensor field can be obtained from the transverse-vector spherical waves through,

$$\Psi^{VB}_{(JM)ab}(\mathbf{x}) = -\frac{i}{k\sqrt{2}}\left(\nabla_a\Psi^{B}_{(JM)b}(\mathbf{x}) + \nabla_b\Psi^{B}_{(JM)a}(\mathbf{x})\right), \qquad \Psi^{VE}_{(JM)ab}(\mathbf{x}) = -\frac{i}{k\sqrt{2}}\left(\nabla_a\Psi^{E}_{(JM)b}(\mathbf{x}) + \nabla_b\Psi^{E}_{(JM)a}(\mathbf{x})\right). \tag{77}$$

The longitudinal mode. The last orthogonal linear combination of the orbital-angular-momentum states,

$$\begin{aligned}\Psi^{L}_{(JM)ab}(\mathbf{x}) =& \left(\frac{3(J-1)J}{2(2J-1)(2J+1)}\right)^{1/2}\Psi^{J-2}_{(JM)ab}(\mathbf{x}) + \left(\frac{J(J+1)}{(2J-1)(2J+3)}\right)^{1/2}\Psi^{J}_{(JM)ab}(\mathbf{x})\\ &+\left(\frac{3(J+1)(J+2)}{2(2J+1)(2J+3)}\right)^{1/2}\Psi^{J+2}_{(JM)ab}(\mathbf{x}),\end{aligned} \tag{78}$$

decries the longitudinal component. To check, we find its divergence to be

$$\frac{1}{k}\nabla^a\Psi^{L}_{(JM)ab}(\mathbf{x}) = i\sqrt{\frac{2}{3}}\Psi^{L}_{(JM)b}(\mathbf{x}). \tag{79}$$

It implies that the longitudinal mode is the only one that has non-vanishing double divergence,

$$\frac{1}{k^2}\nabla^a\nabla^b\Psi^{L}_{(JM)ab}(\mathbf{x}) = \sqrt{\frac{2}{3}}\Psi_{(JM)}(\mathbf{x}), \tag{80}$$

in terms of the scalar spherical wave $\Psi_{(JM)}$. Using this result, it further follows that,

$$\Psi^{L}_{(JM)ab}(\mathbf{x}) = \frac{1}{k^2}\sqrt{\frac{3}{2}}\left(\nabla_a\nabla_b - \frac{1}{3}g_{ab}\nabla^2\right)\Psi_{(JM)}(\mathbf{x}). \tag{81}$$

3. The longitudinal/vector/transverse-traceless basis in terms of derivative operators

As seen in Section IV B 3, the L, B, and E vector waves can be written by applying the vector operators D_a, K_a, and M_a, respectively, to scalar TAM waves. Likewise, we have just seen in Eq. (81) that the TAM wave for the longitudinal component of the tensor field can be written by applying a derivative operator $\nabla_a\nabla_b - (1/3)g_{ab}\nabla^2$ to the scalar spherical wave. We have also seen in Eq. (77) that TAM waves for the vector components of the tensor field can be written by taking a symmetrized gradient of the transverse-vector spherical harmonics; i.e., by applying the operators $\nabla_{(a}K_{b)}$ and $\nabla_{(a}M_{b)}$, to scalar spherical waves. We now present an operator approach so that we have a complete treatment of symmetric traceless tensors in terms of tensor differential operators, including the two transverse-traceless tensor modes.

Using the three vector operators D_a, K_a, and M_a we have proposed as basic building blocks, we construct five tensor operators

$$\begin{aligned}
T^L_{ab} &= -D_aD_b + \frac{1}{3}g_{ab}, \qquad T^{VB}_{ab} = D_{(a}K_{b)}, \qquad T^{VE}_{ab} = D_{(a}M_{b)}, \\
T^{TB}_{ab} &= K_{(a}M_{b)} + M_{(a}K_{b)} + 2D_{(a}K_{b)}, \qquad T^{TE}_{ab} = M_{(a}M_{b)} - K_{(a}K_{b)} + 2D_{(a}M_{b)}.
\end{aligned} \tag{82}$$

These are irreducible tensors since they are symmetric and traceless, and they commute with ∇^2. Therefore, when acting on scalar TAM waves $\Psi^k_{(JM)}$, they generate symmetric tensor TAM waves that solve the tensor Helmholtz equation and have the same total angular momentum JM. We have seen that T^L_{ab} generates longitudinal mode, and T^{VB}_{ab} and T^{VE}_{ab} generate B and E vector modes, respectively. It is straightforward to show that $D^aT^{TB}_{ab} = D^aT^{TE}_{ab} = 0$, so they generate transverse-traceless tensor modes. The parity-odd T^{TB}_{ab} generates the B mode $\Psi^{TB,k}_{(JM)ab}(\mathbf{x}) \propto T^{TB}_{ab}\Psi^k_{(JM)}(\mathbf{x})$, while the parity-even T^{TE}_{ab} generates the E mode, $\Psi^{TE,k}_{(JM)ab}(\mathbf{x}) \propto T^{TE}_{ab}\Psi^k_{(JM)}(\mathbf{x})$. The five operators generate five linearly independent tensor modes, because they are orthogonal according to

$$\left(T^{\alpha'}_{ab}\right)^\dagger T^{\alpha,ab} = 0, \quad \text{if}\ \alpha \neq \alpha', \qquad \text{for} \qquad \alpha,\alpha' = L, VB, VE, TB, TE. \tag{83}$$

To normalize the tensor spherical waves, we calculate the norms of those five tensor operators to be

$$\begin{aligned}
&\left(T^L_{ab}\right)^\dagger T^{L,ab} = \frac{2}{3}, \qquad \left(T^{VB}_{ab}\right)^\dagger T^{VB,ab} = \left(T^{VE}_{ab}\right)^\dagger T^{VE,ab} = \frac{J\,(J+1)}{2}, \\
&\left(T^{TE}_{ab}\right)^\dagger T^{TE,ab} = \left(T^{TB}_{ab}\right)^\dagger T^{TB,ab} = \frac{2(J+2)!}{(J-2)!}.
\end{aligned} \tag{84}$$

To summarize, the decomposition of the traceless symmetric rank-2 tensor into longitudinal, vector, and transverse tensor modes is

$$\begin{aligned}
\text{longitudinal}:\ \Psi^L_{(JM)ab}(\mathbf{x}) &= \sqrt{\frac{3}{2}}T^L_{ab}\Psi_{(JM)}(\mathbf{x}) = \left(\frac{3\,(J-1)\,J}{2\,(2J-1)\,(2J+1)}\right)^{\frac{1}{2}}\Psi^{J-2}_{(JM)ab}(\mathbf{x}) \\
&+ \left(\frac{J\,(J+1)}{(2J-1)\,(2J+3)}\right)^{\frac{1}{2}}\Psi^{J}_{(JM)ab}(\mathbf{x}) + \left(\frac{3\,(J+1)\,(J+2)}{2\,(2J+1)\,(2J+3)}\right)^{\frac{1}{2}}\Psi^{J+2}_{(JM)ab}(\mathbf{x}),
\end{aligned}$$

$$\begin{aligned}
\text{vector B mode}:\ \Psi^{VB}_{(JM)ab}(\mathbf{x}) &= -\sqrt{\frac{2}{J\,(J+1)}}T^{VB}_{ab}\Psi_{(JM)}(\mathbf{x}) \\
&= \left(\frac{J-1}{2J+1}\right)^{\frac{1}{2}}\Psi^{J-1}_{(JM)ab}(\mathbf{x}) + \left(\frac{J+2}{2J+1}\right)^{\frac{1}{2}}\Psi^{J+1}_{(JM)ab}(\mathbf{x}),
\end{aligned}$$

$$\begin{aligned}
\text{vector E mode}:\ \Psi^{VE}_{(JM)ab}(\mathbf{x}) &= -\sqrt{\frac{2}{J\,(J+1)}}T^{VE}_{ab}\Psi_{(JM)}(\mathbf{x}) = \left(\frac{2\,(J-1)\,(J+1)}{(2J-1)\,(2J+1)}\right)^{\frac{1}{2}}\Psi^{J-2}_{(JM)ab}(\mathbf{x}) \\
&+ \left(\frac{3}{(2J-1)\,(2J+3)}\right)^{\frac{1}{2}}\Psi^{J}_{(JM)ab}(\mathbf{x}) - \left(\frac{2J\,(J+2)}{(2J+1)\,(2J+3)}\right)^{\frac{1}{2}}\Psi^{J+2}_{(JM)ab}(\mathbf{x}),
\end{aligned}$$

$$\text{transverse tensor B mode}: \ \Psi^{TB}_{(JM)ab}(\mathbf{x}) \ = \ -\sqrt{\frac{(J-2)!}{2(J+2)!}} T^{TB}_{ab} \Psi_{(JM)}(\mathbf{x})$$
$$= \ \left(\frac{J+2}{2J+1}\right)^{\frac{1}{2}} \Psi^{J-1}_{(JM)ab}(\mathbf{x}) - \left(\frac{J-1}{2J+1}\right)^{\frac{1}{2}} \Psi^{J+1}_{(JM)ab}(\mathbf{x}),$$

$$\text{transverse tensor E mode}: \ \Psi^{TE}_{(JM)ab}(\mathbf{x}) \ = \ -\sqrt{\frac{(J-2)!}{2(J+2)!}} T^{TE}_{ab} \Psi_{(JM)}(\mathbf{x}) = \left(\frac{(J+1)(J+2)}{2(2J-1)(2J+1)}\right)^{\frac{1}{2}} \Psi^{J-2}_{(JM)ab}(\mathbf{x})$$
$$- \left(\frac{3(J-1)(J+2)}{(2J-1)(2J+3)}\right)^{\frac{1}{2}} \Psi^{J}_{(JM)ab}(\mathbf{x}) + \left(\frac{J(J-1)}{2(2J+1)(2J+3)}\right)^{\frac{1}{2}} \Psi^{J+2}_{(JM)ab}(\mathbf{x}). \tag{85}$$

The normalizations are chosen such that

$$16\pi^2 \int d^3x \left[\Psi^{\alpha,k}_{(JM)ab}(\mathbf{x})\right]^* \Psi^{\beta,k \ ab}_{(J'M')}(\mathbf{x}) = \delta_{\alpha\beta}\delta_{JJ'}\delta_{MM'}\frac{(2\pi)^3}{k^2}\delta_D(k-k'), \tag{86}$$

where $\{\alpha,\beta\} = \{L, VB, VE, TB, TE\}$. Again, the five modes are orthogonal as tensor wave functions in the Hilbert space, but their tensor values at any given point are *not* necessarily orthogonal, as we will see below. This orthogonality does hold asymptotically in the far-field limit $kr \to \infty$.

4. Summary and helicity basis

So far we have constructed two sets of TAM-wave bases for symmetric traceless tensors. The OAM basis $\Psi^{l}_{(JM)ab}(\mathbf{x})$, where $l = J-2, J-1, J, J+1, J+2$ for each JM, are eigenstates of the square of orbital angular momentum $\mathbf{L}^2$. We have also defined a second basis $\Psi^{\alpha}_{(JM)ab}(\mathbf{x})$ in terms of a longitudinal mode $\alpha = L$, two vector modes $\alpha = VE, VB$, and two transverse-traceless tensor modes $\alpha = TE, TB$, for each JM. We can furthermore construct a helicity basis $\Psi^{\lambda}_{(JM)ab}(\mathbf{x})$, denoted by helicity $\lambda = \pm 2, \pm 1, 0$, through

$$\begin{aligned}
\Psi^{\pm 2}_{(JM)ab}(\mathbf{x}) &= \frac{1}{\sqrt{2}}\left(\Psi^{TE}_{(JM)ab}(\mathbf{x}) \pm i\Psi^{TB}_{(JM)ab}(\mathbf{x})\right), \\
\Psi^{\pm 1}_{(JM)ab}(\mathbf{x}) &= \frac{1}{\sqrt{2}}\left(\Psi^{VE}_{(JM)ab}(\mathbf{x}) \pm i\Psi^{VB}_{(JM)ab}(\mathbf{x})\right), \\
\Psi^{0}_{(JM)ab}(\mathbf{x}) &= \Psi^{L}_{(JM)ab}(\mathbf{x}).
\end{aligned} \tag{87}$$

These are eigenstates of the helicity operator $H = \mathbf{S}\cdot\hat{\mathbf{p}}$, but for the tensor field, $(S_e)_{ab,cd} \equiv i\epsilon_{aec}g_{bd} + ig_{ac}\epsilon_{bed}$, and $\hat{p}_a = -(i/k)\nabla_a$.

The orbital basis $\Psi^{l}_{(JM)ab}(\mathbf{x})$, for $l = J-2, J-1, J, J+2, J+2$, the longitudinal/vector/transverse-traceless basis $\Psi^{\alpha}_{(JM)ab}(\mathbf{x})$, for $\alpha = L, VE, VB, TE, TB$, and the helicity basis $\Psi^{\lambda}_{(JM)ab}(\mathbf{x})$, for $\lambda = 0, +1, -1, +2, -2$, are related by unitary transformations,

$$U^J_{\alpha l} = \begin{pmatrix} \sqrt{\frac{3(J-1)J}{2(2J-1)(2J+1)}} & 0 & \sqrt{\frac{J(J+1)}{(2J-1)(2J+3)}} & 0 & \sqrt{\frac{3(J+1)(J+2)}{2(2J+1)(2J+3)}} \\ \sqrt{\frac{2(J-1)(J+1)}{(2J-1)(2J+1)}} & 0 & \sqrt{\frac{3}{(2J-1)(2J+3)}} & 0 & -\sqrt{\frac{2J(J+2)}{(2J+1)(2J+3)}} \\ 0 & \sqrt{\frac{J-1}{2J+1}} & 0 & \sqrt{\frac{J+2}{2J+1}} & 0 \\ \sqrt{\frac{(J+1)(J+2)}{2(2J-1)(2J+1)}} & 0 & -\sqrt{\frac{3(J-1)(J+2)}{(2J-1)(2J+3)}} & 0 & \sqrt{\frac{J(J-1)}{2(2J+1)(2J+3)}} \\ 0 & \sqrt{\frac{J+2}{2J+1}} & 0 & -\sqrt{\frac{J-1}{2J+1}} & 0 \end{pmatrix},$$

$$U^J_{\lambda\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & i/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

$$U^J_{\lambda l} = \begin{pmatrix} \sqrt{\frac{3(J-1)J}{2(2J-1)(2J+1)}} & 0 & \sqrt{\frac{J(J+1)}{(2J-1)(2J+3)}} & 0 & \sqrt{\frac{3(J+1)(J+2)}{2(2J+1)(2J+3)}} \\ \sqrt{\frac{(J-1)(J+1)}{(2J-1)(2J+1)}} & i\sqrt{\frac{J-1}{2(2J+1)}} & \sqrt{\frac{3}{2(2J-1)(2J+3)}} & i\sqrt{\frac{J+2}{2(2J+1)}} & -\sqrt{\frac{J(J+2)}{(2J+1)(2J+3)}} \\ \sqrt{\frac{(J-1)(J+1)}{(2J-1)(2J+1)}} & -i\sqrt{\frac{J-1}{2(2J+1)}} & \sqrt{\frac{3}{2(2J-1)(2J+3)}} & -i\sqrt{\frac{J+2}{2(2J+1)}} & -\sqrt{\frac{J(J+2)}{(2J+1)(2J+3)}} \\ \sqrt{\frac{(J+1)(J+2)}{2(2J-1)(2J+1)}} & i\sqrt{\frac{J+2}{2(2J+1)}} & -\sqrt{\frac{3(J-1)(J+2)}{(2J-1)(2J+3)}} & -i\sqrt{\frac{J-1}{2(2J+1)}} & \sqrt{\frac{J(J-1)}{2(2J+1)(2J+3)}} \\ \sqrt{\frac{(J+1)(J+2)}{2(2J-1)(2J+1)}} & -i\sqrt{\frac{J+2}{2(2J+1)}} & -\sqrt{\frac{3(J-1)(J+2)}{(2J-1)(2J+3)}} & i\sqrt{\frac{J-1}{2(2J+1)}} & \sqrt{\frac{J(J-1)}{2(2J+1)(2J+3)}} \end{pmatrix}. \tag{88}$$

5. Projection onto Tensor Spherical Harmonics

We now describe the projection of three-dimensional traceless tensor TAM waves onto the two-sphere. We will begin by reviewing the projection onto the familiar E/B tensor spherical harmonics [8] in the $\hat{\theta}$-$\hat{\phi}$ space perpendicular to $\hat{\mathbf{n}}$. We will then generalize these two basis tensor spherical harmonics to include three more that will constitute a complete orthonormal basis for three-dimensional traceless tensors that live on the two-sphere, parametrized by $\hat{\mathbf{n}}$.

The usual E/B tensor spherical harmonics are defined by

$$Y^E_{(JM)AB}(\hat{\mathbf{n}}) = \sqrt{\frac{2}{J\,(J+1)\,(J-1)\,(J+2)}}\left(-\nabla_A\nabla_B + \frac{1}{2}g_{AB}\nabla^C\nabla_C\right)Y_{(JM)}(\hat{\mathbf{n}}),$$
$$Y^B_{(JM)AB}(\hat{\mathbf{n}}) = \sqrt{\frac{1}{2J\,(J+1)\,(J-1)\,(J+2)}}\left(\epsilon_B{}^C\nabla_C\nabla_A + \epsilon_A{}^C\nabla_C\nabla_B\right)Y_{(JM)}(\hat{\mathbf{n}}), \tag{89}$$

where here $\{A,B\} = \{\theta,\phi\}$, and ∇_A is a covariant derivative on the two-sphere, with metric $g_{AB} = \mathrm{diag}(1, \sin^2\theta)$ and antisymmetric tensor ϵ_{AB}.

The operator approach developed in Section IV B 5 for vector spherical harmonics can be generalized to tensor spherical harmonics. Recall that the operators $\{N_a, K_a, M_{\perp a}\}$ satisfy the same algebra as the operators $\{D_a, K_a, M_a\}$ do. Therefore, following the same line of reasoning we construct five tensor operators,

$$W^L_{ab} = -N_aN_b + \frac{1}{3}g_{ab}, \qquad W^{VB}_{ab} = N_{(a}K_{b)}, \qquad W^{VE}_{ab} = N_{(a}M_{\perp b)},$$
$$W^{TB}_{ab} = K_{(a}M_{\perp b)} + M_{\perp(a}K_{b)} + 2N_{(a}K_{b)}, \qquad W^{TE}_{ab} = M_{\perp(a}M_{\perp b)} - K_{(a}K_{b)} + 2N_{(a}M_{\perp b)}, \tag{90}$$

for tensor spherical harmonics. These symmetric and traceless operators conserve total angular momentum, since they are irreducible tensors. The last two operators are perpendicular to the radial direction, $\hat{n}^aW^{TE}_{ab} = \hat{n}^aW^{TB}_{ab} = 0$, so they must generate the E/B tensor spherical harmonics, $\hat{n}^aW^{TE}_{ab}Y_{(JM)}(\hat{\mathbf{n}}) \propto Y^{TE}_{(JM)ab}(\hat{\mathbf{n}})$ and $\hat{n}^aW^{TB}_{ab}Y_{(JM)}(\hat{\mathbf{n}}) \propto Y^{TB}_{(JM)ab}(\hat{\mathbf{n}})$, respectively. Here $Y^{TE}_{(JM)ab}(\hat{\mathbf{n}})$ and $Y^{TB}_{(JM)ab}(\hat{\mathbf{n}})$ correspond to the E/B tensor harmonics $Y^E_{(JM)AB}(\hat{\mathbf{n}}), Y^B_{(JM)AB}(\hat{\mathbf{n}})$, but under a three-dimensional orthonormal basis, denoted by lower-case indices $a, b, \ldots$. In addition, the W^{VE}_{ab} and W^{VB}_{ab}, when acting on ordinary spherical harmonics $Y_{(JM)}(\hat{\mathbf{n}})$, generate VE/VB tensor spherical harmonics $Y^{VE}_{(JM)ab}(\hat{\mathbf{n}})$ and $Y^{VB}_{(JM)ab}(\hat{\mathbf{n}})$, respectively, with components in both the tangential plane and normal. Similarly, W^L_{ab} generates the longitudinal tensor spherical harmonics $Y^L_{(JM)ab}(\hat{\mathbf{n}})$. For the tensor spherical harmonics, the terms "longitudinal," "vector," and "transverse," refer to the nature of the tensor components with respect to the normal $\hat{n}_a$. To be more specific, we define

$$\begin{aligned}
Y^L_{(JM)ab}(\hat{\mathbf{n}}) &= \sqrt{\frac{3}{2}} W^L_{ab} Y_{(JM)}(\hat{\mathbf{n}}), \\
Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) &= -\sqrt{\frac{2}{J(J+1)}} W^{VE}_{ab} Y_{(JM)}(\hat{\mathbf{n}}), \qquad Y^{VB}_{(JM)ab}(\hat{\mathbf{n}}) = -\sqrt{\frac{2}{J(J+1)}} W^{VB}_{ab} Y_{(JM)}(\hat{\mathbf{n}}), \\
Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}) &= -\sqrt{\frac{(J-2)!}{2(J+2)!}} W^{TE}_{ab} Y_{(JM)}(\hat{\mathbf{n}}), \qquad Y^{TB}_{(JM)ab}(\hat{\mathbf{n}}) = -\sqrt{\frac{(J-2)!}{2(J+2)!}} W^{TB}_{ab} Y_{(JM)}(\hat{\mathbf{n}}).
\end{aligned} \tag{91}$$

In terms of the OAM tensor spherical harmonics $Y^l_{(JM)ab}(\hat{\mathbf{n}})$, for $l = J, J\pm1, J\pm2$, the longitudinal/vector/transverse-traceless basis $Y^\alpha_{(JM)ab}(\hat{\mathbf{n}}), \alpha = L, VE, VB, TE, TB$, are found to be

$$\begin{aligned}
\text{longitudinal}: \quad Y^L_{(JM)ab}(\hat{\mathbf{n}}) &= -\left(\frac{3J(J-1)}{2(2J-1)(2J+1)}\right)^{\frac{1}{2}} Y^{J-2}_{(JM)ab}(\hat{\mathbf{n}}) \\
&+ \left(\frac{J(J+1)}{(2J-1)(2J+3)}\right)^{\frac{1}{2}} Y^{J}_{(JM)}(\hat{\mathbf{n}}) - \left(\frac{3(J+1)(J+2)}{2(2J+1)(2J+3)}\right)^{\frac{1}{2}} Y^{J+2}_{(JM)}(\hat{\mathbf{n}}),
\end{aligned}$$

$$\begin{aligned}
\text{vector E mode}: \quad Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) &= -\left(\frac{2(J-1)(J+1)}{(2J-1)(2J+1)}\right)^{\frac{1}{2}} Y^{J-2}_{(JM)ab}(\hat{\mathbf{n}}) \\
&+ \left(\frac{3}{(2J-1)(2J+3)}\right)^{\frac{1}{2}} Y^{J}_{(JM)ab}(\hat{\mathbf{n}}) + \left(\frac{2J(J+2)}{(2J+1)(2J+3)}\right)^{\frac{1}{2}} Y^{J+2}_{(JM)ab}(\hat{\mathbf{n}}),
\end{aligned}$$

$$\text{vector B mode}: \quad Y^{VB}_{(JM)ab}(\hat{\mathbf{n}}) = i\left(\frac{J-1}{2J+1}\right)^{\frac{1}{2}} Y^{J-1}_{(JM)ab}(\hat{\mathbf{n}}) - i\left(\frac{J+2}{2J+1}\right)^{\frac{1}{2}} Y^{J+1}_{(JM)ab}(\hat{\mathbf{n}}),$$

$$\begin{aligned}
\text{transverse tensor E mode}: \quad Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}) &= -\left(\frac{(J+1)(J+2)}{2(2J-1)(2J+1)}\right)^{\frac{1}{2}} Y^{J-2}_{(JM)ab}(\hat{\mathbf{n}}) \\
&- \left(\frac{3(J-1)(J+2)}{(2J-1)(2J+3)}\right)^{\frac{1}{2}} Y^{J}_{(JM)ab}(\hat{\mathbf{n}}) - \left(\frac{J(J-1)}{2(2J+1)(2J+3)}\right)^{\frac{1}{2}} Y^{J+2}_{(JM)ab}(\hat{\mathbf{n}}),
\end{aligned}$$

$$\text{transverse tensor B mode}: \quad Y^{TB}_{(JM)ab}(\hat{\mathbf{n}}) = i\left(\frac{J+2}{2J+1}\right)^{\frac{1}{2}} Y^{J-1}_{(JM)ab}(\hat{\mathbf{n}}) + i\left(\frac{J-1}{2J+1}\right)^{\frac{1}{2}} Y^{J+1}_{(JM)ab}(\hat{\mathbf{n}}). \tag{92}$$

These are normalized to

$$\int d^2\hat{\mathbf{n}} \left[Y^{\alpha,ab}_{(JM)}(\hat{\mathbf{n}})\right]^* Y^{\beta}_{(J'M')ab}(\hat{\mathbf{n}}) = \delta_{JJ'}\delta_{MM'}\delta_{\alpha\beta}, \tag{93}$$

for $\alpha, \beta = \{L, VE, VB, TE, TB\}$. Note again that although the transformations between the OAM and $L/VE/VE/TB/TE$ bases for the tensor spherical harmonics resemble those for the transformations between the analogous bases for TAM waves, there are important sign differences.

In terms of the tensor spherical harmonics, the TAM waves $\Psi^{k,\alpha}_{(JM)ab}(\mathbf{x})$ can be written,

$$
\begin{aligned}
\Psi^{k,L}_{(JM)ab}(\mathbf{x}) =& -\frac{1}{2}\left(j_J(kr)+3g_J(kr)\right)Y^{L}_{(JM)ab}(\hat{\mathbf{n}}) - \sqrt{3J(J+1)}f_J(kr)\,Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) \\
& - \frac{\sqrt{3J(J+1)(J-1)(J+2)}}{2}\frac{j_J(kr)}{(kr)^2}Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{k,VE}_{(JM)ab}(\mathbf{x}) =& -\sqrt{3J(J+1)}f_J(kr)\,Y^{L}_{(JM)ab}(\hat{\mathbf{n}}) - \left(j_J(kr)+2g_J(kr)+2f_J(kr)\right)Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) \\
& - \sqrt{(J-1)(J+2)}\left(f_J(kr)+2\frac{j_J(kr)}{(kr)^2}\right)Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{k,TE}_{(JM)ab}(\mathbf{x}) =& -\frac{\sqrt{3J(J+1)(J-1)(J+2)}}{2}\frac{j_J(kr)}{(kr)^2}Y^{L}_{(JM)ab}(\hat{\mathbf{n}}) - \sqrt{(J-1)(J+2)}\left(f_J(kr)+2\frac{j_J(kr)}{(kr)^2}\right)Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) \\
& - \frac{1}{2}\left(-j_J(kr)+g_J(kr)+4f_J(kr)+6\frac{j_J(kr)}{(kr)^2}\right)Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{k,VB}_{(JM)ab}(\mathbf{x}) =& -i\left(j'_J(kr)-\frac{j_J(kr)}{kr}\right)Y^{VB}_{(JM)ab}(\hat{\mathbf{n}}) - i\sqrt{(J-1)(J+2)}\frac{j_J(kr)}{kr}Y^{TB}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{k,TB}_{(JM)ab}(\mathbf{x}) =& -i\sqrt{(J-1)(J+2)}\frac{j_J(kr)}{kr}Y^{VB}_{(JM)ab}(\hat{\mathbf{n}}) - i\left(j'_J(kr)+2\frac{j_J(kr)}{kr}\right)Y^{TB}_{(JM)ab}(\hat{\mathbf{n}}).
\end{aligned}
\tag{94}
$$

Here we have introduced radial profiles,

$$
f_J(x) \equiv \frac{d}{dx}\frac{j_J(x)}{x}, \qquad \text{and} \qquad g_J(x) \equiv -j_J(x) - 2f_J(x) + (J-1)(J+2)\frac{j_J(x)}{x^2}. \tag{95}
$$

The components proportional to the transverse $Y^{TE}_{(JM)ab}(\hat{\mathbf{n}})$ and $Y^{TB}_{(JM)ab}(\hat{\mathbf{n}})$ harmonics are projections onto the 2-sphere. These are the components of principal interest for angular measurements on the sky. Eq. (94) shows that the different tensor TAM waves are not everywhere orthogonal, even though they are orthonormal, although they do become asymptotically orthogonal in the $kr \gg 1$ limit. Both the VB and TB tensor TAM waves have projections onto the VB and TB tensor spherical harmonics. The L, VE, and TE TAM waves have projections onto the L, VE, and TE tensor spherical harmonics. The phases in our definitions of the tensor spherical harmonics are chosen so that a rotation of a TE (VB) mode by 45° (90°) about $\hat{\mathbf{n}}$ produces a TB (VB) mode.

6. *Plane-wave expansion for traceless tensor fields*

We now determine the transformation between the tensor plane-wave basis and the tensor TAM-wave bases. We start with the OAM basis. The $\Psi^{l,k}_{(JM)ab}$ constitute a complete basis, and we can write

$$
\hat{\varepsilon}_{ab}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{lJM} 4\pi i^l B^l_{(JM)}(\hat{\mathbf{k}})\Psi^{l,k}_{(JM)ab}(\mathbf{x}) = \sum_{lJM} 4\pi i^l B^l_{(JM)}(\hat{\mathbf{k}}) j_l(kr) Y^l_{(JM)ab}(\hat{\mathbf{n}}). \tag{96}
$$

Here $\hat{\varepsilon}_{ab}(\mathbf{k})$ is a normalized polarization tensor for the plane wave. We use orthonormality of the $Y^l_{(JM)a}(\hat{\mathbf{n}})$ to obtain the expansion coefficients,

$$
B^l_{(JM)}(\hat{\mathbf{k}}) = \hat{\varepsilon}^{ab}(\mathbf{k}) Y^{l\,*}_{(JM)ab}(\hat{\mathbf{k}}). \tag{97}
$$

We can similarly expand in terms of $\Psi^{k,\alpha}_{(JM)ab}$, for $\alpha = L, VE, VB, TE, TB$ modes, or helicity modes, as

$$
\hat{\varepsilon}_{ab}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\alpha}\sum_{JM} 4\pi i^J B^{\alpha}_{(JM)}(\hat{\mathbf{k}})\Psi^{k,\alpha}_{(JM)ab}(\mathbf{x}), \qquad \hat{\varepsilon}_{ab}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\lambda=0,\pm1,\pm2}\sum_{JM} 4\pi i^J B^{\lambda}_{(JM)}(\hat{\mathbf{k}})\Psi^{k,\lambda}_{(JM)ab}(\mathbf{x}), \tag{98}
$$

where the expansion coefficients are

$$
B^{\alpha}_{(JM)}(\hat{\mathbf{k}}) = \hat{\varepsilon}^{ab}(\mathbf{k}) Y^{\alpha\,*}_{(JM)ab}(\hat{\mathbf{k}}), \qquad B^{\lambda}_{(JM)}(\hat{\mathbf{k}}) = \hat{\varepsilon}^{ab}(\mathbf{k}) Y^{\lambda\,*}_{(JM)ab}(\hat{\mathbf{k}}). \tag{99}
$$

Here, the spin-2 tensor spherical harmonics are defined as $Y^{\lambda=\pm2}_{(JM)ab} = 2^{-1/2}\left[Y^{TE}_{(JM)ab} \pm iY^{TB}_{(JM)ab}\right]$, $Y^{\lambda=\pm1}_{(JM)ab} = 2^{-1/2}\left[Y^{VE}_{(JM)ab} \pm iY^{VB}_{(JM)ab}\right]$, and $Y^{\lambda=0}_{(JM)ab} = Y^{L}_{(JM)ab}$. These are related to the spin-2 spherical harmonics ${}_{\lambda}Y_{(JM)}(\hat{\mathbf{n}})$ [26, 27] by

$$\hat{\varepsilon}^{ab}_{\lambda'}(\mathbf{k})Y^{\lambda}_{(JM)ab}(\hat{\mathbf{k}}) = {}_{-\lambda}Y_{(JM)}\left(\hat{\mathbf{k}}\right)\delta_{\lambda\lambda'}, \qquad \text{for} \qquad \lambda,\lambda' = 0,\pm1,\pm2. \tag{100}$$

Here, $\hat{\varepsilon}^{ab}_{\lambda}(\mathbf{k})$, for $\lambda = 0,\pm1,\pm2$, are the polarization tensors for a tensor plane wave with wavevector $\mathbf{k}$ and helicity λ. This equation defines our phase convention for $\hat{\varepsilon}^{ab}_{\lambda}$. In terms of basis vectors in spherical coordinates, these are defined as

$$\hat{\varepsilon}^{ab}_{\pm1} = \frac{1}{\sqrt{2}}\left[\hat{\varepsilon}^{a}_{\pm1}\hat{n}^{b} + \hat{\varepsilon}^{b}_{\pm1}\hat{n}^{a}\right], \qquad \hat{\varepsilon}^{ab}_{\pm2} = -\hat{\varepsilon}^{a}_{\pm1}\hat{\varepsilon}^{b}_{\pm1}, \qquad \hat{\varepsilon}^{ab}_{0} = \sqrt{\frac{3}{2}}\left(\frac{1}{3}\delta_{ab} - \hat{n}^{a}\hat{n}^{b}\right), \tag{101}$$

where $\hat{\varepsilon}^{a}_{0}$ and $\hat{\varepsilon}^{a}_{\pm1}$ are defined in Eq. (53).

7. *Expansion of tensor fields and power spectra*

An arbitrary symmetric traceless tensor field $h_{ab}(\mathbf{x})$ can be expanded in the orbital-angular-momentum basis by

$$h_{ab}(\mathbf{x}) = \sum_{JM}\sum_{l=J,J\pm1,J\pm2}\int\frac{k^2dk}{(2\pi)^3}h^{l}_{(JM)}(k)4\pi i^{l}\Psi^{l,k}_{(JM)ab}(\mathbf{x}), \tag{102}$$

with expansion coefficients,

$$h^{l}_{(JM)}(k) = \int d^3\mathbf{x}\, h^{ab}(\mathbf{x})\left[4\pi i^{l}\Psi^{l,k}_{(JM)ab}(\mathbf{x})\right]^{*}. \tag{103}$$

These can also be written as tensor-spherical-harmonic transforms [30],

$$T^{l,k}_{(JM)} = \int d^2\hat{\mathbf{k}}\,\tilde{T}^{ab}(\mathbf{k})Y^{l\,*}_{(JM)ab}(\hat{\mathbf{k}}), \tag{104}$$

of the tensor Fourier amplitudes. Again, similar relations hold for the $L/VE/VB/TE/TB$ and helicity bases as well.

The expansion coefficients for the $L/VE/VB/TE/TB$ basis can also be re-written, by integrating by parts, as

$$\begin{aligned} h^{\alpha}_{(JM)}(k) &= \int d^3\mathbf{x}\, h^{ab}(\mathbf{x})\left[4\pi i^{J}\Psi^{\alpha,k}_{(JM)ab}(\mathbf{x})\right]^{*} = \int d^3\mathbf{x}\, h^{ab}(\mathbf{x})\left[4\pi i^{J}T^{\alpha}_{ab}\Psi^{k}_{(JM)}(\mathbf{x})\right]^{*} \\ &= \int d^3\mathbf{x}\left[\left(T^{\alpha}_{ab}\right)^{\dagger}h^{ab}(\mathbf{x})\right]\left[4\pi i^{J}\Psi^{k}_{(JM)}(\mathbf{x})\right]^{*}, \end{aligned} \tag{105}$$

where T^{α}_{ab} are the operators defined in Eq. (82). Their hermitian conjugates are given by

$$\left(T^{L}_{ab}\right)^{\dagger} = -D_aD_b + \frac{1}{3}g_{ab}, \qquad \left(T^{VE}_{ab}\right)^{\dagger} = -M_{(a}D_{b)} + 2D_aD_b, \qquad \left(T^{VB}_{ab}\right)^{\dagger} = -K_{(a}D_{b)}, \tag{106}$$

$$\left(T^{TE}_{ab}\right)^{\dagger} = M_{(a}M_{b)} - K_{(a}K_{b)} - 4M_{(a}D_{b)} - 2D_{(a}M_{b)} + 8D_aD_b, \tag{107}$$

$$\left(T^{TB}_{ab}\right)^{\dagger} = M_{(a}K_{b)} + K_{(a}M_{b)} - 2D_{(a}K_{b)} - 4K_{(a}D_{b)}. \tag{108}$$

We can thus write the expansion coefficients for tensor TAM waves as coefficients of the scalar TAM wave for the following scalar functions:

$$h^L(\mathbf{x}) = \left[-D_aD_b + \frac{1}{3}g_{ab}\right]h^{ab}(\mathbf{x}) = \frac{1}{k^2}\left[\nabla_a\nabla_b - \frac{1}{3}g_{ab}\nabla^2\right]h_{ab}(\mathbf{x}), \tag{109}$$

$$h^{VE}(\mathbf{x}) = -\frac{1}{k^2}\left[\frac{1}{2}\epsilon_{ac}{}^{d}\nabla^c K_d\nabla_b + \frac{1}{2}\epsilon_{bc}{}^{d}\nabla^c K_d\nabla_a + 2\nabla_a\nabla_b\right]h^{ab}(\mathbf{x}), \tag{110}$$

$$h^{VB}(\mathbf{x}) = -\frac{i}{k}K_{(a}\nabla_{b)}h^{ab}(\mathbf{x}), \tag{111}$$

$$h^{TE}(\mathbf{x}) = \left[\frac{1}{k^2}\left\{-\frac{1}{2}\left(\epsilon_{ac}{}^{d}\nabla^c K_d\epsilon_{be}{}^{f}\nabla^e K_f + \epsilon_{bc}{}^{d}\nabla^c K_d\epsilon_{ae}{}^{f}\nabla^e K_f\right) + 2\epsilon_{ac}{}^{d}\nabla^c K_d\nabla_b + 2\epsilon_{bc}{}^{d}\nabla^c K_d\nabla_a\right.\right. \tag{112}$$

$$\left.\left. +\nabla_a\epsilon_{bc}{}^{d}\nabla^c K_d + \nabla_b\epsilon_{ac}{}^{d}\nabla^c K_d - 8\nabla_a\nabla_b\right\} - K_{(a}K_{b)}\right]h^{ab}(\mathbf{x}), \tag{113}$$

$$h^{TB}(\mathbf{x}) = -\frac{1}{2}\frac{i}{k}\left[\epsilon_{ac}{}^{d}\nabla^c K_d K_b + \epsilon_{bc}{}^{d}\nabla^c K_d K_a + K_a\epsilon_{bc}{}^{d}\nabla^c K_d + K_b\epsilon_{ac}{}^{d}\nabla^c K_d + 2\nabla_{(a}K_{b)} + 4K_{(a}\nabla_{b)}\right]h^{ab}(\mathbf{x}). \tag{114}$$

Here, $K_a = \epsilon_{abc}x^b\nabla^c$.

If $h_{ab}(\mathbf{x})$ is written in terms of longitudinal/vector/transverse-traceless parts, and if these have power spectra $P_L(k)$, $P_V(k)$, and $P_T(k)$, then

$$\left\langle\left[h^{\alpha}_{(JM)}(k)\right]^* h^{\beta}_{(J'M')}(k')\right\rangle = P_T(k)\delta_{JJ'}\delta_{MM'}\delta_{\alpha\beta}\frac{(2\pi)^3}{k^2}\delta_D(k-k'), \qquad \alpha,\beta = TE, TB. \tag{115}$$

$$\left\langle\left[h^{\alpha}_{(JM)}(k)\right]^* h^{\beta}_{(J'M')}(k')\right\rangle = P_V(k)\delta_{JJ'}\delta_{MM'}\delta_{\alpha\beta}\frac{(2\pi)^3}{k^2}\delta_D(k-k'), \qquad \alpha,\beta = VE, VB. \tag{116}$$

$$\left\langle\left[h^{L}_{(JM)}(k)\right]^* h^{L}_{(J'M')}(k')\right\rangle = P_L(k)\delta_{JJ'}\delta_{MM'}\delta_{\alpha\beta}\frac{(2\pi)^3}{k^2}\delta_D(k-k'). \tag{117}$$

For both the vector and transverse-traceless modes, the E modes and B modes have the same power spectra, a consequence of statistical homogeneity.

VI. CALCULATION OF LENSING POWER SPECTRA

In this Section we provide as an example of the TAM-wave formalism the calculation of lensing power spectra by density perturbations and by gravitational waves. We will reproduce results from previous work [31–35], which were obtained with the Fourier expansion. For clarity, we only take into account the weak-lensing contribution from the deflectors (density perturbations or gravitational waves) along the line of sight. However, the measured weak-lensing signal also contains other contributions. In particular, metric shear, gravitational-wave effects at the source location [31], and tidal alignment dominate the power spectrum for lensing by gravitational waves [36].

Our aim here will be to calculate the lensing deflection field,

$$\Delta_a(\hat{\mathbf{n}}) = \frac{\Pi_{ae}}{\eta_0 - \eta}\int_{\eta_0}^{\eta} d\eta' \left[\hat{n}_b h_{eb} - \frac{1}{2}(\eta' - \eta)\hat{n}_c\hat{n}_d\nabla_e h_{cd}\right]_{(\eta', (\eta_0-\eta')\hat{\mathbf{n}})}, \tag{118}$$

where $\Pi_{ab} \equiv g_{ab} - \hat{n}_a\hat{n}_b$ is the projection tensor onto the tangential plane. Thus, the deflection field has no radial component and can be viewed as a two-dimensional vector field on the two-sphere. Here, $h_{ab}(\eta, \mathbf{x})$ is a (rank-2) tensor metric perturbation evaluated at conformal time η.

The deflection field $\Delta_a(\hat{\mathbf{n}})$ on the two-sphere can be decomposed into gradient and curl components,

$$\Delta_a(\hat{\mathbf{n}}) = M_{\perp a}\varphi(\hat{\mathbf{n}}) + K_a\Omega(\hat{\mathbf{n}}), \tag{119}$$

where $M_{\perp a}$ and K_a are the two transverse-vector operators in Eq. (39), and $\varphi(\hat{\mathbf{n}})$ and $\Omega(\hat{\mathbf{n}})$ are projected lensing potentials. Here we will calculate the contribution $\Delta_a(\hat{\mathbf{n}})$ that arises from (a) a scalar TAM wave of angular momentum JM; (b) an E mode transverse-traceless TAM wave of JM; and (c) a B mode transverse-traceless TAM wave of JM. We will then be able to reproduce the power spectra $C_J^{\varphi\varphi}$ and $C_J^{\Omega\Omega}$ for density perturbations and gravitational waves that have been obtained earlier by considering individual Fourier modes. These power spectra are defined by,

$$\langle\varphi_{(JM)}\varphi^*_{(J'M')}\rangle = \delta_{JJ'}\delta_{MM'}C_J^{\varphi\varphi}, \qquad \langle\Omega_{(JM)}\Omega^*_{(J'M')}\rangle = \delta_{JJ'}\delta_{MM'}C_J^{\Omega\Omega}, \tag{120}$$

where $\varphi_{(JM)}$ and $\Omega_{(JM)}$ are spherical-harmonic coefficients for $\varphi(\hat{\mathbf{n}})$ and $\Omega(\hat{\mathbf{n}})$, respectively. The deflection field can be expanded in terms of vector spherical waves by,

$$\Delta_a(\hat{\mathbf{n}}) = \sum_{JM} \sqrt{J(J+1)} \left\{ \varphi_{(JM)} Y^E_{(JM)a}(\hat{\mathbf{n}}) + \Omega_{(JM)} Y^B_{(JM)a}(\hat{\mathbf{n}}) \right\}. \tag{121}$$

A. Scalar metric perturbation

Suppose we have a single TAM wave for a density perturbation. This is described by a metric perturbation,

$$h_{ab}(\eta', (\eta_0 - \eta')\hat{\mathbf{n}}) = 4\frac{9}{10} T^{\rm sca}(k) \frac{D_1(\eta')}{a(\eta')} \Phi^{k,p}_{(JM)} \Psi^k_{(JM)}((\eta_0 - \eta')\hat{\mathbf{n}}) g_{ab}, \tag{122}$$

where $D_1(\eta)$ is a linear-theory growth factor, $a(\eta)$ is the scale factor, and $\Phi^{k,p}_{(JM)}$ is the primordial amplitude of the Newtonian potential for wavenumber k and total angular momentum JM. We then have

$$\begin{aligned} -\frac{1}{2}(\eta_0 - \eta')\hat{n}^b \hat{n}^c \nabla_a h_{bc} &= -2(\eta' - \eta)\frac{9}{10} T^{\rm sca}(k) \frac{D_1(\eta')}{a(\eta')} \left[\nabla_a \Psi^k_{(JM)}((\eta_0 - \eta')\hat{\mathbf{n}}) \right] \\ &= 2i(\eta' - \eta)\frac{9}{10} T^{\rm sca}(k) \frac{D_1(\eta')}{a(\eta')} k \Psi^{k,L}_{(JM)a}((\eta_0 - \eta')\hat{\mathbf{n}}). \end{aligned} \tag{123}$$

Using Eq. (46), which provides the projection of $\Psi^{k,L}_{(JM)}(\mathbf{x})$ onto the plane normal to $\hat{n}^a$, we find that this single kJM mode of the scalar field gives rise to a deflection field,

$$\Delta^{\rm sca}_a(\hat{\mathbf{n}}) = \sqrt{J(J+1)} \Phi^{k,p}_{(JM)} F^{\rm sca}_J(k) Y^E_{(JM)a}(\hat{\mathbf{n}}), \tag{124}$$

with

$$F^{\rm sca}_J(k) = \frac{9}{5} T^{\rm sca}(k) \int_{\eta_0}^{\eta} d\eta' \frac{\eta' - \eta}{(\eta_0 - \eta)(\eta_0 - \eta')} \frac{D_1(\eta')}{a(\eta')} j_J(k(\eta_0 - \eta')). \tag{125}$$

We thus see that a given TAM wave of JM gives rise only to spherical harmonics in the deflection field of the same JM. The absence of a curl (B) mode is a consequence of the fact that the longitudinal TAM wave $\Psi^L_{(JM)a}(\mathbf{x})$ has no projection onto $Y^B_{(JM)}(\hat{\mathbf{n}})$ [cf. Eq. (46)]. We can equivalently conclude that this particular kJM TAM mode of the potential Φ gives rise to a spherical-harmonic coefficient $a^E_{(JM)}(k) = \Phi^{k,p}_{(JM)} F^{\rm sca}_J(k)$. The E mode deflection-angle power spectrum from the complete random field is then given by summing,

$$C^{EE}_J = J(J+1) C^{\varphi\varphi}_J \frac{2}{\pi} \int k^2 \, dk \, P_\Phi(k) \left| F_J(k) \right|^2, \tag{126}$$

over all k modes with this JM, in agreement with results obtained by summing over Fourier waves, rather than TAM waves.

B. Tensor Metric Perturbations

The TAM formalism will have more power, however, for tensor metric perturbations. So consider now a TAM wave,

$$h_{ab}(\eta', (\eta_0 - \eta')\hat{\mathbf{n}}) = h^{k,X}_{(JM)} T(k, \eta') \Psi^{k,X}_{(JM)ab}((\eta_0 - \eta')\hat{\mathbf{n}}), \tag{127}$$

of a transverse-traceless metric perturbation. Here, we will take X to be either E or B (although we could have alternatively considered $\lambda = \pm 2$ modes in the helicity basis), $T(k, \eta')$ gives the time evolution of modes of wavenumber k, and $h^{k,X}_{(JM)}$ is the primordial amplitude of the mode. From Eq. (118), we will need to calculate the tangential projections of $\hat{n}^a \Psi^{k,X}_{(JM)ab}(\mathbf{x})$ and $\hat{n}^a \hat{n}^c \nabla_b \Psi^{k,X}_{(JM)ac}(\mathbf{x})$. Using the transformation in Eq. (88) between the OAM and

$L/VE/VB/TE/TB$ bases and the relations, Eq. (94), between the tensor TAM waves and the tensor spherical harmonics, one can show that

$$\hat{n}^a\Psi^l_{(JM)ab}(\mathbf{x}) = \begin{cases} \sqrt{\frac{J-1}{2J-1}} j_{J-2}(kr) Y^{J-1}_{(JM)b}(\hat{\mathbf{n}}), & l=J-2,\\ \sqrt{\frac{J-1}{2(2J+1)}} j_{J-1}(kr) Y^{J}_{(JM)b}(\hat{\mathbf{n}}), & l=J-1,\\ j_J(kr)\left(-\sqrt{\frac{(J+1)(2J+3)}{6(2J-1)(2J+1)}} Y^{J-1}_{(JM)b}(\hat{\mathbf{n}}) + \sqrt{\frac{J(2J-1)}{6(2J+1)(2J+3)}} Y^{J+1}_{(JM)b}(\hat{\mathbf{n}})\right), & l=J,\\ -\sqrt{\frac{J+2}{2(2J+1)}} j_{J+1}(kr) Y^{J}_{(JM)b}(\hat{\mathbf{n}}), & l=J+1,\\ -\sqrt{\frac{J+2}{2J+3}} j_{J+2}(kr) Y^{J+1}_{(JM)b}(\hat{\mathbf{n}}), & l=J+2. \end{cases} \tag{128}$$

It then follows that,

$$\begin{aligned} \hat{n}^a\Psi^B_{(JM)ab}(\mathbf{x}) &= \sqrt{\frac{(J-1)(J+2)}{2(2J+1)^2}}\left(j_{J-1}(kr)+j_{J+1}(kr)\right) Y^J_{(JM)b}(\hat{\mathbf{n}}) \\ &= -i\sqrt{\frac{(J-1)(J+2)}{2}} \frac{j_J(kr)\,(kr)}{kr} Y^B_{(JM)b}(\hat{\mathbf{n}}), \end{aligned} \tag{129}$$

which lies entirely in the plane normal to $\hat{\mathbf{n}}$, and

$$\begin{aligned} \hat{n}^a\Psi^E_{(JM)ab}(\mathbf{x}) &= \sqrt{\frac{(J-1)(J+1)(J+2)}{2\,(2J-1)^2\,(2J+1)}}\left(j_{J-2}(kr)+j_J(kr)\right) Y^{J-1}_{(JM)b}(\hat{\mathbf{n}}) \\ &\quad - \sqrt{\frac{J\,(J-1)(J+2)}{2\,(2J+3)^2\,(2J+1)}}\left(j_J(kr)+j_{J+2}(kr)\right) Y^{J+1}_{(JM)b}(\hat{\mathbf{n}}) \\ &= \sqrt{\frac{(J-1)(J+1)(J+2)}{2(2J+1)}} \frac{j_{J-1}(kr)}{kr} Y^{J-1}_{(JM)b}(\hat{\mathbf{n}}) - \sqrt{\frac{J\,(J-1)(J+2)}{2(2J+1)}} \frac{j_{J+1}(kr)}{kr} Y^{J+1}_{(JM)b}(\hat{\mathbf{n}}). \end{aligned} \tag{130}$$

This vector has components both in the normal direction and in the tangent space. The projection onto the tangent space is

$$\Pi_{bc} n^a \Psi^E_{(JM)ac}(\mathbf{x}) = -\sqrt{\frac{(J-1)(J+2)}{2}} \frac{1}{kr}\left[j'_J(kr) + \frac{j_J(kr)}{kr}\right] Y^E_{(JM)b}(\hat{\mathbf{n}}). \tag{131}$$

The first term in Eq. (118) contributes

$$\Delta_a^{GW,X(1)}(\hat{\mathbf{n}}) = -h^{k,X}_{(JM)}\sqrt{\frac{(J-1)(J+2)}{2}} Y^X_{(JM)a}(\hat{\mathbf{n}}) \int_{\eta_0}^{\eta} d\eta' \frac{T(k,\eta')}{(\eta_0-\eta)\,k\,(\eta_0-\eta')} f^X_J\left(k(\eta_0-\eta')\right), \tag{132}$$

where the radial functions are

$$f^X_J(kr) = \begin{cases} i j_J(kr), & \text{for } X=B,\\ j'_J(kr) + \frac{j_J(kr)}{kr}, & \text{for } X=E. \end{cases} \tag{133}$$

We now turn to the second term in Eq. (118), that proportional to $\hat{n}_c\hat{n}_d\nabla_e h_{cd}$. Using $\nabla_a\hat{n}_b = (g_{ab}-\hat{n}_a\hat{n}_b)/r = \Pi_{ab}/r$, we can write

$$\begin{aligned} \hat{n}_c\hat{n}_d\nabla_e h_{cd} &= \nabla_e\left(\hat{n}_c\hat{n}_d h_{cd}\right) - \left(\nabla_e\hat{n}_c\right)\hat{n}_d h_{cd} - \hat{n}_c\left(\nabla_e\hat{n}_d\right)h_{cd} \\ &= \nabla_e\left(\hat{n}_c\hat{n}_d h_{cd}\right) - 2\left(\nabla_e\hat{n}_c\right)\hat{n}_d h_{cd} \\ &= \nabla_e\left(\hat{n}_c\hat{n}_d h_{cd}\right) - \frac{2}{(\eta_0-\eta')}\Pi_{ec}\hat{n}_c h_{cd}. \end{aligned} \tag{134}$$

Given that $\Pi_{ae}\Pi^e{}_c = \Pi_{ac}$, we see that the second term here is similar to what we calculated before. It thus contributes,

$$\Delta_a^{GW,X(2)}(\hat{\mathbf{n}}) = -h^{k,X}_{(JM)}\sqrt{\frac{(J-1)(J+2)}{2}} Y^X_{(JM)a}(\hat{\mathbf{n}}) \int_{\eta_0}^{\eta} d\eta' \frac{(\eta'-\eta)\,T(k,\eta')}{(\eta_0-\eta)\,k\,(\eta_0-\eta')^2} f^X_J\left(k(\eta_0-\eta')\right), \tag{135}$$

to the deflection field. Now consider the first term in Eq. (134). We have already seen for the B mode that $\hat{n}^a \Psi^B_{(JM)ab}(\mathbf{x})$ is perpendicular to $\hat{n}^b$. Thus, $\hat{n}^a \hat{n}^b \Psi^B_{(JM)ab}(\mathbf{x}) = 0$, and this term does not contribute to the curl (B mode). Using Eq. (130) and

$$\hat{n}^a Y^l_{(JM)a}(\hat{\mathbf{n}}) = \begin{cases} \sqrt{\frac{J}{2J+1}} Y_{(JM)}(\hat{\mathbf{n}}), & l = J-1, \\ 0, & l = J, \\ -\sqrt{\frac{J+1}{2J+1}} Y_{(JM)}(\hat{\mathbf{n}}), & l = J+1, \end{cases} \tag{136}$$

we find

$$\hat{n}^a \hat{n}^b \Psi^E_{(JM)ab}(\mathbf{x}) = \sqrt{\frac{(J+2)!}{2(J-2)!}} \frac{j_J\left(kr\right)}{\left(kr\right)^2} Y_{JM}(\hat{\mathbf{n}}). \tag{137}$$

When we now take the gradient $-M_{\perp a} = r \Pi_{ab} \nabla^b$ of this in the tangential plane, the operator acts only on the spherical harmonic. Using $M_{\perp a} Y_{(JM)} = \sqrt{J(J+1)} Y^E_{(JM)a}$, we find,

$$\Pi_{ab} \nabla^b \hat{n}^c \hat{n}^d \Psi^{TE}_{(JM)cd}(\mathbf{x}) = -\sqrt{\frac{(J-1)\,(J+2)}{2}} \frac{J\,(J+1)}{k^2 \left(\eta_0 - \eta'\right)^3} j_J\left(k\left(\eta_0 - \eta'\right)\right) Y^E_{(JM)a}(\hat{\mathbf{n}}). \tag{138}$$

We thus find that this term contributes

$$\Delta_a^{GW,E(3)}\left(\hat{\mathbf{n}}\right) = \frac{1}{2}\sqrt{\frac{(J-1)\,(J+2)}{2}} J\,(J+1)\, Y^E_{(JM)a}\left(\hat{\mathbf{n}}\right) \int_{\eta_0}^{\eta} d\eta' \frac{\left(\eta' - \eta\right) T\left(k, \eta'\right)}{\left(\eta_0 - \eta\right) k^2 \left(\eta_0 - \eta'\right)^3} j_J\left(k\left(\eta_0 - \eta'\right)\right). \tag{139}$$

In summary, a single B- or E-mode TAM wave contributes a deflection field,

$$\Delta_a^{\mathrm{GW}}\left(\hat{\mathbf{n}}\right) = h^{k,X}_{(JM)} \sqrt{J(J+1)} F_J^{\mathrm{GW},X}(k) Y^X_{(JM)a}(\hat{\mathbf{n}}), \tag{140}$$

for $X = \{E, B\}$, with

$$F_J^{\mathrm{GW},B}(k) = -i\sqrt{\frac{(J-1)\,(J+2)}{2J(J+1)}} \frac{1}{\eta_0 - \eta} \int_{\eta_0}^{\eta} d\eta' \frac{T\left(k, \eta'\right)}{k\left(\eta_0 - \eta'\right)^2} j_J\left(k\left(\eta_0 - \eta'\right)\right), \tag{141}$$

and

$$\begin{aligned} F_J^{\mathrm{GW},E}(k) = -&\sqrt{\frac{(J-1)\,(J+2)}{2J(J+1)}} \int_{\eta_0}^{\eta} d\eta' \frac{T\left(k, \eta'\right)}{k\left(\eta_0 - \eta'\right)^2} \\ &\times \left\{ \left[j_J'\left(k\left(\eta_0 - \eta'\right)\right) + \frac{j_J\left(k\left(\eta_0 - \eta'\right)\right)}{k(\eta_0 - \eta')} \right] - \frac{J(J+1)}{2} \frac{\eta' - \eta}{\eta_0 - \eta} \frac{j_J\left(k(\eta_0 - \eta')\right)}{k(\eta_0 - \eta')} \right\}. \end{aligned} \tag{142}$$

The power spectra are then obtained from

$$C_J^{XX} = \frac{2}{\pi} \int k^2\, dk\, \frac{P_h(k)}{2} \left| F_J^{\mathrm{GW},X}(k) \right|^2, \tag{143}$$

for $X = E, B$, by summing over all k modes with this JM.

VII. BOLTZMANN EQUATIONS FOR CMB FLUCTUATIONS

TAM waves can also be used to provide an alternative set of Boltzmann equations to calculate CMB power spectra. Our discussion here is preliminary; we leave the full calculation to future work [18]. There is some overlap, although not complete, with what we discuss here and work in Ref. [17], and also in Ref. [37].

The radiation perturbation $\Theta(\mathbf{x}, \hat{\mathbf{q}}; \eta)$ is most generally a function of position $\mathbf{x}$, the photon direction $\hat{\mathbf{q}}$, and conformal time η. This perturbation satisfies a Boltzmann equation, a partial differential equation in time, space, and in photon direction $\hat{\mathbf{q}}$. In the standard treatment [38], one considers a single Fourier mode $\Phi(\mathbf{x}, \eta) = \Phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$ of wavevector $\mathbf{k}$ of the gravitational potential (or of the gravitational-wave field). The spatial dependence of $\Theta(\mathbf{x}, \hat{\mathbf{q}}; \eta)$

must also then be $\propto e^{i\mathbf{k}\cdot\mathbf{x}}$. The $\hat{\mathbf{q}}$ dependence is, however, then expanded in spherical harmonics. Since the end result, the power spectrum C_J, is a rotational invariant, one generally then chooses $\mathbf{k} \parallel \hat{\mathbf{z}}$ so that the spherical-harmonic expansion for the $\hat{\mathbf{q}}$ dependence of $\Theta_{\mathbf{k}}(\hat{\mathbf{q}};\eta)$ becomes in practice an expansion in Legendre polynomials $P(\cos\theta_p) \propto Y_{(J0)}(\hat{\mathbf{q}})$.

Alternatively, though, the gravitational potential can be expanded $\Phi(\mathbf{x},\eta) = \sum_{kJM} \Phi^k_{(JM)}(\eta)\Psi^k_{(JM)}(\mathbf{x})$ in terms of scalar TAM waves $\Psi^k_{(JM)}(\mathbf{x}) = j_J(kx)Y_{(JM)}(\hat{\mathbf{x}})$ (or for tensor perturbations, in terms of tensor TAM waves). The most general radiation perturbation associated with this scalar perturbation can then be expanded in terms of states of TAM JM,

$$\Theta(\mathbf{x},\hat{\mathbf{q}};\eta) = \sum_{k,JM,ll'} \Theta^{k,JM}_{ll'}(\eta)\Xi^{k,JM}_{ll'}(\mathbf{x},\hat{\mathbf{q}}), \tag{144}$$

where the total-angular-momentum eigenfunctions (which are also eigenfunctions, of quantum numbers l and l', of $\mathbf{x}$ and $\hat{\mathbf{q}}$ angular momentum, respectively) are

$$\Xi^{k,JM}_{ll'}(\mathbf{x},\hat{\mathbf{q}}) = \sum_{mm'} \langle lml'm'|JM\rangle\, j_l(kx)Y_{(lm)}(\hat{\mathbf{x}})Y_{(l'm')}(\hat{\mathbf{q}}). \tag{145}$$

It now follows that the angular dependence of the observed radiation from a spherical wave with quantum numbers kJM will be proportional to $Y_{(JM)}(\hat{\mathbf{q}})$. We take the observer to be at the origin. We then note that the radial eigenfunctions $j_l(kr)$ all vanish at the origin unless $l=0$. Thus,

$$\begin{aligned}\Theta(\hat{\mathbf{x}}=0,\hat{\mathbf{q}};\eta) &= \sum_{ll'} \Theta^{k,JM}_{ll'}\Xi^{k,JM}_{ll'}(\mathbf{x}=0,\hat{\mathbf{q}}) = \Theta^{k,JM}_{0J}(\eta)\Xi^{k,JM}_{0J}(\mathbf{x}=0,\hat{\mathbf{q}})\\ &= \Theta^{k,JM}_{0J}\,\langle 00JM|JM\rangle\, Y_{(00)}(\hat{\mathbf{x}})j_0(0)Y_{(JM)}(\hat{\mathbf{q}}).\end{aligned} \tag{146}$$

In the TAM approach, therefore, calculation of the CMB temperature fluctuation boils down to calculation of $\Theta^{k,JM}_{0J}(\eta)$. The Boltzmann equation for this particular coefficient, however, will be coupled to those for all $\Theta^{k,JM}_{ll'}$. We thus trade the infinite tower of equations for the $l=0,1,2,\ldots$ coefficients $\Theta_l(k)$ for each wavenumber k for an infinite tower $l,l'=0,1,2,\ldots$ for the coefficients $\Theta^{k,JM}_{ll'}$ for a particular J. The advantage, though, is that each TAM wave of JM contributes only to C_J. Thus, the power spectrum C_J can be evaluated for a single J. There may also be conceptual advantages to this approach, even if there are no immediate numerical advantages.

VIII. CONCLUSIONS

In this paper we have obtained complete sets of basis functions, specified by their total angular momentum JM, for scalar, vector, and tensor fields on $\mathbb{R}^3$. We have written three such sets of basis functions, one in terms of orbital-angular-momentum states, one in terms of an $L/E/B$ or $L/VE/VB/TE/TB$ decomposition of the vector and tensor fields, and a third in terms of states of definite helicity. In the process, we have also shown how all five components of a rank-2 traceless symmetric tensor field, including the transverse-traceless components, can be written in terms of derivative operators acting on scalar fields, a result that may be useful for basis functions beyond those, based on spherical coordinates, that we have derived here.

We have shown how the projections of these three-dimensional vector and tensor fields onto the two-sphere yield the familiar E/B vector and tensor spherical harmonics. We found that an E mode on the two-sphere may arise from either a longitudinal vector or tensor mode or from E mode vector or tensor TAM waves. Conversely a B mode on the two-sphere is seen to arise from a projection of a vector or tensor B mode. We also generalized the two usual E/B tensor spherical harmonics to account for the three other possible polarizations of a traceless three-dimensional tensor field. We showed how the five TAM waves project onto these five tensor spherical harmonics.

A realization of a random scalar, vector, or tensor field is usually described as a collection of plane waves with amplitudes selected from some distribution. We have shown, however, that a random field can also be realized as a collection of TAM waves, and we have shown how the power spectra for these TAM-wave amplitudes are related to the power spectra for the more familiar plane-wave amplitudes. The advantage of TAM waves over the simpler but more naive outer product of the tensor spherical harmonics with radial wave functions is that such basis functions are not necessarily eigenfunctions of the Laplacian. They therefore do not follow a simple evolution during the linear regime at late times, and they are not normal modes during inflation.

The utility of TAM waves in cosmology is apparent given that most observations are performed on a spherical sky. Many calculations of cosmological observables, which are usually performed by considering the projection of a single Fourier mode onto a spherical sky, can be performed alternatively by considering a single TAM wave. The angular dependence of any observable on a TAM wave of total angular momentum JM must then be a scalar, vector, or tensor harmonic (depending on the observable) of that same JM. We showed, as one example, how the calculation of power spectra for the lensing-deflection field for gravitational lensing by density perturbations and gravitational waves is carried out in the TAM formalism, and we made preliminary remarks on the possible utility of the TAM formalism in numerical evaluation of CMB power spectra. The full power of the TAM formalism will be manifest most clearly, though, in the calculation of higher-order correlations (e.g., angular bispectra) in models with non-Gaussianity, particularly those involving vector and/or tensor fields. The basic idea here is that the Wigner-Eckart theorem guarantees that angular correlations of three TAM waves must be proportional to a Clebsch-Gordan coefficient, along with some prefactor that will depend on the tensorial nature of the waves. This will be presented in Ref. [30].

The development of the TAM-wave formalism requires considerable technical detail. However, once completed, understood, and mastered, it may facilitate the calculation of many cosmological observables.

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Appendix A: Divergence of the vector harmonics

1. Gradient of scalar TAM waves

First, we calculate the gradient,

$$\nabla_a \Psi^k_{(JM)}(\mathbf{x}) = \nabla_a j_J(kr) Y_{(JM)}(\hat{\mathbf{n}}), \tag{A1}$$

of scalar TAM waves. Our derivation is based on the Fourier transform. From Rayleigh's formula [Eq. (5)], we find the Fourier transform of scalar harmonics $j_J(kr)Y_{(JM)}(\hat{\mathbf{n}})$ as

$$\int d^3x\, j_J(kr) Y_{(JM)}(\hat{\mathbf{n}}) e^{-i\mathbf{q}\cdot\mathbf{x}} = 2\pi^2 (-i)^J \frac{\delta_D(q-k)}{q^2} Y_{(JM)}(\hat{\mathbf{q}}), \tag{A2}$$

and re-write the gradient as an inverse Fourier transform:

$$\begin{aligned} \nabla_a \Psi^k_{(JM)}(\mathbf{x}) &= \nabla_a \int \frac{d^3q}{(2\pi^3)} \left[2\pi^2 (-i)^J \frac{\delta_D(q-k)}{q^2} Y_{(JM)}(\hat{\mathbf{q}}) \right] e^{i\mathbf{q}\cdot\mathbf{x}} \\ &= \sum_{lm} i^{l-J+1} \left[\int q^2 dq \frac{q}{q^2} j_l(qr) \delta_D(q-k) \right] \left[\int d^2\hat{\mathbf{q}}\, \hat{\mathbf{q}}_a Y_{(JM)}(\hat{\mathbf{q}}) Y^*_{(lm)}(\hat{\mathbf{q}}) \right] Y_{(lm)}(\hat{\mathbf{n}}). \end{aligned} \tag{A3}$$

The radial integral trivially reads

$$\int q^2 dq \frac{q}{q^2} j_l(qr) \delta_D(q-k) = k j_l(kr), \tag{A4}$$

and the angular integral can be written as a Gaunt integral by decomposing

$$\hat{n}_a = \sqrt{\frac{4\pi}{3}} \sum_{\bar{m}} Y_{(1\bar{m})}(\hat{\mathbf{n}}) (-1)^{\bar{m}} e_a^{-\bar{m}}. \tag{A5}$$

Then,

$$\begin{aligned} \int d^2\hat{\mathbf{q}}\, \hat{\mathbf{q}}_a Y_{(JM)}(\hat{\mathbf{q}}) Y^*_{(lm)}(\hat{\mathbf{q}}) &= \sqrt{\frac{4\pi}{3}} \sum_{\bar{m}} (-1)^{\bar{m}} e_a^{-\bar{m}} \int d^2\hat{\mathbf{q}} Y_{(1\bar{m})}(\hat{\mathbf{q}}) Y_{(JM)}(\hat{\mathbf{q}}) Y^*_{(lm)}(\hat{\mathbf{q}}) \\ &= \sqrt{\frac{4\pi}{3}} \sum_{\bar{m}} (-1)^{\bar{m}+m} e_a^{-\bar{m}} \sqrt{\frac{(3(2J+1)(2l+1)}{4\pi}} \begin{pmatrix} 1 & J & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & J & l \\ \bar{m} & M & -m \end{pmatrix} \\ &= \sqrt{\frac{4\pi}{3}} \sum_{\bar{m}} (-1)^{M} e_a^{-\bar{m}} \sqrt{\frac{3(2J+1)(2l+1)}{4\pi}} \begin{pmatrix} 1 & J & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & l & J \\ -\bar{m} & m & -M \end{pmatrix}. \end{aligned} \tag{A6}$$

Here,

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \equiv \frac{1}{\sqrt{2l_3+1}}(-1)^{l_1-l_2+m_3} \langle l_1 m_1 l_2 m_2 | l_3 m_3 \rangle \,, \tag{A7}$$

is the Wigner-3j symbol. Combining all, we find the gradient of the scalar TAM wave to be

$$\begin{aligned} \nabla_a \Psi^k_{(JM)}(\mathbf{x}) &= k \sum_{lm} i^{l-J+1} j_l(kr) \sum_{\bar{m}} (-1)^M e_a^{-\bar{m}} \sqrt{(2J+1)(2l+1)} \begin{pmatrix} l & 1 & J \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & l & J \\ -\bar{m} & m & -M \end{pmatrix} Y_{(lm)}(\hat{\mathbf{n}}) \\ &= k \sum_l i^{l-J+1} j_l(kr) \sqrt{\frac{2l+1}{2J+1}} \langle l010|J0\rangle \sum_{m\bar{m}} e_a^{-\bar{m}} \langle 1-\bar{m} lm|JM\rangle Y_{(lm)}(\hat{\mathbf{n}}) \\ &= k \left[\sqrt{\frac{J}{2J+1}} \Psi^{J-1}_{(JM)a}(\mathbf{x}) + \sqrt{\frac{J+1}{2J+1}} \Psi^{J+1}_{(JM)a}(\mathbf{x}) \right], \end{aligned} \tag{A8}$$

where we have used that the relevant Clebsch-Gordan coefficient is non-zero only for

$$\langle l010|J0\rangle = \begin{cases} \sqrt{\frac{l+1}{2l+1}}, & l = J-1, \\ -\sqrt{\frac{l}{2l+1}}, & l = J+1. \end{cases} \tag{A9}$$

2. Divergence of vector TAM waves

We can now derive the divergence, given in Eq. (27), of the vector TAM waves. We start with

$$\begin{aligned} r\nabla^a \Psi^{l,k}_{(JM)a}(\mathbf{x}) &= r\nabla^a \left[j_l(kr) \sum_{m,\bar{m}} \langle 1\bar{m} lm|JM\rangle Y_{(lm)}(\hat{\mathbf{n}}) e_a^{\bar{m}} \right] \\ &= \sum_{m,\bar{m}} \langle 1\bar{m} lm|JM\rangle r e_a^{\bar{m}} \nabla^a j_l(kr) Y_{(lm)}(\hat{\mathbf{n}}), \end{aligned} \tag{A10}$$

and then from Eq. (A8), we have

$$r\nabla^a \Psi^{l,k}_{(JM)a}(\mathbf{x}) = kr \sum_{m,\bar{m}} \langle 1\bar{m} lm|JM\rangle e_a^{\bar{m}} \left[\sqrt{\frac{l}{2l+1}} \Psi^{l-1}_{(lm)a}(\mathbf{x}) + \sqrt{\frac{l+1}{2l+1}} \Psi^{l+1}_{(lm)a}(\mathbf{x}) \right]. \tag{A11}$$

The sum can be simplified as

$$\begin{aligned} \sum_{m,\bar{m}} & \langle 1\bar{m} lm|JM\rangle e_a^{\bar{m}} \Psi^{l\pm1}_{(lm)a}(\mathbf{x}) = j_{l\pm1}(kr) \sum_{m,\bar{m}} \langle 1\bar{m} lm|JM\rangle e_a^{\bar{m}} \sum_{m',\bar{m}'} \langle 1, l\pm1; \bar{m}' m'|lm\rangle Y_{(l\pm1,m')}(\hat{\mathbf{n}}) e_a^{\bar{m}'} \\ &= j_{l\pm1}(kr) \sum_{m,\bar{m}} \sum_{m',\bar{m}'} \langle 1\bar{m} lm|JM\rangle \langle 1, l\pm1; \bar{m}' m'|lm\rangle Y_{(l\pm1,m')}(\hat{\mathbf{n}}) (-1)^{\bar{m}} \delta_{\bar{m},-\bar{m}'} \\ &= j_{l\pm1}(kr) \sum_{m,\bar{m}} \sum_{m'} (-1)^{\bar{m}} \langle 1\bar{m} lm|JM\rangle \langle 1, l\pm1; -\bar{m} m'|lm\rangle Y_{(l\pm1,m')}(\hat{\mathbf{n}}) \\ &= j_{l\pm1}(kr) \sum_{m,\bar{m}} \sum_{m'} \sqrt{(2J+1)(2l+1)} (-1)^{\bar{m}+M+m-1} \begin{pmatrix} 1 & l & J \\ \bar{m} & m & -M \end{pmatrix} \begin{pmatrix} 1 & l\pm1 & l \\ -\bar{m} & m' & -m \end{pmatrix} Y_{(l\pm1,m')}(\hat{\mathbf{n}}) \\ &= j_{l\pm1}(kr) \sum_{m,\bar{m}} \sum_{m'} \sqrt{(2J+1)(2l+1)} (-1)^{\bar{m}+-1+M+m} \begin{pmatrix} 1 & l & J \\ \bar{m} & m & -M \end{pmatrix} \begin{pmatrix} 1 & l & l\pm1 \\ \bar{m} & m & -m' \end{pmatrix} Y_{(l\pm1,m')}(\hat{\mathbf{n}}) \\ &= -j_{l\pm1}(kr) \sum_{m'} \sqrt{2J+1}\sqrt{2l+1} \left[\sum_{m,\bar{m}} \begin{pmatrix} 1 & l & J \\ \bar{m} & m & -M \end{pmatrix} \begin{pmatrix} 1 & l & l\pm1 \\ \bar{m} & m & -m' \end{pmatrix} \right] Y_{(l\pm1,m')}(\hat{\mathbf{n}}) \\ &= -j_{l\pm1}(kr) \sum_{m'} \sqrt{2J+1}\sqrt{2l+1} \frac{\delta_{J,l\pm1}\delta_{Mm'}}{2J+1} Y_{(l\pm1,m')}(\hat{\mathbf{n}}) = -j_{l\pm1}(kr) \sqrt{\frac{2l+1}{2J+1}} \delta_{J,l\pm1} Y_{(l\pm1,M)}(\hat{\mathbf{n}}), \end{aligned} \tag{A12}$$

from which follows Eq. (27).

Appendix B: Divergence of tensor harmonics

In this Appendix we return to the use of our usual index notation for vectors and tensors so that vectors and tensors can be distinguished by the number of indices. The divergence of the tensor TAM waves is,

$$\frac{1}{k}\nabla^a\Psi^l_{(JM)ab}(\mathbf{x}) = \sum_{\tilde{m}m}\langle 2\tilde{m}lm|JM\rangle\frac{1}{k}\left(\nabla^a j_l(kr)Y_{(lm)}(\hat{\mathbf{x}})\right)t^{\tilde{m}}_{ab} = \sum_{\tilde{m}m}\langle 2\tilde{m}lm|JM\rangle\Psi^{L\,a}_{(lm)}(\mathbf{x})t^{\tilde{m}}_{ab}. \tag{B1}$$

From Eq. (A8), we calculate

$$\begin{aligned}\Psi^{l'\,a}_{(lm)}t^{\tilde{m}}_{ab} &= \sum_{m'\bar{m}}\sum_{\bar{m}_1\bar{m}_2}\langle 1\bar{m}l'm'|lm\rangle\Psi_{(l'm')}(\mathbf{x})\langle 1\bar{m}_1 1\bar{m}_2|2\tilde{m}\rangle e^{\bar{m}\,a}\left(e^{\bar{m}_1}_a e^{\bar{m}_2}_b\right)\\ &= \sum_{\bar{m}_2}\sum_{m'\bar{m}}(-1)^{\bar{m}}\langle 1,-\bar{m},1\bar{m}_2|2\tilde{m}\rangle\langle 1\bar{m}l'm'|lm\rangle\Psi_{(l'm')}(\mathbf{x})e^{\bar{m}_2}_b.\end{aligned} \tag{B2}$$

We first work out the sums over m, $\bar{m}$, and $\tilde{m}$. We trade Clebsch-Gordan coefficients for Wigner-3j symbols,

$$\begin{aligned}&\sum_{m\bar{m}\tilde{m}}(-1)^{\bar{m}}\langle 2\tilde{m}lm|JM\rangle\langle 1,-\bar{m},1\bar{m}_2|2\tilde{m}\rangle\langle 1\bar{m}l'm'|lm\rangle\\ &= \sum_{m\bar{m}\tilde{m}}(-1)^{1+l+l'+M+\bar{m}+m+\tilde{m}}\sqrt{5\,(2l+1)\,(2J+1)}\begin{pmatrix}2 & l & J\\ \tilde{m} & m & -M\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ -\bar{m} & \bar{m}_2 & -\tilde{m}\end{pmatrix}\begin{pmatrix}1 & l' & l\\ \bar{m} & m' & -m\end{pmatrix}\\ &= (-1)^{M+J+1}\sqrt{5\,(2l+1)\,(2J+1)}\sum_{m\bar{m}\tilde{m}}(-1)^{l+m+1+\bar{m}+2+\tilde{m}}\begin{pmatrix}1 & 2 & 1\\ \bar{m}_2 & -\tilde{m} & -\bar{m}\end{pmatrix}\begin{pmatrix}l & l' & 1\\ -m & m' & \bar{m}\end{pmatrix}\begin{pmatrix}l & 2 & J\\ m & \tilde{m} & -M\end{pmatrix}\\ &= (-1)^{M+J+1}\sqrt{5\,(2l+1)\,(2J+1)}\begin{Bmatrix}1 & l' & J\\ l & 2 & 1\end{Bmatrix}\begin{pmatrix}1 & l' & J\\ \bar{m}_2 & m' & -M\end{pmatrix},\end{aligned} \tag{B3}$$

where we have used the definition,

$$\begin{aligned}&\begin{Bmatrix}L_1 & L_2 & L_3\\ l_1 & l_2 & l_3\end{Bmatrix}\begin{pmatrix}L_1 & L_2 & L_3\\ M_1 & M_2 & M_3\end{pmatrix}\\ &\qquad\equiv \sum_{m_1m_2m_3}(-1)^{l_1+l_2+l_3+m_1+m_2+m_3}\begin{pmatrix}L_1 & l_2 & l_3\\ M_1 & m_2 & -m_3\end{pmatrix}\begin{pmatrix}l_1 & L_2 & l_3\\ -m_1 & M_2 & m_3\end{pmatrix}\begin{pmatrix}l_1 & l_2 & L_3\\ m_1 & -m_2 & M_3\end{pmatrix},\end{aligned} \tag{B4}$$

for the Wigner-6j symbol. We are then left with

$$\begin{aligned}&\sum_{\bar{m}_2m'}(-1)^{M+J+1}\sqrt{5\,(2l+1)\,(2J+1)}\begin{Bmatrix}1 & l' & J\\ l & 2 & 1\end{Bmatrix}\begin{pmatrix}1 & l' & J\\ \bar{m}_2 & m' & -M\end{pmatrix}\Psi_{(l'm')}(\mathbf{x})e^{\bar{m}_2}_a\\ &\qquad= -(-1)^{J+l'+1}\sqrt{5\,(2l+1)}\begin{Bmatrix}1 & l' & J\\ l & 2 & 1\end{Bmatrix}\sum_{\bar{m}_2m'}\langle 1\bar{m}_2l'm'|JM\rangle\Psi_{(l'm')}(\mathbf{x})e^{\bar{m}_2}_a\\ &\qquad= -(-1)^{J+l}\sqrt{5\,(2l+1)}\begin{Bmatrix}1 & l' & J\\ l & 2 & 1\end{Bmatrix}\Psi^{l'}_{(JM)a}(\mathbf{x}).\end{aligned} \tag{B5}$$

We evaluate the 6j-symbols explicitly and find

$$\begin{aligned}\frac{1}{k}\nabla^a\Psi^l_{(JM)ab}(\mathbf{x}) &= -\sqrt{5}(-1)^{J+l}\left(\begin{Bmatrix}1 & l-1 & J\\ l & 2 & 1\end{Bmatrix}\sqrt{l}\,\Psi^{l-1}_{(JM)b}(\mathbf{x}) + \begin{Bmatrix}1 & l+1 & J\\ l & 2 & 1\end{Bmatrix}\sqrt{l+1}\,\Psi^{l+1}_{(JM)b}(\mathbf{x})\right)\\ &= -\begin{cases}\sqrt{\frac{J-1}{2J-1}}\Psi^{J-1}_{(JM)b}(\mathbf{x}), & l=J-2,\\ \sqrt{\frac{J-1}{2(2J+1)}}\Psi^{J}_{(JM)b}(\mathbf{x}), & l=J-1,\\ \sqrt{\frac{(J+1)(2J+3)}{6(2J-1)(2J+1)}}\Psi^{J-1}_{(JM)b}(\mathbf{x}) + \sqrt{\frac{J(2J-1)}{6(2J+1)(2J+3)}}\Psi^{J+1}_{(JM)b}(\mathbf{x}), & l=J,\\ \sqrt{\frac{J+2}{2(2J+1)}}\Psi^{J}_{(JM)b}(\mathbf{x}), & l=J+1,\\ \sqrt{\frac{J+2}{2J+3}}\Psi^{J+1}_{(JM)b}(\mathbf{x}), & l=J+2.\end{cases}\end{aligned} \tag{B6}$$

Note that we automatically obtain eigenfunctions of total angular momentum as a consequence of acting with ∇_a, an irreducible-vector operator, on a tensor eigenfunction of total angular momentum.

Appendix C: Irreducible tensors

In this Appendix, we show that irreducible-tensor operators, when acting on a wavefunction, conserve the total angular momentum, even though the spin might change. To be more precise, let us consider the spherical harmonics $Y_{(JM)}(\hat{\mathbf{n}})$, which are eigenfunctions of orbital angular momentum $\mathbf{L}^2$ and L_z, for given JM; i.e.,

$$\mathbf{L}^2 Y_{(JM)} = J\left(J+1\right) Y_{(JM)}, \qquad L_z Y_{(JM)} = M Y_{(JM)}. \tag{C1}$$

Assume we have a group of irreducible-tensor operators $\mathcal{O}^l_m$, for $m = -l, -l+1, \ldots, l-1, l$, that transform as a representation of order l under rotations. There are $2l+1$ such operators, and $\mathcal{O}^l_m Y_{(JM)}$ is a spin-l object, a tensor wavefunction of higher rank. There will be spin operators S_a, for $a = x, y, z$, that act on such spin-l objects. The total angular momentum $J_a = L_a + S_a$ is then defined as the sum of the orbital angular momentum and the spin. We would like to prove that

$$\mathbf{J}^2 \mathcal{O}^l_m Y_{(JM)} = J\left(J+1\right) \mathcal{O}^l_m Y_{(JM)}, \qquad J_z \mathcal{O}^l_m Y_{(JM)} = M \mathcal{O}^l_m Y_{(JM)}. \tag{C2}$$

Consider a rotation $\mathcal{R} = e^{i\Theta^a J_a}$ acting on the Hilbert space of spin-l wavefunctions, where Θ^a parametrize rotation angles. The orbital angular momentum L_a generates rotations of configuration space, and the spin S_a generates rotations of the internal tensor space; i.e. mixing of the tensor components. Using the fact that L_a and S_a commute, we have

$$e^{i\Theta^a J_a} \mathcal{O}^l_m Y_{(JM)} = e^{i\Theta^a L_a} \left(e^{i\Theta^b S_b} \mathcal{O}^l_m \right) e^{-i\Theta^c L_c} e^{i\Theta^d L_d} Y_{(JM)}. \tag{C3}$$

First we note that spin operators rotate $\mathcal{O}^l_m$ contravariantly by acting on the tensor index m (if the tensor basis transforms covariantly),

$$e^{i\Theta^a S_a} \mathcal{O}^l_m = \sum_{m_1} \mathcal{O}^l_{m_1} \mathcal{D}^{l*}_{mm_1}\left(\mathcal{R}\right), \tag{C4}$$

where we have introduced the Wigner $\mathcal{D}$ rotation matrices $\mathcal{D}^l_{m_1 m_2}\left(\mathcal{R}\right)$. Meanwhile, the orbital angular momentum rotates $Y_{(JM)}$ covariantly,

$$e^{i\Theta^a L_a} Y_{(JM)} = \sum_{M_1} Y_{(JM_1)} \mathcal{D}^J_{M_1 M}\left(\mathcal{R}\right). \tag{C5}$$

Finally, $\mathcal{O}^l_m$, being irreducible-tensor operators, are rotated by the orbital-angular-momentum operators according to,

$$e^{i\Theta^a L_a} \mathcal{O}^l_{m_1} e^{-i\Theta^b L_b} = \sum_{m_2} \mathcal{O}^l_{m_2} \mathcal{D}^l_{m_2 m_1}\left(\mathcal{R}\right). \tag{C6}$$

From the above three equations, plus the unitarity of Wigner $\mathcal{D}$-matrices,

$$\sum_{m_1} \mathcal{D}^{l*}_{mm_1}\left(\mathcal{R}\right) \mathcal{D}^l_{m_2 m_1}\left(\mathcal{R}\right) = \delta_{mm_2}, \tag{C7}$$

we find

$$e^{i\Theta^a J_a} \mathcal{O}^l_m Y_{(JM)} = \sum_{M_1} \mathcal{O}^l_m Y_{(JM_1)} \mathcal{D}^J_{M_1 M}\left(\mathcal{R}\right). \tag{C8}$$

The derivation holds for any rotation $\mathcal{R}$, so we conclude that spin-l objects $\mathcal{O}^l_m Y_{(JM)}$ transform as a representation of order J under rotations, and hence must be eigenfunctions of total angular momentum, as described in Eq.(C2).

The proof is easily generalized if the $Y_{(JM)}$ are replaced by spherical harmonics of higher spin. Then the orbital angular momentum L_a is replaced by total angular momentum, while J_a will be the new total angular momentum which is obtained by adding the additional spin carried by irreducible-tensor operators O^l_m.

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