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Disorder-driven transition in a chain with power-law hopping

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We study a 1D system with a power-law quasiparticle dispersion $\propto |k|^\alpha \operatorname{sign} k$ in the presence of a short-range-correlated random potential and demonstrate that for $\alpha < 1/2$ it exhibits a disorder-driven quantum phase transition with the critical properties similar to those of the localisation transition near the edge of the band of a semiconductor in high dimensions, studied in Refs. [1] and [2]. Despite the absence of localisation in the considered 1D system, the disorder-driven transition manifests itself, for example, in a critical form of the disorder-averaged density of states. We confirm the existence of the transition by numerical simulations and find the critical exponents and the critical disorder strength as a function of α. The proposed system thus presents a convenient platform for numerical studies of the recently predicted unconventional high-dimensional localisation effects and has potential for experimental realisations in chains of ultracold atoms in optical traps.

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It is generally believed that increasing disorder strength in a conducting material in dimensions $d > 2$ leads to the Anderson localisation transition[3] with universal properties that depend only on the space dimensionality d.

However, as we have demonstrated recently[1, 2], in high dimensions $d > d_c$ (with $d_c = 4$ for conventional weakly doped semiconductors and $d_c = 2$ for Dirac semimetals) the phenomenology is significantly richer. Namely, a material with a power-law quasiparticle spectrum $\xi_{\mathbf{k}} \propto k^\alpha$ in the presence of short-range random potential in high dimensions $d > 2\alpha$ exhibits an unconventional disorder-driven quantum phase transition in the bottom of the band, that lies in a universality class distinct from the Anderson transition[1, 2]. Almost 30 years ago[4–6] the existence of such transition was suggested for the specific case $\alpha = 1, d = 3$ for 3D Dirac materials, that have been later extensively studied in the literature[1, 2, 7–11] establishing a consensus for the existence of this novel transition in 3D Dirac semimetals. Recently[1, 2] we have shown the existence of such an unconventional disorder-driven transition and studied its properties for arbitrary α and d, such that $d > 2\alpha$, demonstrating that it is a generic property of high dimensions and is not specific just to Dirac semimetals.

For materials in $d > 2$ dimensions in the symmetry classes that allow for localisation, this transition coincides with the localisation transition for the states in the bottom of the band[2]. However, it exists even if all states are always localised (e.g., in $d < 2$ dimensions) or if localisation is disallowed by symmetry [e.g., in 3D Weyl semimetals (WSMs) with sufficiently smooth disorder[12, 13]] and manifests itself, for example, in the critical behaviour of the density of states and conductivity.

So far such unconventional disorder-driven transition has not been observed experimentally. Perhaps the main obstacle in 3D Dirac materials is the long-range Coulomb nature of quenched disorder distinct from short-range random potential required to observe the critical behaviour of the conductivity[1] (although the transition in the density of states is still observable in the presence of Coulomb impurities). Another possible platform for studying high-dimensional localisation phenomena is periodically-kicked quantum-rotor systems, that can be mapped[14, 15] onto high-dimensional semiconductors with quadratic spectra. Such systems have been used to simulate 1D[16], 2D[17], and 3D[18, 19] Anderson localisation, but the case of higher dimensions still remains to be realised.

Numerical simulations in high dimensions may be extremely demanding in terms of computing power. For instance, quadratic spectrum of long-wave excitations, generic for lattice models with short-range hopping and inversion symmetry, corresponds to $d_c = 4$ and thus requires simulations in $d \geq 5$ dimensions, with the number of sites growing rapidly $\propto L^d$ as a function of the linear size L of the system.

In this paper we suggest and study a new playground for unconventional disorder-driven transitions, which is rather convenient for numerical simulations and is also currently accessible for experiments: 1D systems with long-range hopping.

Because the concept of high dimensions d is defined[1, 2] relative to the quasiparticle spectrum, the physics of high-dimensional disorder-driven transitions can be observed in any dimension d by appropriately designing the inter-site hopping; e.g., realising the quasiparticle spectra $\propto |k|^\alpha$ with $\alpha < d/2$. For instance, the spectrum $\propto |k|^\alpha$ with $\alpha < 1$ in $d = 1$ requires the inter-site hopping $\propto r^{-1-\alpha}$ which has already been realised in 1D chains[20, 21] and 2D arrays[22] of ultracold trapped ions.

Utilising fractional $\alpha < 1/2$ in 1D systems also allows one to compare the properties of the unconventional disorder-driven transition for $|\alpha - 1/2| \ll 1$ with theoretical predictions[1, 2] based on the RG approaches controlled by the small parameter $\varepsilon = d - 2\alpha$.

Model. In this paper we focus on a chiral (lacking reflection symmetry) 1D system described by the Hamiltonian

$$\hat{\mathcal{H}} = |k|^{\alpha} \operatorname{sign} k + U(x), \tag{1}$$

where k and x are momentum and coordinate and $U(x)$ is a short-range-correlated random potential.

We emphasise that "chiral system" hereinafter means a system without reflection symmetry of the quasiparticle dispersion ($k \to -k$) and should not be confused with the concept of a system in a chiral symmetry class[23, 24].

In some sense, such system is a 1D analogue of a 3D Weyl semimetal. Indeed, the quasiparticle spectrum consists of two bands; the conduction band ($k > 0$) and the valence band ($k < 0$), touching at the node $k = 0$. Quasiparticles in such a system cannot be localised due to the absence of backscattering; the velocity $v(k) = \alpha|k|^{\alpha-1}$ never changes sign. In principle, the spectrum of a realistic 1D system on a lattice contains an equal number of branches with left- and right- movers, due to the continuity and periodicity of the velocity $v(k)$ (analogously, Weyl semimetal has an even number of Weyl points[12]). However, for sufficiently smooth disorder elastic scattering of long-wavelength quasiparticles to states far from the node ($k = 0$) can be neglected and the quasiparticle dynamics near the node can be described by the model (1).

Quenched disorder potential $U(x)$ with zero mean and a symmetric distribution function preservers the $E \to -E$ symmetry of the quasiparticle spectrum and the density of states $\rho(E)$ (with the energy E measured from the node), which makes the chiral system particularly convenient for numerical studies of the disorder-driven transition near the node. In contrast, in a system with a single band (corresponding, e.g., to the spectrum $|k|^{\alpha}$) quenched disorder generically leads to the renormalisation of the band edge and to the formation of Lifshitz tails below the band[2], making it hard to define and to identify numerically the renormalised edge of the band.

Disorder-driven transition. For α slightly smaller than $1/2$ the effects of disorder can be analysed using a renormalisation-group (RG) approach, controlled by the small parameter $\varepsilon = 2\alpha - 1$. This RG, previously applied to Dirac materials[1, 2, 7, 9, 25–28] and to high-dimensional semiconductors in the orthogonal symmetry class[2], repeatedly removes the highest momenta from the system, renormalising its properties at lower momenta. Depending on whether or not the characteristic amplitude W of the random potential exceeds a critical value W_c, the dimensionless strength of disorder $\gamma \sim 1/(k\ell)$ flows to larger or smaller values under the RG, where ℓ is the (flowing) mean free path. Such behaviour of the renormalised disorder strength signifies a phase transition between the weak- and strong-disorder phases at $E = 0$. In this paper we verify numerically that such a transition persists at all $\alpha < 1/2$.

The behaviour of the low-energy density of states near a critical point has the generic scaling form (first proposed for 3D Dirac materials in Ref. [10])

$$\rho(E, W) = E^{\frac{d}{z}-1} \Phi\left[(W - W_c)/E^{\frac{1}{z\nu}}\right], \tag{2}$$

with ν and z being the correlation-length and dynamical critical exponents. In the weak-disorder phase ($W < W_c$) the density of states has the same energy dependency as free quasiparticles $\rho(E, W) \propto (W_c - W)^{-\nu\left(\frac{z}{\alpha}-1\right)} E^{\frac{1}{\alpha}-1}$, while for strong disorder ($W > W_c$) the density of states is smeared and thus energy independent: $\rho(E, W) \propto (W - W_c)^{(1-z)\nu}$. For finite energy E these two regimes are separated by a critical region near $W = W_c$ with $\rho(E) \propto E^{1/z-1}$.

The RG analysis similar to that of Refs. [1] and [2] for the model (1) in the one-loop approximation with the small parameter $\varepsilon = 2\alpha - 1$ yields

$$\nu = (1 - 2\alpha)^{-1}, \tag{3}$$

$$z = 1/2. \tag{4}$$

We emphasise that the dynamical exponent z, Eq. (4), is independent of α only in the first order in ε and only for the 1D chiral system under consideration. For arbitrary α, not necessarily close to $1/2$, the values of the critical exponents can be found numerically.

Numerical results. In what follows we present the results of the numerical simulations that demonstrate the

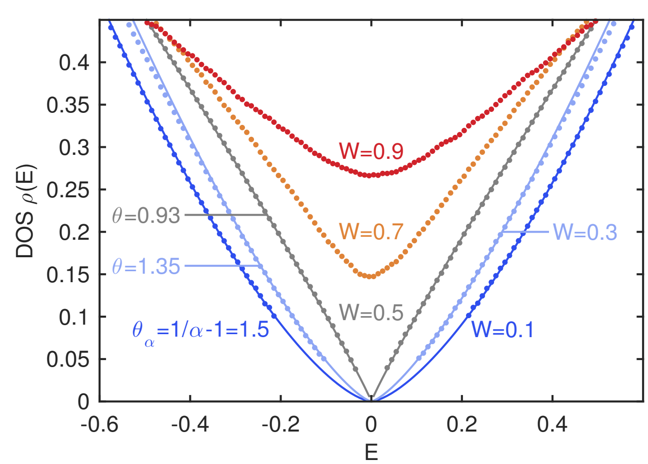

FIG. 1: (Colour online) The disorder-averaged density of states $\rho(E)$ vs. energy E for $\alpha = 0.4$ and various disorder amplitudes W. For subcritical disorder strength, $W < W_c \approx 0.5$, the density of states vanishes at $E = 0$, whereas for stronger disorder, $W > W_c$, the density of states is finite for all energies. At the critical disorder strength, $W = W_c \approx 0.5$, the density of states is linear in energy, in agreement with the analytical predictions based on one-loop RG calculations.

existence of the above described disorder-driven transition for $\alpha < 1/2$ and its absence for $\alpha > 1/2$. We also analyse the critical behaviour near the transition and obtain numerically the values of the critical exponents.

To simulate the model (1), we use its lattice version

$$\hat{\mathcal{H}} = \sum_{x,x'} J_{xx'}\hat{a}^\dagger_x \hat{a}_{x'} + \sum_x U_x \hat{a}^\dagger_x \hat{a}_x, \tag{5}$$

of finite size N with periodic boundary conditions, where distances (momenta) are measured in (inverse) lattice spacings; $J_{x,x'} = J_{x-x'} = \sum_k e^{ik(x-x')}|k|^\alpha \operatorname{sign} k$ is the inter-site hopping element (that we find numerically), for long distances $|x-x'| \gg 1$ given by the odd power-law function[35]

$$J_{x-x'} = i\,\frac{\operatorname{sign}(x-x')}{|x-x'|^{1+\alpha}}\,\frac{\Gamma(1+\alpha)}{2\pi}\,\sin\left[\frac{\pi}{2}(1+\alpha)\right]; \tag{6}$$

and U_x is the random disorder potential, uncorrelated on different sites and described by the Gaussian on-site distribution with standard deviation W, $P(U_x) = (W\sqrt{2\pi})^{-1}\exp[-U_x^2/(2W^2)]$. For the on-site-correlated disorder under consideration the ultraviolet momentum cutoff[2] $K_0 \sim 1$ is determined by the lattice spacing.

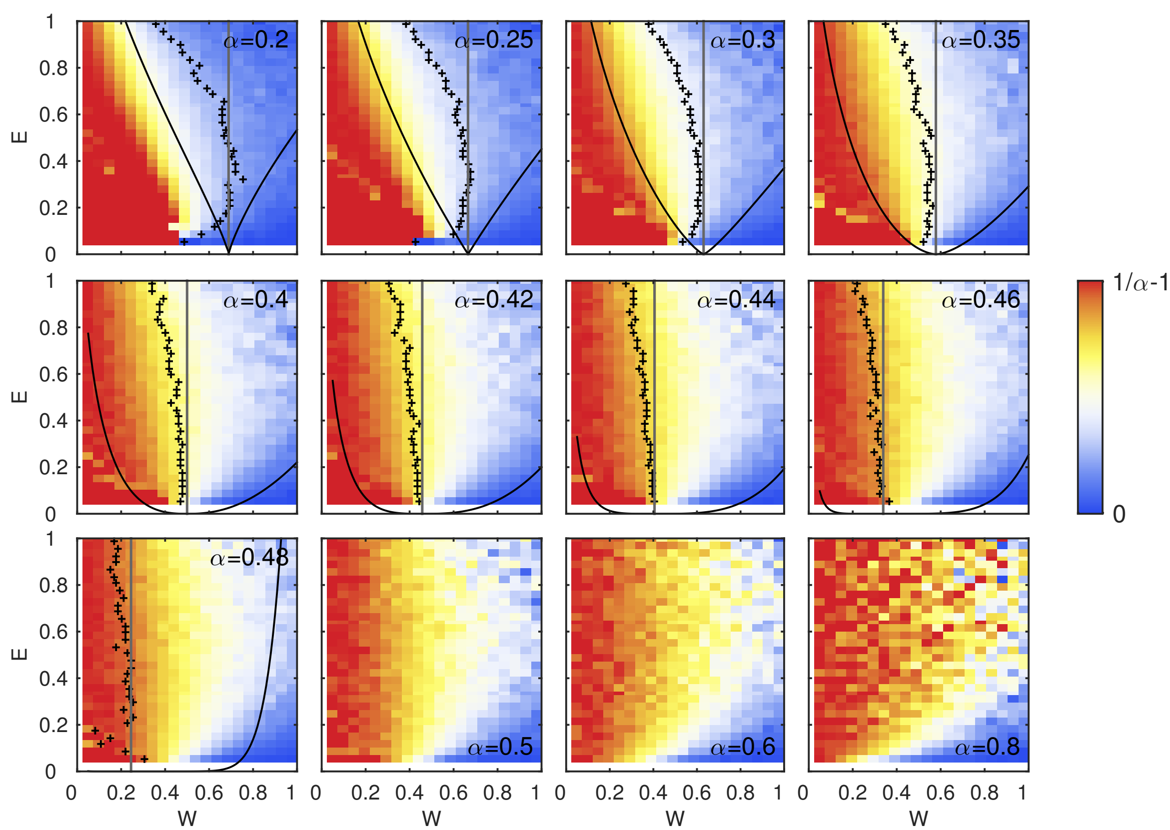

FIG. 2: (Colour online) Energy (E) vs. disorder amplitude (W) diagram for the low-energy density of states in a 1D chiral system with the quasiparticle spectrum $\xi_k = a|k|^\alpha \operatorname{sign} k$. The colour shows the exponent θ of the density of states $\rho(E) \propto E^\theta$ [see the colourbar and Eq. 7]. For $\alpha < 1/2$ the density of states has a critical point ($E = 0$, $W = W_c$) that separates the weak-disorder ($\theta_\alpha = \alpha^{-1} - 1$) and strong-disorder ($\theta = 0$) phases at $E = 0$. The transition disappears for $\alpha > 1/2$. The grey vertical line shows the analytical value of the critical disorder amplitude obtained assuming $0 < 1 - 2\alpha \ll 1$. The black crosses show the isoline $\theta = 1$. The black solid lines show the theoretical crossover energies $E = (|W - W_c|/W_c)^{z\nu}$ between the critical region ($\theta = 1/z - 1$) near $W \approx W_c$ and the weak- ($\theta_\alpha = \alpha^{-1} - 1$) and strong- ($\theta = 0$) disorder phases, with the exponents ν and z given by Eqs. (3) and (4).

For each value of α and disorder strength we use the exact diagonalisation method to obtain the spectrum of the system for 100 disorder realisations on a lattice with

$N = 4000$ sites and find the parameter

$$\theta(E,W) = \frac{\partial \ln \rho(E,W)}{\partial \ln E} \tag{7}$$

as a function of the random potential amplitude W and energy E, with the results summarised in Fig. 2.

For sufficiently small α we observe a disorder-driven quantum phase transition; the zero-energy density of states vanishes[36] for disorder amplitudes smaller than a critical value, $W < W_c$, and has a finite value otherwise [the dependency $\rho(E)$ for $\alpha = 0.4$ is shown in Fig. 1].

For sufficiently small α the density of states displays three regions with qualitatively different behaviours, that touch at a critical point $E = 0$, $W = W_c$, Fig. 2; (i) for low energies and disorder strengths the density of states has the energy dependency $\rho(E) \propto E^{1/\alpha - 1}$ of free quasiparticles with the spectrum $\propto |k|^\alpha \operatorname{sign} k$ ($\theta = 1/\alpha - 1$, red colour in Fig. 2); (ii) for low energies and sufficiently strong disorder the density of states is constant, $\theta = 0$, (blue colour in Fig. 2); (iii) near the critical disorder strength, $W = W_c$, there is an intermediate critical region with an almost constant intermediate value of θ. These results confirm the existence of the disorder-driven phase transition for sufficiently small α.

In order to accurately verify that the criticality disappears for $\alpha > 1/2$, we utilise the theoretical predictions of the one-loop RG analysis for the critical properties of the transition near $\alpha = 1/2$. Indeed, such RG analysis is controlled by the small parameter $\varepsilon = 2\alpha - 1$ and thus becomes exact when this parameter vanishes.

The one-loop RG analysis predicts the critical exponents (3) and (4) and the density of states $\rho(E) \propto E^{1/z-1} = E$ at the critical disorder strength ($W = W_c$), corresponding to $\theta = 1$, which can be used to accurately identify the critical point. In Fig. 2 the points with $\theta = 1$ are shown by black crosses that form a line which contains the critical point and is vertical at low energies.

The respective value of the critical disorder amplitude W_c matches well the result (shown by the grey vertical solid line in Fig. 2)

$$W_c = [\pi(1-2\alpha)/2]^{\frac{1}{2}}\, K_0^{\alpha - \frac{1}{2}} \tag{8}$$

of the one-loop RG calculation that we obtain under the assumption $0 < 1 - 2\alpha \ll 1$ with the ultraviolett momentum cutoff $K_0 = \pi$ (for details of the scheme of the perturbative RG calculation see Ref. [2]), even for α significantly below 0.5.

These results demonstrate the existence of the criticality of the density of states for $\alpha < 0.5$ and its disappearance for $\alpha > 0.5$. For all values of α in the interval $0.2 \ldots 0.5$ the critical properties of the transition are well described by the results of the one-loop perturbative RG analysis.

Conclusion and outlook. In summary, we have demonstrated that a chiral 1D system with the quasiparticle spectrum $|k|^\alpha \operatorname{sign} k$ with $\alpha < 1/2$ displays the phenomenology of high-dimensional disorder-driven phase transition. Although all the states in the proposed system are delocalised, it exhibits a disorder-driven transition that manifests itself in the density of states and is analogous to the localisation transition near the edge of the band of a high-dimensional semiconductor. In terms of its symmetries and the critical behaviour of observables, the system under consideration presents a 1D analogue of a 3D Weyl semimetal. The numerical values of the critical exponents and the critical disorder strength are well described by the results of a one-loop perturbative RG calculation. Such system presents a convenient platform for studying high-dimensional localisation physics and can be used to further investigate strong-disorder conduction in semimetals with delocalised states, the interplay of disorder with interactions, effects of various disorder symmetries on the transition, etc.

We emphasise, that the unconventional disorder-driven transition we studied is not specific to chiral 1D systems and can be observed (with different critical exponents) in any 1D chain with sufficiently long-range hopping of the excitations, e.g., corresponding to the even dispersion $\xi_k \propto |k|^\alpha$ with $\alpha < 1/2$. However, for systems with non-odd dispersions ξ_k the transition may be harder to observe numerically and experimentally due to the renormalisation of band edges or nodal points by disorder.

Implementing the specific 1D chiral model in experiments still remains a future research direction. Natural candidates are chains of trapped ions, since in those systems a power-law excitation spectrum $\propto |k|^\alpha$ with tunable $0 < \alpha < 1.5$ has already been demonstrated[20, 21]. However, ways to generate chiral excitations in these chains still have to be investigated. In principle, non-chiral power-law spectrum $\propto |k|^\alpha$ is also suitable for the observations of the high-dimensional localisation physics, but is less convenient for numerical simulations and is more sensitive to finite-size effects, that we expect to obscure the disorder-driven transition for non-chiral spectra for small numbers of ions (< 20) used in the current experiments. Another candidate for the observation of high-dimensional localisation physics is 2D arrays of ions in Penning-trap experiments[22], where a tunable power-law spectrum has been demonstrated for about 500 ions arranged in a triangular lattice. We leave the analysis of finite-size effects and possible realisations of chiral excitations in such systems for future studies.

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[35] We emphasise, that the model (5) with power-law hopping $J_{xx'} \propto |x-x'|^{-(1+\alpha)}\,\mathrm{sign}(x-x')$, that we consider, should not be confused with the power-law random banded matrix model[29], characterised by random Gaussian hopping with the power-law dependence on distance.
[36] We emphasise, that exponentially rare strong fluctuations of the disorder potential may lead to a finite density of states[30],[2] even for $W < W_c$, similarly to the formation of Lifshitz tails in semiconductors[31–34]. We believe, the accuracy of our numerical simulations is insufficient to observe such rare-regions contributions, if they exist for the system under consideration.